Machine Learning Department
School of Computer Science
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Perceptron

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Lecture 6
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Q&A

Q: We pick the best hyperparameters by learning on the training data and evaluating error on the validation error. For our final model, should we then learn from training + validation?

A: Yes.

Let's assume that {train-original} is the original training data, and {test} is the provided test dataset.

1. Split {train-original} into {train-subset} and {validation}.
2. Pick the hyperparameters that when training on {train-subset} give the lowest error on {validation}. Call these hyperparameters {best-hyper}.
3. Retrain a new model using {best-hyper} on {train-original} = {train-subset} ∪ {validation}.
4. Report test error by evaluating on {test}.

Alternatively, you could replace Step 1/2 with the following:

Pick the hyperparameters that give the lowest cross-validation error on {train-original}. Call these hyperparameters {best-hyper}.

Reminders

- Homework 2: Decision Trees
  - Out: Wed, Sep 05
  - Due: Wed, Sep 19 at 11:59pm
- Homework 3: KNN, Perceptron, Lin.Reg.
  - Out: Wed, Sep 19
  - Due: Wed, Sep 26 at 11:59pm

THE PERCEPTRON ALGORITHM

Perceptron: History

Imagine you are trying to build a new machine learning technique… your name is Frank Rosenblatt…and the year is 1957

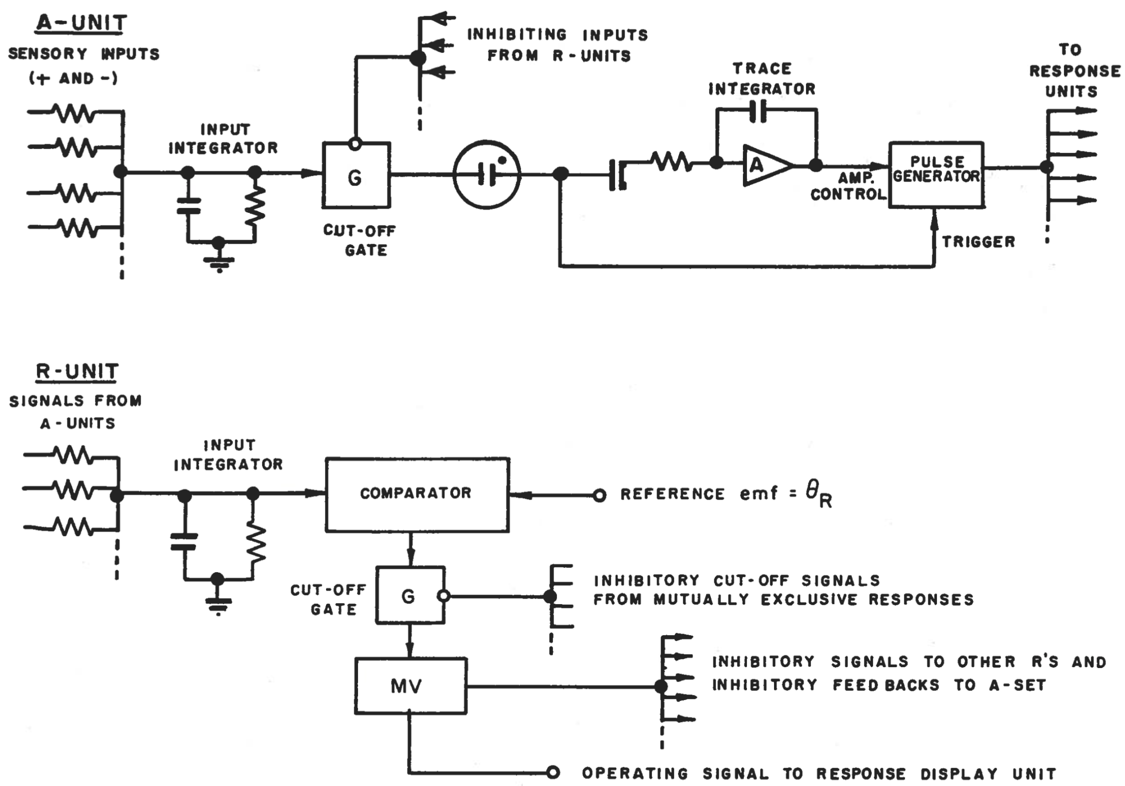

FIGURE 5
DESIGN OF TYPICAL UNITS

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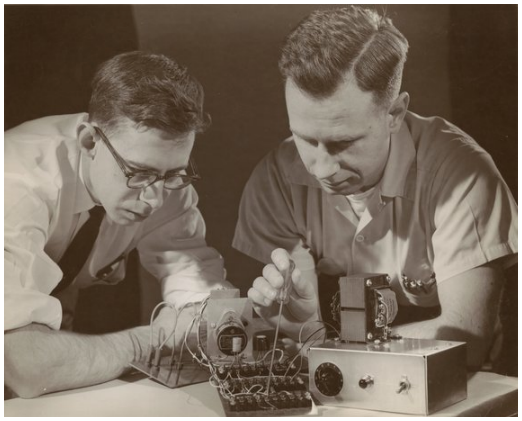
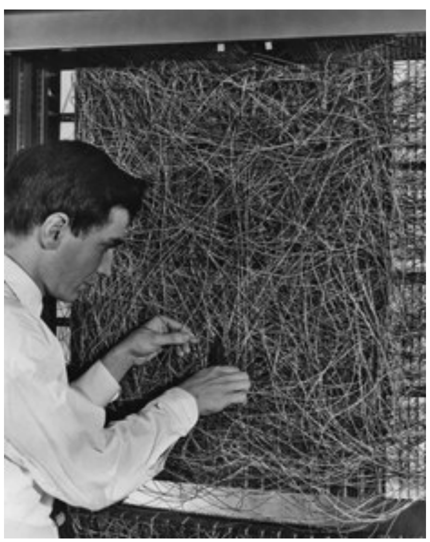
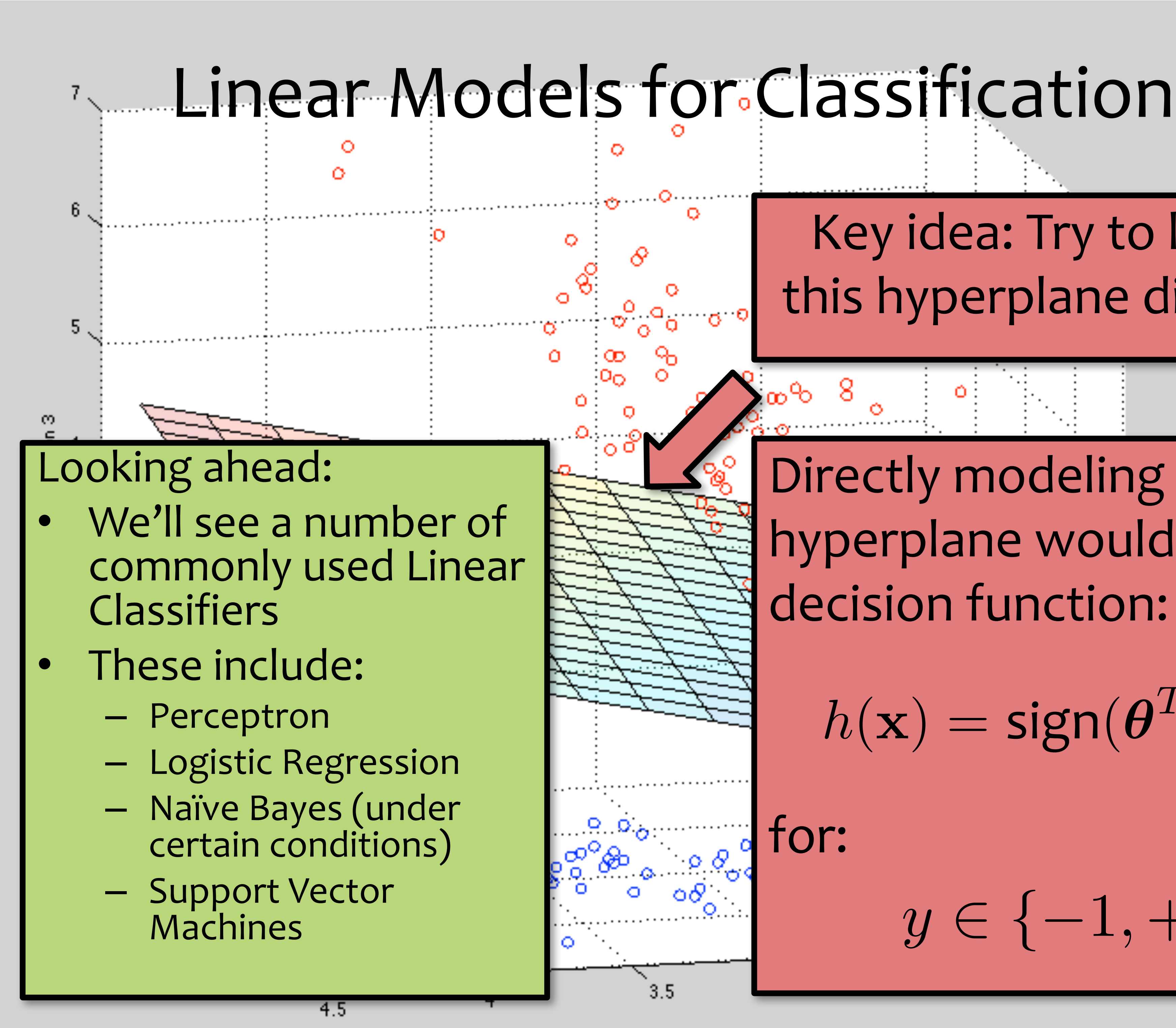




The New Yorker, December 6, 1958 P. 44

Talk story about the perceptron, a new electronic brain which hasn't been built, but which has been successfully simulated on the I.B.M. 704. Talk with Dr. Frank Rosenblatt, of the Cornell Aeronautical Laboratory, who is one of the two men who developed the prodigy; the other man is Dr. Marshall C. Yovits, of the Office of Naval Research, in Washington. Dr. Rosenblatt defined the perceptron as the first non-biological object which will achieve an organization o its external environment in a meaningful way. It interacts with its environment, forming concepts that have not been made ready for it by a human agent. If a triangle is held up, the perceptron's eye picks up the image & conveys it along a random succession of lines to the response units, where the image is registered. It can tell the difference betw. a cat and a dog, although it wouldn't be able to tell whether the dog was to theleft or right of the cat. Right now it is of no practical use, Dr. Rosenblatt conceded, but he said that one day it might be useful to send one into outer space to take in impressions for us.

Linear Models for Classification



Key idea: Try to learn this hyperplane directly

Looking ahead:

- We'll see a number of commonly used Linear Classifiers
- These include:
  - Perceptron
  - Logistic Regression
  - Naïve Bayes (under certain conditions)
  - Support Vector Machines

Directly modeling the hyperplane would use a decision function:

$$h(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

for:

$$y \in \{-1, +1\}$$

Geometry

In-Class Exercise

Draw a picture of the region corresponding to:

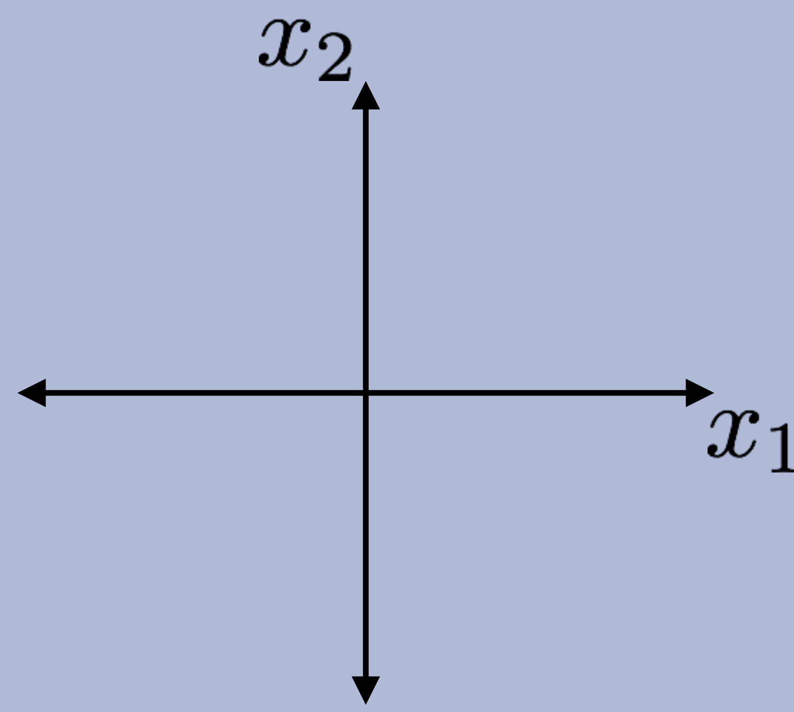
$$w_1 x_1 + w_2 x_2 + b > 0$$

where $w_1 = 2, w_2 = 3, b = 6$

Draw the vector

$\mathbf{w} = [w_1, w_2]$

Answer Here:

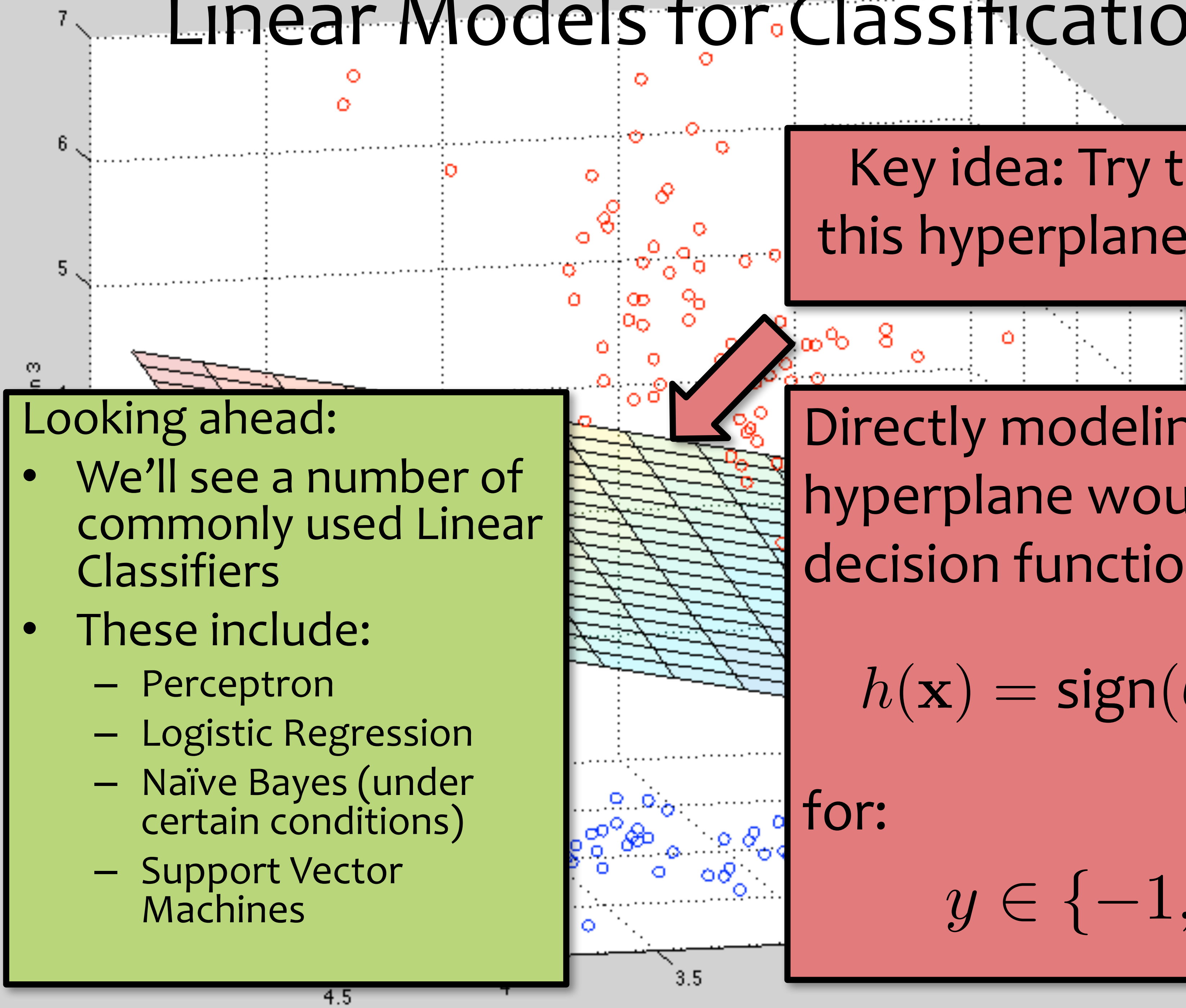


Visualizing Dot-Products

Chalkboard:

- vector in 2D
- line in 2D
- adding a bias term
- definition of orthogonality
- vector projection
- hyperplane definition
- half-space definitions

Linear Models for Classification



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Online vs. Batch Learning

Batch Learning

Learn from all the examples at once

Online Learning

Gradually learn as each example is received

Online Learning

Examples

1. **Stock market** prediction (what will the value of Alphabet Inc. be tomorrow?)
2. **Email** classification (distribution of both spam and regular mail changes over time, but the target function stays fixed - last year's spam still looks like spam)
3. **Recommendation** systems. Examples: recommending movies; predicting whether a user will be interested in a new news article
4. **Ad placement** in a new market

Slide adapted from Nina Balcan

Online Learning

For i = 1, 2, 3, … **:**

- **Receive** an unlabeled instance $\mathbf{x}^{(i)}$
- **Predict** $y' = h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)})$
- **Receive** true label $y^{(i)}$
- **Suffer loss** if a mistake was made, $y' \neq y^{(i)}$
- **Update** parameters $\boldsymbol{\theta}$

Goal:

- **Minimize** the number of **mistakes**

Perceptron

Chalkboard:

- (Online) Perceptron Algorithm
- Why do we need a bias term?
- Inductive Bias of Perceptron
- Limitations of Linear Models

Perceptron Algorithm: Example

Example: $(-1,2)$ − ✗

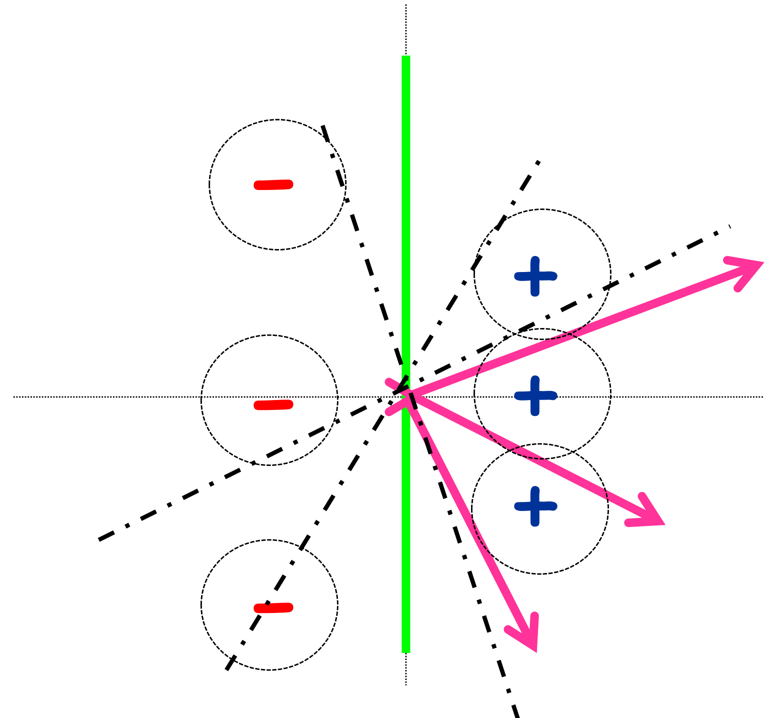
$(1,0)$ + ✓

$(1,1)$ + ✗

$(-1,0)$ − ✓

$(-1,-2)$ − ✗

$(1,-1)$ + ✓



Perceptron Algorithm: (without the bias term)

- Set t=1, start with all-zeroes weight vector w_1.
- Given example x, predict positive iff $w_t \cdot x \geq 0$.
- On a mistake, update as follows:
  - Mistake on positive, update $w_{t+1} \leftarrow w_t + x$
  - Mistake on negative, update $w_{t+1} \leftarrow w_t - x$

$w_1 = (0,0)$

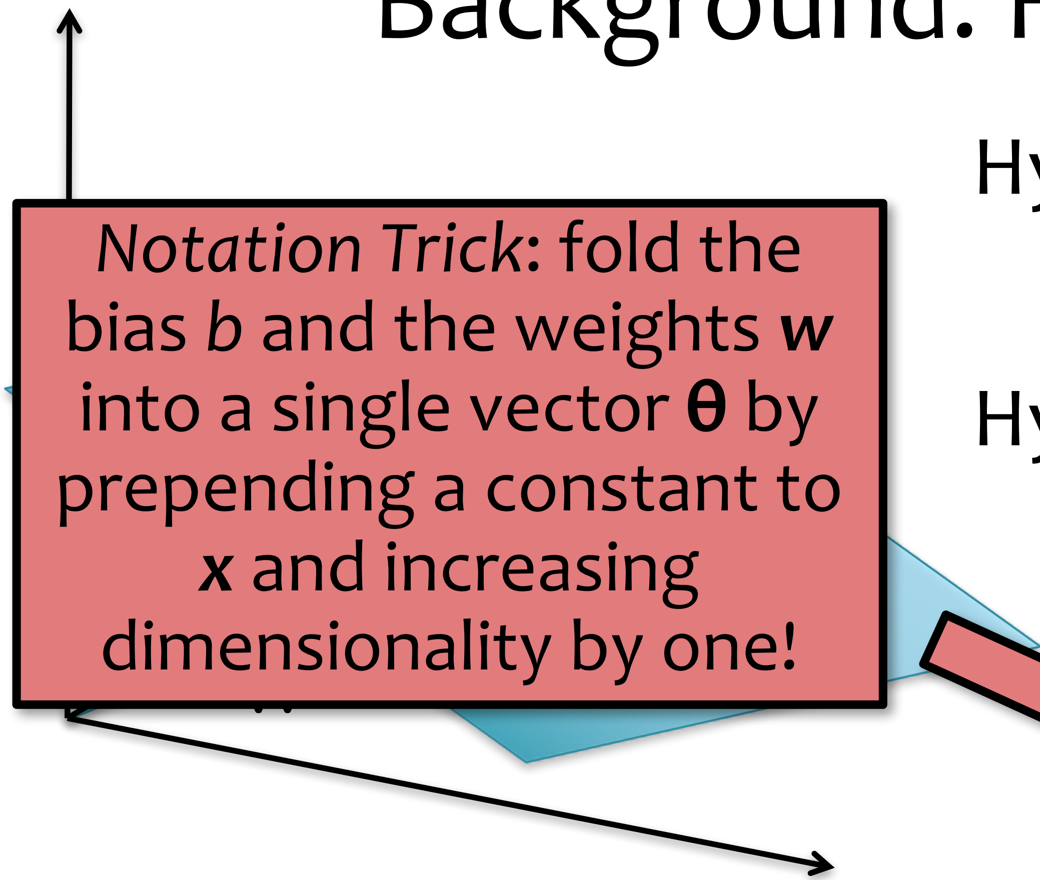
$w_2 = w_1 - (-1,2) = (1,-2)$

$w_3 = w_2 + (1,1) = (2,-1)$

$w_4 = w_3 - (-1,-2) = (3,1)$

Slide adapted from Nina Balcan

Background: Hyperplanes



Notation Trick: fold the bias *b* and the weights ***w*** into a single vector **θ** by prepending a constant to ***x*** and increasing dimensionality by one!

Hyperplane (Definition 1):

$$\mathcal{H} = \{\mathbf{x} : \mathbf{w}^T\mathbf{x} = b\}$$

Hyperplane (Definition 2):

$$\mathcal{H} = \{\mathbf{x} : \boldsymbol{\theta}^T\mathbf{x} = 0 \text{ and } x_0 = 1\}$$

$$\boldsymbol{\theta} = [b, w_1, \ldots, w_M]^T$$

Half-spaces:

$$\mathcal{H}^+ = \{\mathbf{x} : \boldsymbol{\theta}^T\mathbf{x} > 0 \text{ and } x_0 = 1\}$$

$$\mathcal{H}^- = \{\mathbf{x} : \boldsymbol{\theta}^T\mathbf{x} < 0 \text{ and } x_0 = 1\}$$

(Online) Perceptron Algorithm

Data: Inputs are continuous vectors of length *M*. Outputs are discrete.

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots$$

where $\mathbf{x} \in \mathbb{R}^M$ and $y \in \{+1, -1\}$

Prediction: Output determined by hyperplane.

$$\hat{y} = h_{\boldsymbol{\theta}}(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

$$\text{sign}(a) = \begin{cases} 1, & \text{if } a \geq 0 \\ -1, & \text{otherwise} \end{cases}$$

Assume $\boldsymbol{\theta} = [b, w_1, \ldots, w_M]^T$ and $x_0 = 1$

Learning: Iterative procedure:

- **initialize** parameters to vector of all zeroes
- **while** not converged
  - **receive** next example $(\mathbf{x}^{(i)}, y^{(i)})$
  - **predict** $y' = h(\mathbf{x}^{(i)})$
  - **if** positive mistake: **add** $\mathbf{x}^{(i)}$ to parameters
  - **if** negative mistake: **subtract** $\mathbf{x}^{(i)}$ from parameters

(Online) Perceptron Algorithm

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$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots$$

where $\mathbf{x} \in \mathbb{R}^M$ and $y \in \{+1, -1\}$

Prediction: Output determined

$$\hat{y} = h_{\boldsymbol{\theta}}(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

Assume $\boldsymbol{\theta} = [b, w_1, \ldots, w_M]$

Implementation Trick: same behavior as our "*add on positive mistake and subtract on negative mistake*" version, because $y^{(i)}$ takes care of the sign

Learning:

Algorithm 1 Perceptron Learning Alg

```
procedure PERCEPTRON(𝒟 = {(x
    θ ← 0                              ▷ Initialize parameters
    for i ∈ {1, 2, ...} do             ▷ For each example
        ŷ ← sign(θ^T x^(i))            ▷ Predict
        if ŷ ≠ y^(i) then              ▷ If mistake
            θ ← θ + y^(i) x^(i)        ▷ Update parameters
    return θ
```

(Batch) Perceptron Algorithm

Learning for Perceptron also works if we have a fixed training dataset, D. We call this the "batch" setting in contrast to the "online" setting that we've discussed so far.

Algorithm 1 Perceptron Learning Algorithm (Batch)

```
procedure PERCEPTRON(𝒟 = {(x^(1), y^(1)), ..., (x^(N), y^(N))})
    θ ← 0                                   ▷ Initialize parameters
    while not converged do
        for i ∈ {1, 2, ..., N} do           ▷ For each example
            ŷ ← sign(θ^T x^(i))             ▷ Predict
            if ŷ ≠ y^(i) then               ▷ If mistake
                θ ← θ + y^(i) x^(i)         ▷ Update parameters
    return θ
```

(Batch) Perceptron Algorithm

Learning for Perceptron also works if we have a fixed training dataset, D. We call this the "batch" setting in contrast to the "online" setting that we've discussed so far.

Discussion:

The Batch Perceptron Algorithm can be derived in two ways.

1. By extending the online Perceptron algorithm to the batch setting (as mentioned above)
2. By applying **Stochastic Gradient Descent (SGD)** to minimize a so-called **Hinge Loss** on a linear separator

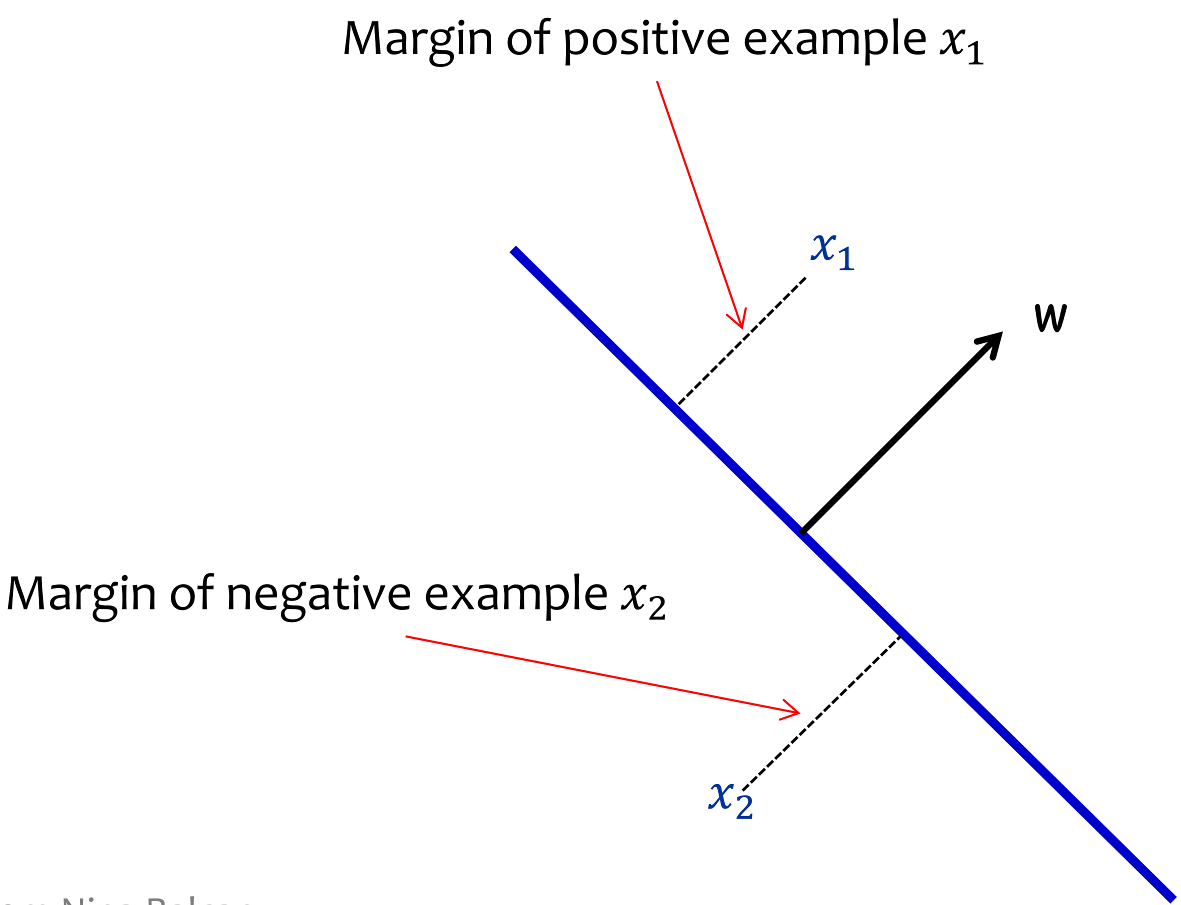
Extensions of Perceptron

- **Voted Perceptron**
  - generalizes better than (standard) perceptron
  - memory intensive (keeps around every weight vector seen during training, so each one can vote)
- **Averaged Perceptron**
  - empirically similar performance to voted perceptron
  - can be implemented in a memory efficient way (running averages are efficient)
- **Kernel Perceptron**
  - Choose a kernel K(x', x)
  - Apply the **kernel trick** to Perceptron
  - Resulting algorithm is **still very simple**
- **Structured Perceptron**
  - Basic idea can also be applied when **y** ranges over an exponentially large set
  - Mistake bound **does not** depend on the size of that set

ANALYSIS OF PERCEPTRON

Geometric Margin

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$ (or the negative if on wrong side)



Slide from Nina Balcan

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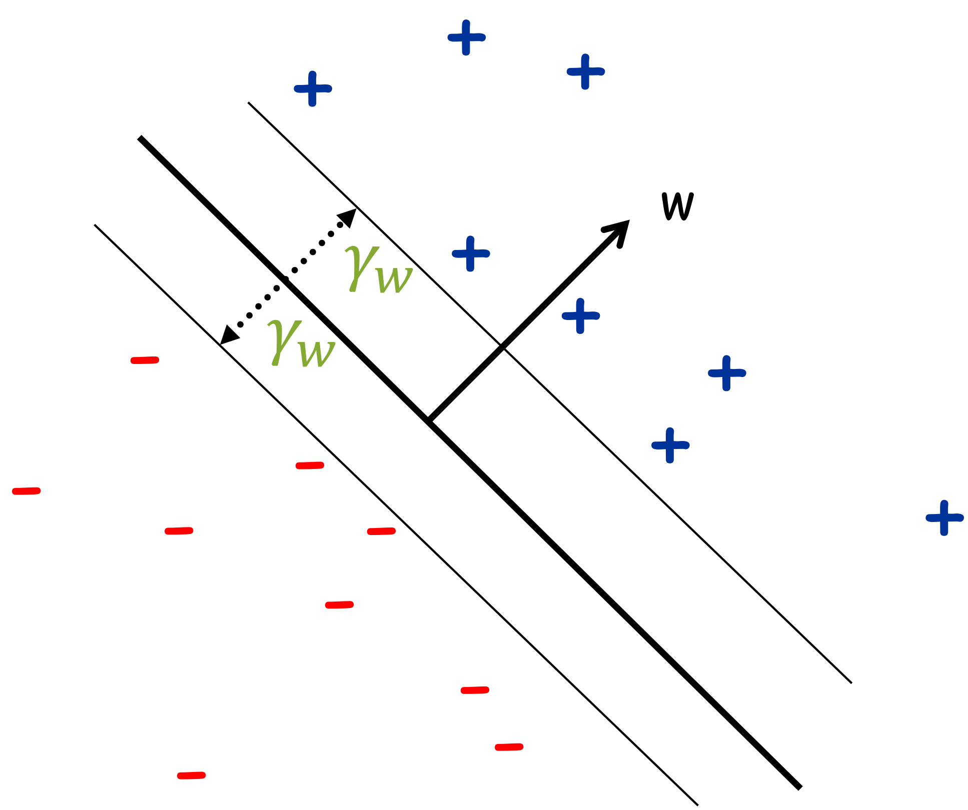

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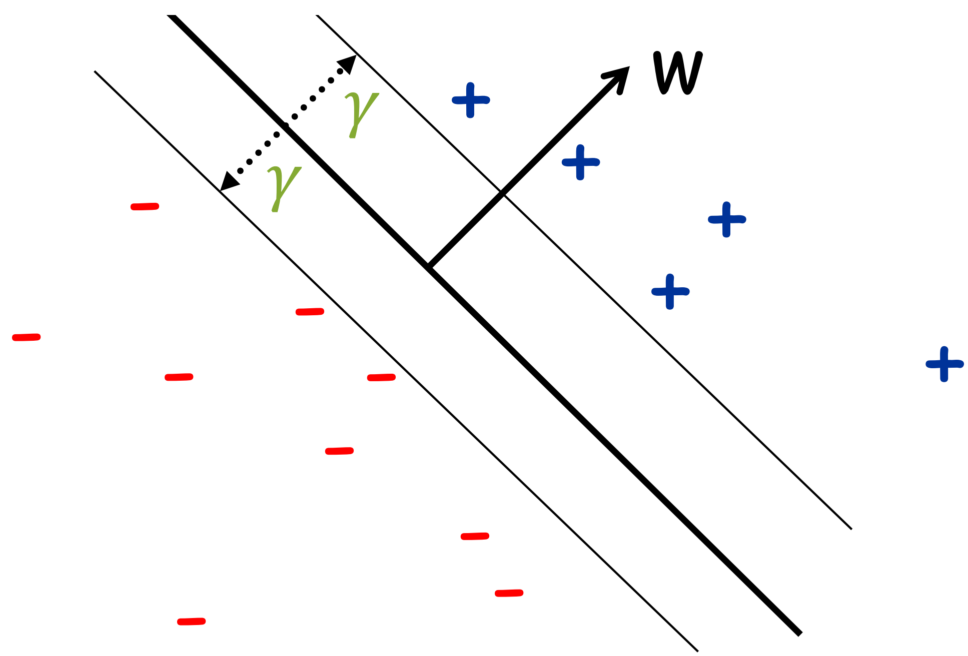
Definition: The margin γ of a set of examples S is the maximum γ_w over all linear separators w.

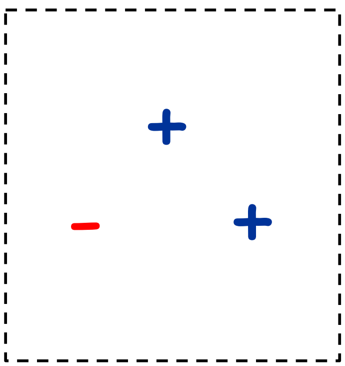


Slide from Nina Balcan

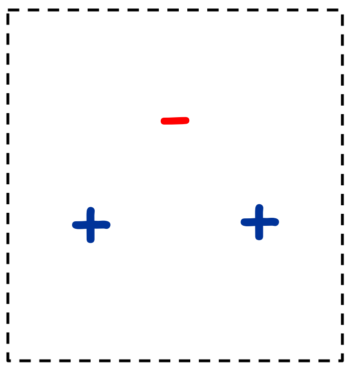
Linear Separability

Def: For a **binary classification** problem, a set of examples S is **linearly separable** if there exists a linear decision boundary that can separate the points

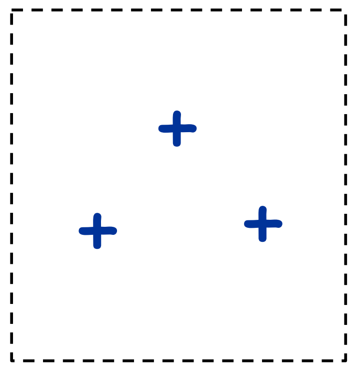
Case 1:



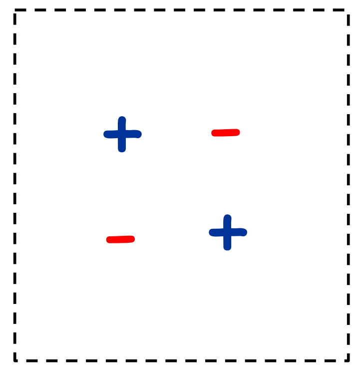
Case 2:



Case 3:



Case 4:

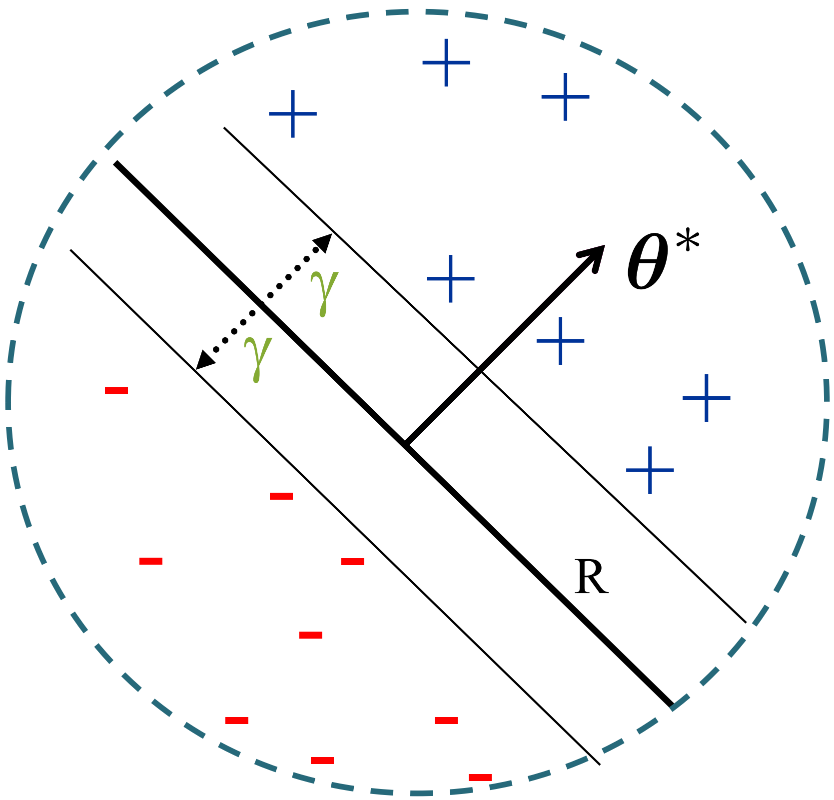


Analysis: Perceptron

Perceptron Mistake Bound

Guarantee: If data has margin γ and all points inside a ball of radius R, then Perceptron makes $\leq (R/\gamma)^2$ mistakes.

(Normalized margin: multiplying all points by 100, or dividing all points by 100, doesn't change the number of mistakes; algo is invariant to scaling.)



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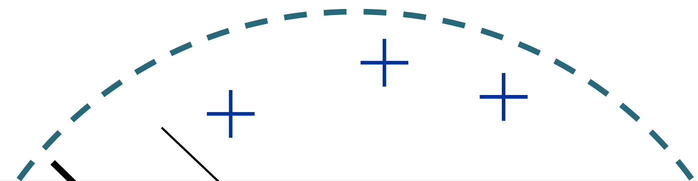


Def: We say that the (batch) perceptron algorithm has **converged** if it stops making mistakes on the training data (perfectly classifies the training data).

Main Takeaway: For **linearly separable** data, if the perceptron algorithm cycles repeatedly through the data, it will **converge** in a finite # of steps.

Slide adapted from Nina Balcan

Analysis: Perceptron

Perceptron Mistake Bound

Theorem 0.1 (Block (1962), Novikoff (1962)).

Given dataset: $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^{N}$.

Suppose:

1. *Finite size inputs:* $||x^{(i)}|| \leq R$
2. *Linearly separable data:* $\exists \boldsymbol{\theta}^*$ *s.t.* $||\boldsymbol{\theta}^*|| = 1$ *and* $y^{(i)}(\boldsymbol{\theta}^* \cdot \mathbf{x}^{(i)}) \geq \gamma, \forall i$

Then: The number of mistakes made by the Perceptron algorithm on this dataset is

$$k \leq (R/\gamma)^2$$

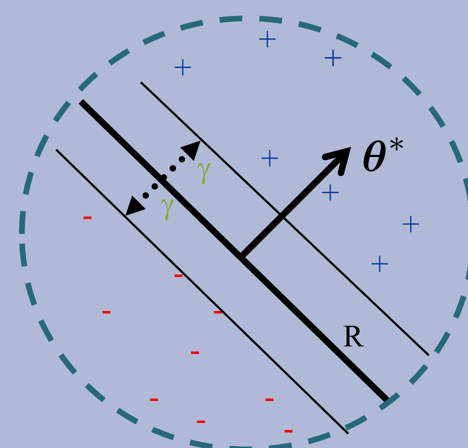


Figure from Nina Balcan

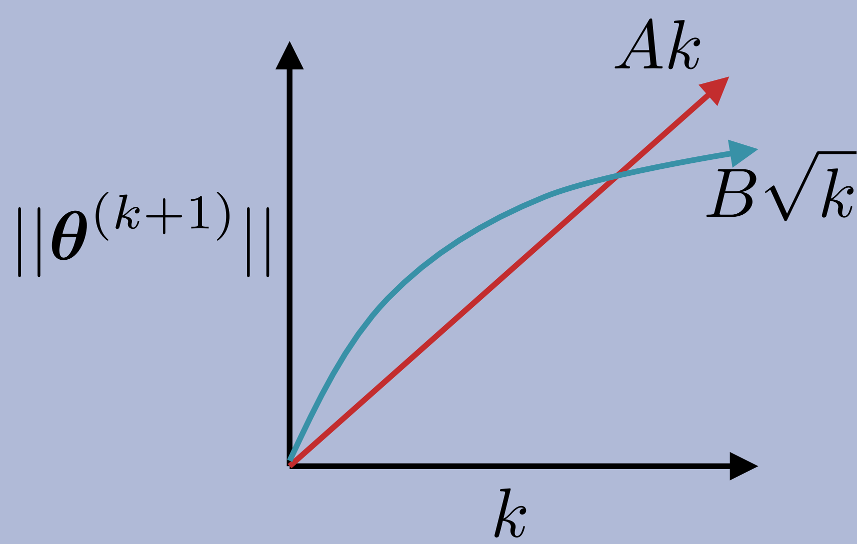
Analysis: Perceptron

Covered in Recitation

Proof of Perceptron Mistake Bound:

We will show that there exist constants A and B s.t.

$$Ak \leq ||\boldsymbol{\theta}^{(k+1)}|| \leq B\sqrt{k}$$



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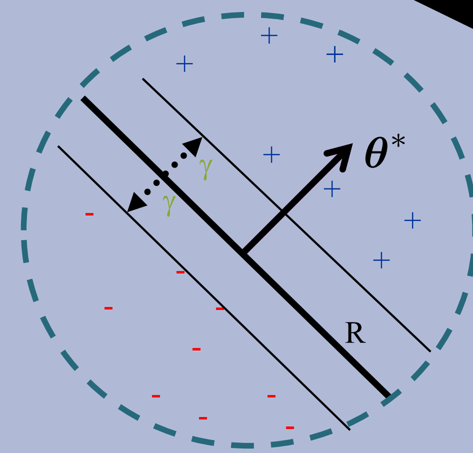

Algorithm 1 Perceptron Learning Algorithm (Online)

procedure PERCEPTRON($\mathcal{D} = \{(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots\}$)
    $\boldsymbol{\theta} \leftarrow \mathbf{0}, k = 1$ ▷ Initialize parameters
    for $i \in \{1, 2, \ldots\}$ **do** ▷ For each example
        if $y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)}) \leq 0$ **then** ▷ If mistake
            $\boldsymbol{\theta}^{(k+1)} \leftarrow \boldsymbol{\theta}^{(k)} + y^{(i)}\mathbf{x}^{(i)}$ ▷ Update parameters
            $k \leftarrow k + 1$
    return $\boldsymbol{\theta}$

Analysis: Perceptron

Covered in Recitation

Proof of Perceptron Mistake Bound:

Part 1: for some A, $Ak \leq ||\boldsymbol{\theta}^{(k+1)}||$

$$\boldsymbol{\theta}^{(k+1)} \cdot \boldsymbol{\theta}^* = (\boldsymbol{\theta}^{(k)} + y^{(i)}\mathbf{x}^{(i)})\boldsymbol{\theta}^*$$

by Perceptron algorithm update

$$= \boldsymbol{\theta}^{(k)} \cdot \boldsymbol{\theta}^* + y^{(i)}(\boldsymbol{\theta}^* \cdot \mathbf{x}^{(i)})$$

$$\geq \boldsymbol{\theta}^{(k)} \cdot \boldsymbol{\theta}^* + \gamma$$

by assumption

$$\Rightarrow \boldsymbol{\theta}^{(k+1)} \cdot \boldsymbol{\theta}^* \geq k\gamma$$

by induction on k since $\theta^{(1)} = \mathbf{0}$

$$\Rightarrow ||\boldsymbol{\theta}^{(k+1)}|| \geq k\gamma$$

since $\underbrace{||\mathbf{w}|| \times ||\mathbf{u}|| \geq \mathbf{w} \cdot \mathbf{u}}_{\text{Cauchy-Schwartz inequality}}$ and $||\theta^*|| = 1$

Cauchy-Schwartz inequality

Analysis: Perceptron

Covered in Recitation

Proof of Perceptron Mistake Bound:

Part 2: for some B, $||\boldsymbol{\theta}^{(k+1)}|| \leq B\sqrt{k}$

$$||\boldsymbol{\theta}^{(k+1)}||^2 = ||\boldsymbol{\theta}^{(k)} + y^{(i)}\mathbf{x}^{(i)}||^2$$

by Perceptron algorithm update

$$= ||\boldsymbol{\theta}^{(k)}||^2 + (y^{(i)})^2||\mathbf{x}^{(i)}||^2 + 2y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)})$$

$$\leq ||\boldsymbol{\theta}^{(k)}||^2 + (y^{(i)})^2||\mathbf{x}^{(i)}||^2$$

since kth mistake $\Rightarrow y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)}) \leq 0$

$$= ||\boldsymbol{\theta}^{(k)}||^2 + R^2$$

since $(y^{(i)})^2||\mathbf{x}^{(i)}||^2 = ||\mathbf{x}^{(i)}||^2 = R^2$ by assumption and $(y^{(i)})^2 = 1$

$$\Rightarrow ||\boldsymbol{\theta}^{(k+1)}||^2 \leq kR^2$$

by induction on k since $(\theta^{(1)})^2 = 0$

$$\Rightarrow ||\boldsymbol{\theta}^{(k+1)}|| \leq \sqrt{k}R$$

Analysis: Perceptron

Covered in Recitation

Proof of Perceptron Mistake Bound:

Part 3: Combining the bounds finishes the proof.

$$k\gamma \leq ||\boldsymbol{\theta}^{(k+1)}|| \leq \sqrt{k}R$$

$$\Rightarrow k \leq \underbrace{(R/\gamma)^2}_{\text{The total number of mistakes must be less than this}}$$

Analysis: Perceptron

What if the data is *not* linearly separable?

1. Perceptron will **not converge** in this case (it can't!)
2. However, Freund & Schapire (1999) show that by projecting the points (hypothetically) into a higher dimensional space, we can achieve a similar bound on the number of mistakes made on **one pass** through the sequence of examples

Theorem 2. *Let* $\langle(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)\rangle$ *be a sequence of labeled examples with* $\|\mathbf{x}_i\| \leq R$*. Let* $\mathbf{u}$ *be any vector with* $\|\mathbf{u}\| = 1$ *and let* $\gamma > 0$*. Define the deviation of each example as*

$$d_i = \max\{0, \gamma - y_i(\mathbf{u} \cdot \mathbf{x}_i)\},$$

and define $D = \sqrt{\sum_{i=1}^m d_i^2}$*. Then the number of mistakes of the online perceptron algorithm on this sequence is bounded by*

$$\left(\frac{R + D}{\gamma}\right)^2.$$

Summary: Perceptron

- Perceptron is a **linear classifier**
- **Simple learning algorithm**: when a mistake is made, add / subtract the features
- Perceptron will converge if the data are **linearly separable**, it will **not** converge if the data are **linearly inseparable**
- For linearly separable and inseparable data, we can **bound the number of mistakes** (geometric argument)
- **Extensions** support nonlinear separators and structured prediction

Perceptron Learning Objectives

You should be able to…

- Explain the difference between online learning and batch learning
- Implement the perceptron algorithm for binary classification [CIML]
- Determine whether the perceptron algorithm will converge based on properties of the dataset, and the limitations of the convergence guarantees
- Describe the inductive bias of perceptron and the limitations of linear models
- Draw the decision boundary of a linear model
- Identify whether a dataset is linearly separable or not
- Defend the use of a bias term in perceptron