10-301/10-601 Introduction to Machine Learning

Machine Learning Department
School of Computer Science
Carnegie Mellon University

Neural Networks
+
Backpropagation

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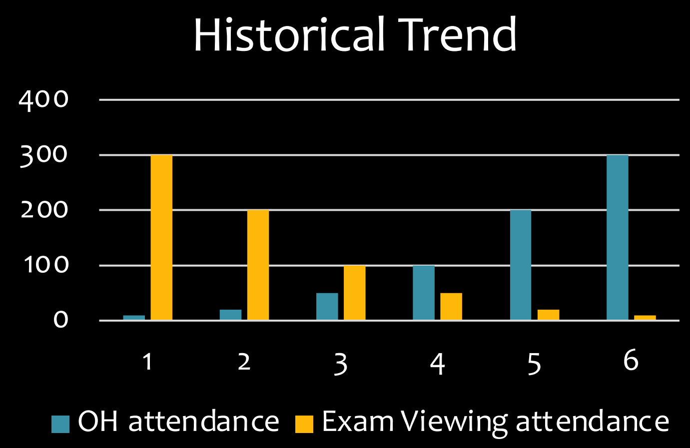
Lecture 12
Oct. 1, 2025

Reminders

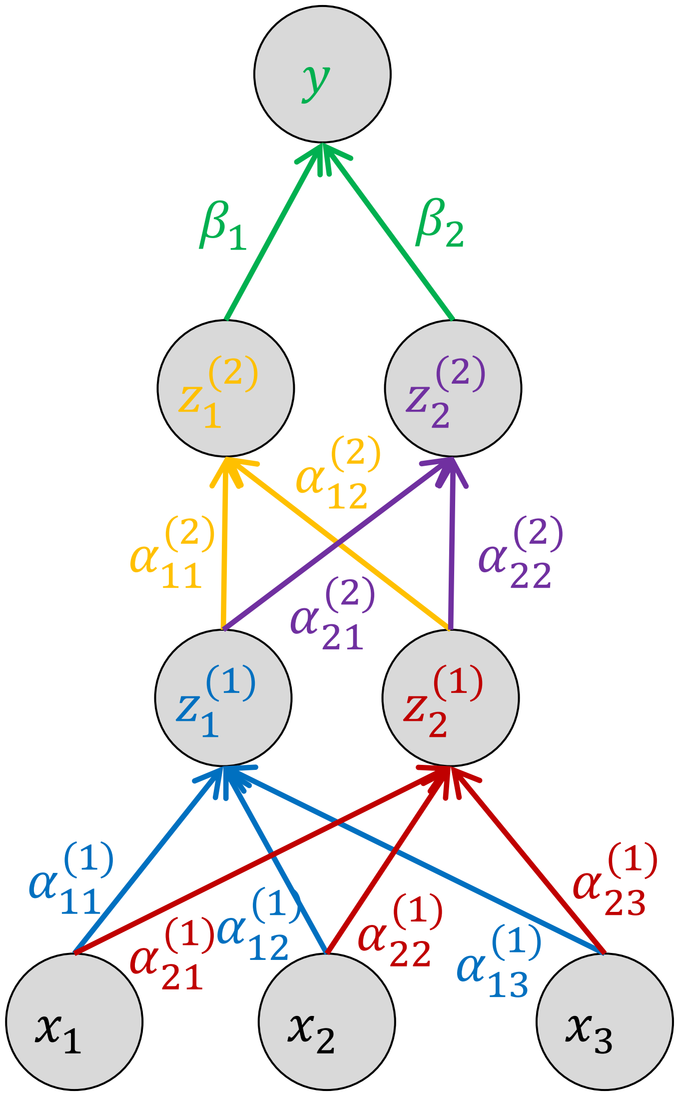
- **Post-Exam/Quiz Followup:**
  - Exam/Quiz Viewing
  - Exit Poll: Exam 1
  - Grade Summary 1
- **Homework 4: Logistic Regression**
  - Out: Mon, Sep 29
  - Due: Wed, Oct 8 at 11:59pm



BUILDING DEEPER NETWORKS

Neural Network

Recall…



Example: Neural Network with 2 Hidden Layers and 2 Hidden Units

$$z_1^{(1)} = \sigma(\alpha_{11}^{(1)} x_1 + \alpha_{12}^{(1)} x_2 + \alpha_{13}^{(1)} x_3 + \alpha_{10}^{(1)})$$

$$z_2^{(1)} = \sigma(\alpha_{21}^{(1)} x_1 + \alpha_{22}^{(1)} x_2 + \alpha_{23}^{(1)} x_3 + \alpha_{20}^{(1)})$$

$$z_1^{(2)} = \sigma(\alpha_{11}^{(2)} z_1^{(1)} + \alpha_{12}^{(2)} z_2^{(1)} + \alpha_{10}^{(2)})$$

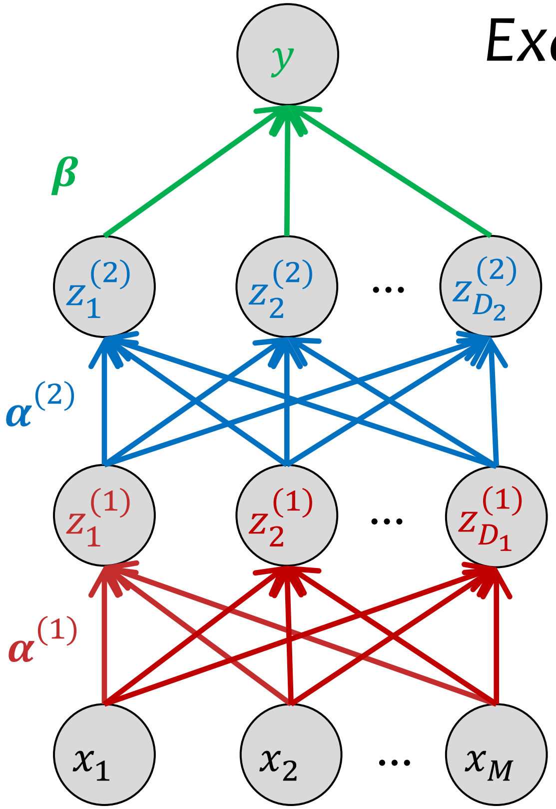
$$z_2^{(2)} = \sigma(\alpha_{21}^{(2)} z_1^{(1)} + \alpha_{22}^{(2)} z_2^{(1)} + \alpha_{20}^{(2)})$$

$$y = \sigma(\beta_1 \, z_1^{(2)} + \beta_2 \, z_2^{(2)} + \beta_0 \,)$$

Neural Network (Matrix Form)

Recall…

Example: Arbitrary Feed-forward Neural Network



$\boldsymbol{\beta} \in \mathbb{R}^{D_2}$

$\beta_0 \in \mathbb{R}$

$\boldsymbol{\alpha}^{(2)} \in \mathbb{R}^{D_1 \times D_2}$

$\boldsymbol{b}^{(2)} \in \mathbb{R}^{D_2}$

$\boldsymbol{\alpha}^{(1)} \in \mathbb{R}^{M \times D_1}$

$\boldsymbol{b}^{(1)} \in \mathbb{R}^{D_1}$

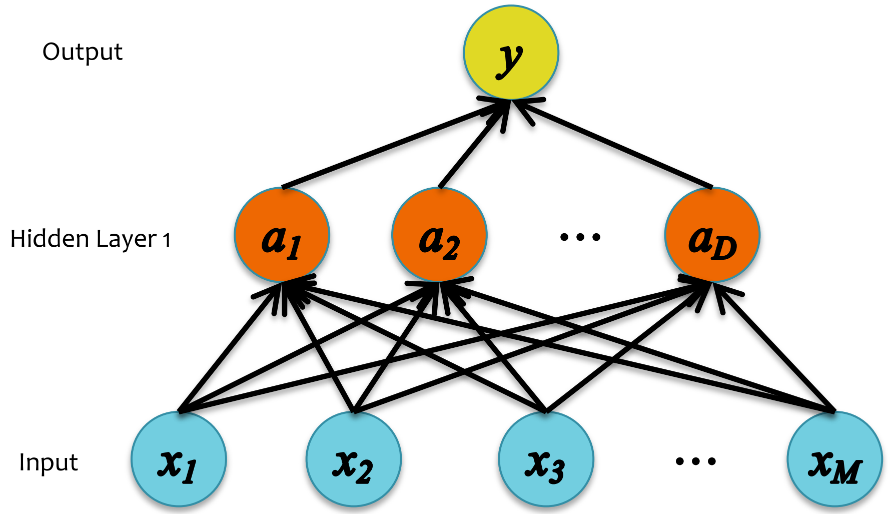
$$y = \sigma((\boldsymbol{\beta})^T \boldsymbol{z}^{(2)} + \beta_0)$$

$$\boldsymbol{z}^{(2)} = \sigma((\boldsymbol{\alpha}^{(2)})^T \boldsymbol{z}^{(1)} + \boldsymbol{b}^{(2)})$$

$$\boldsymbol{z}^{(1)} = \sigma((\boldsymbol{\alpha}^{(1)})^T \boldsymbol{x} + \boldsymbol{b}^{(1)})$$

Deeper Networks

Q: How many layers should we use?



Deeper Networks

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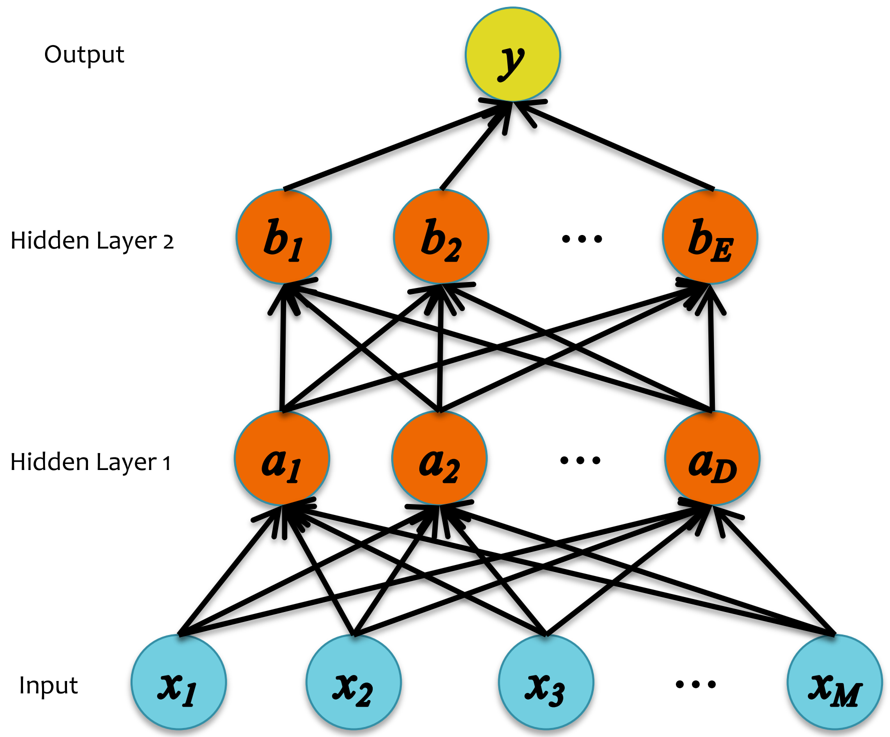

Deeper Networks

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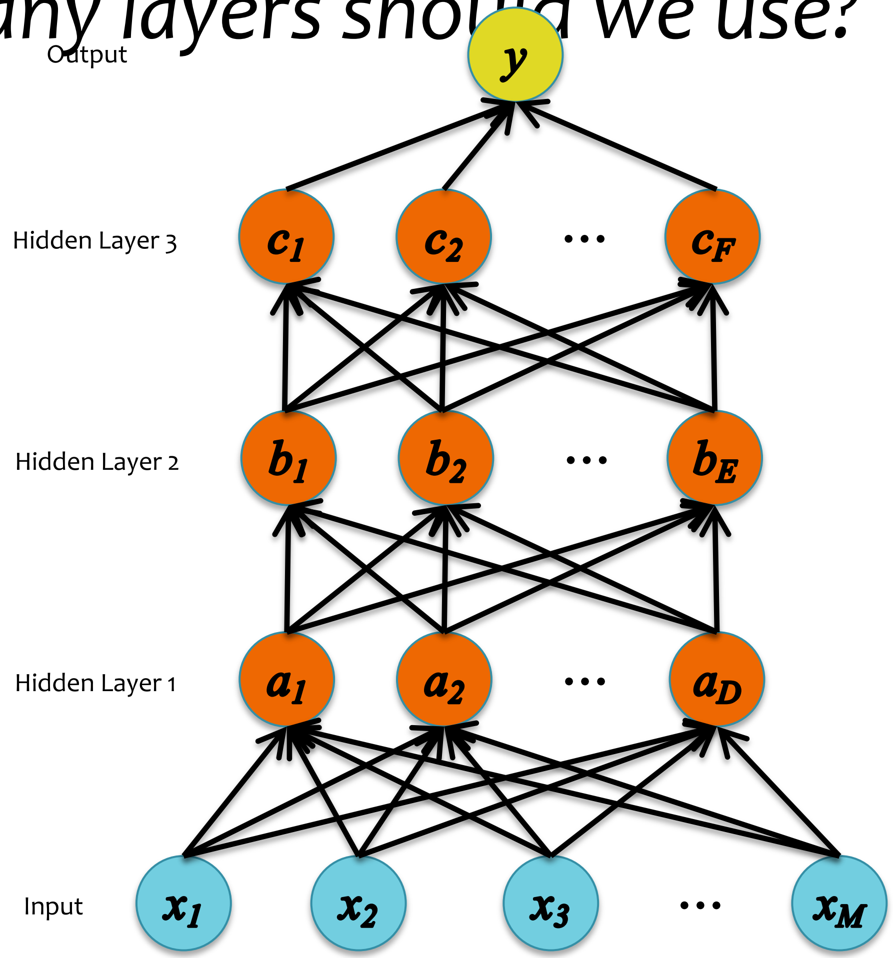


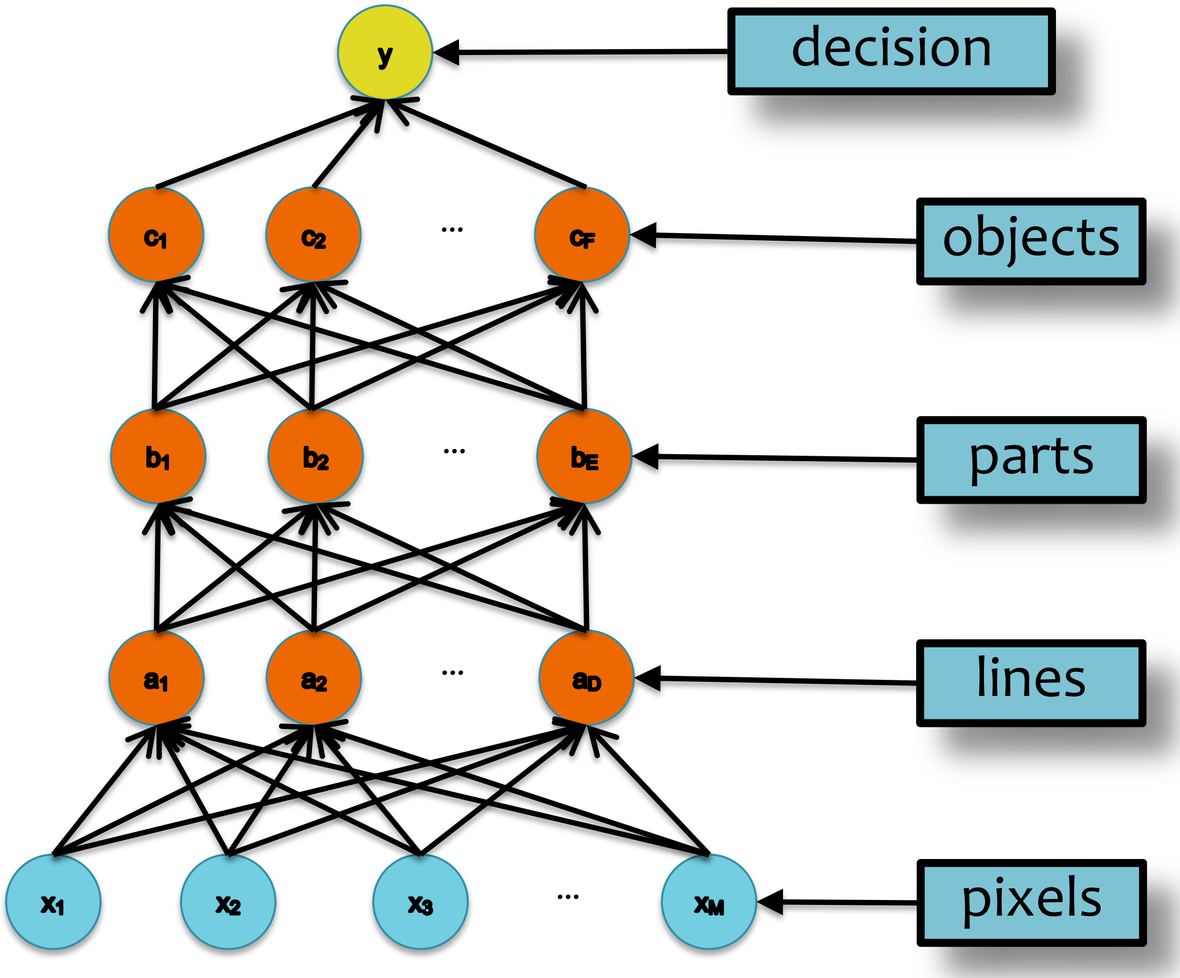
Deeper Networks

Q: How many layers should we use?

- **Theoretical answer:**
  - A neural network with 1 hidden layer is a **universal function approximator**
  - Cybenko (1989): For any continuous function g(**x**), there exists a 1-hidden-layer neural net $h_\theta(\mathbf{x})$ s.t. $| h_\theta(\mathbf{x}) - g(\mathbf{x}) | < \epsilon$ for all **x**, assuming sigmoid activation functions
- **Empirical answer:**
  - Before 2006: "Deep networks (e.g. 3 or more hidden layers) are too hard to train"
  - After 2006: "Deep networks are easier to train than shallow networks (e.g. 2 or fewer layers) for many problems"

    Big caveat: You need to know and use the right tricks.

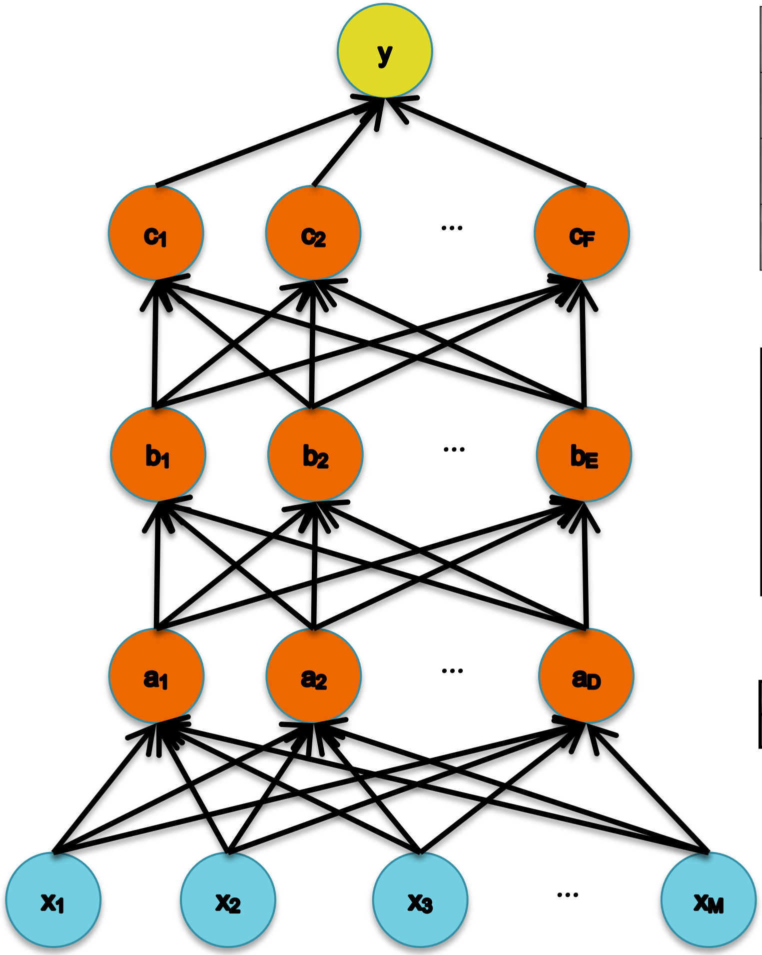
Feature Learning



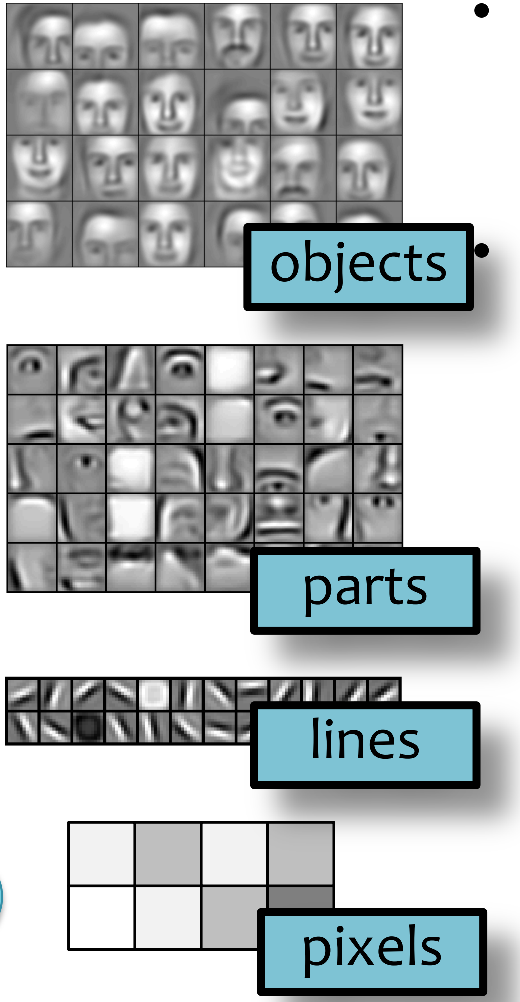
- **Traditional feature engineering:** build up levels of abstraction by hand
- **Deep networks** (e.g. convolution networks): learn the increasingly higher levels of abstraction from data
  - each layer is a learned feature representation
  - sophistication increases in higher layers

Figures from Lee et al. (ICML 2009)

Feature Learning



CBDN on Faces



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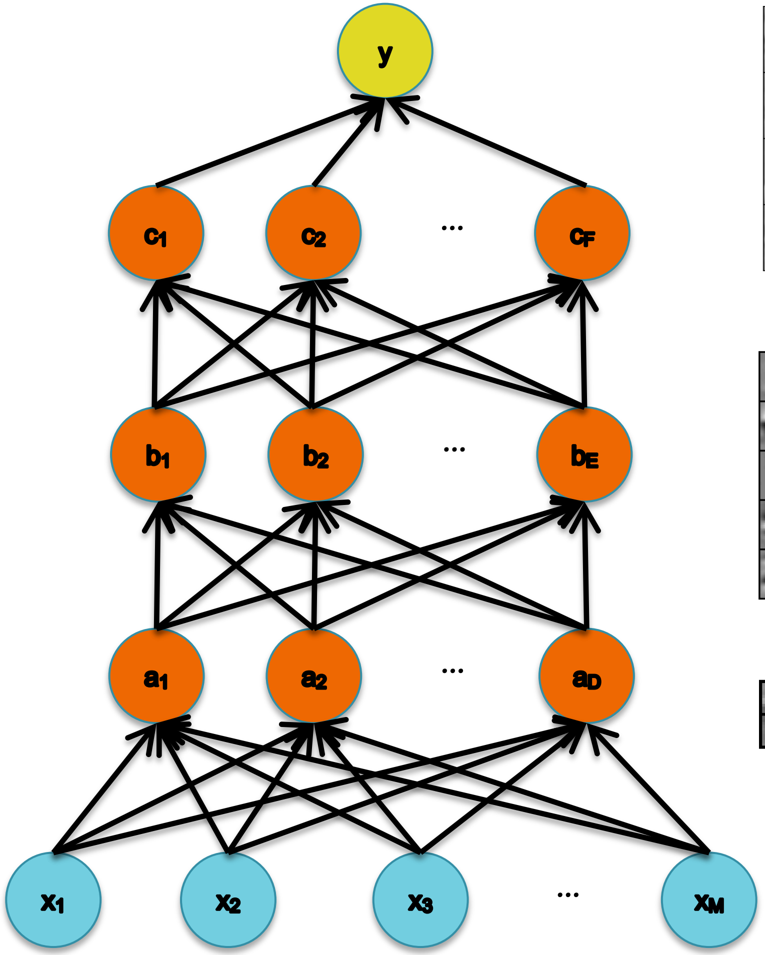
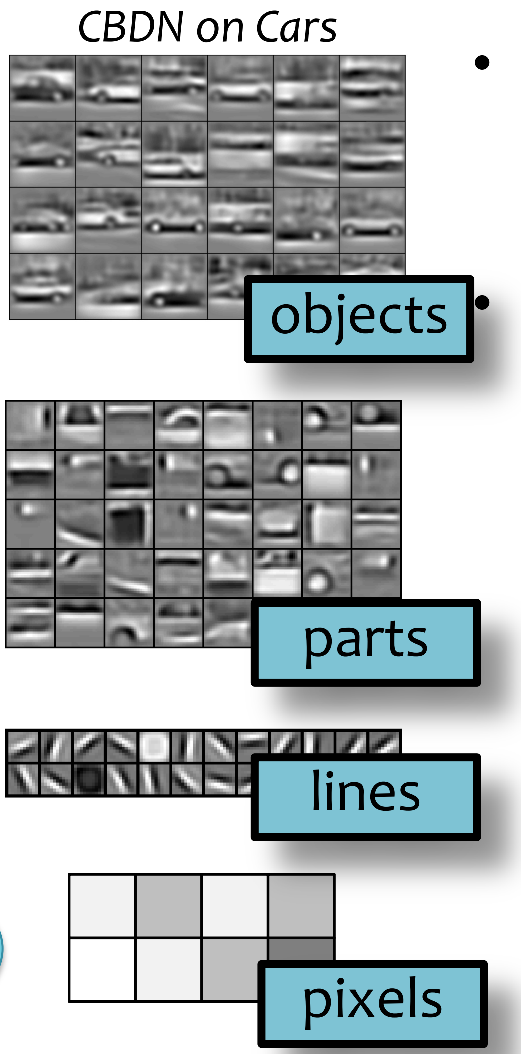
Figures from Lee et al. (ICML 2009)

Feature Learning





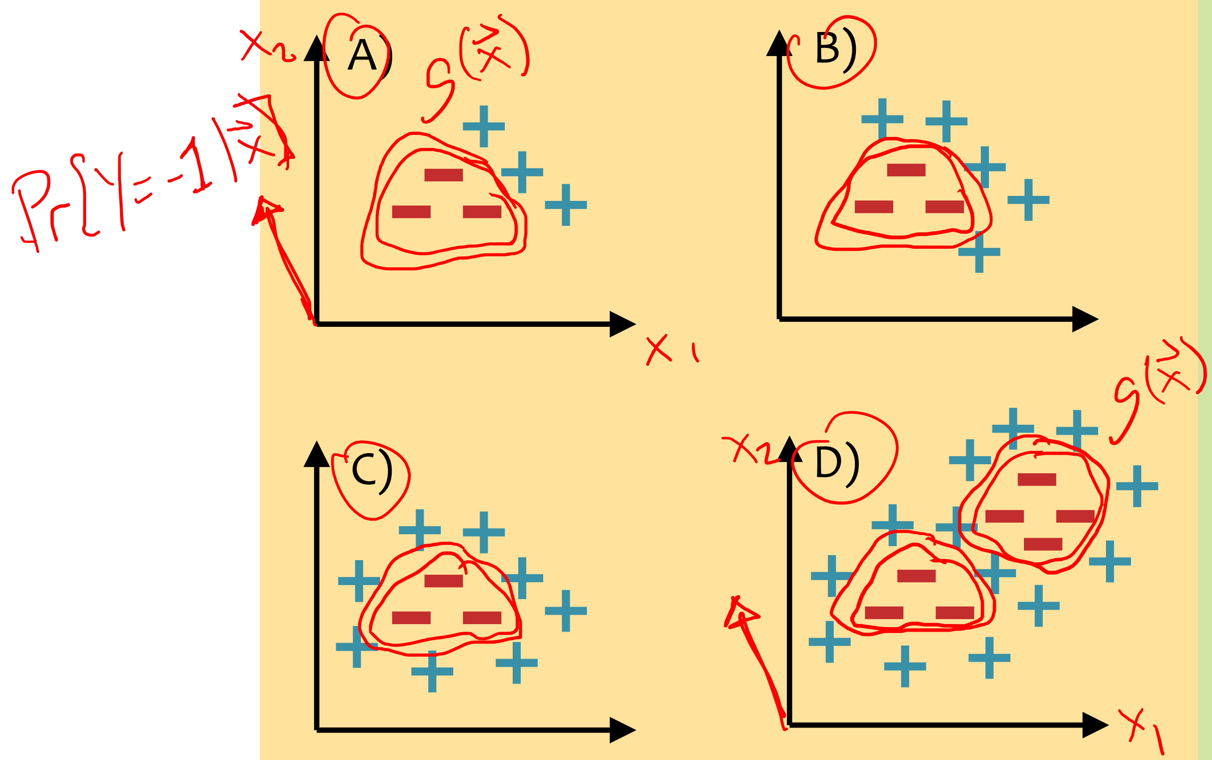
- **Traditional feature engineering:** build up levels of abstraction by hand
- **Deep networks** (e.g. convolution networks): learn the increasingly higher levels of abstraction from data
  - each layer is a learned feature representation
  - sophistication increases in higher layers

Figures from Lee et al. (ICML 2009)

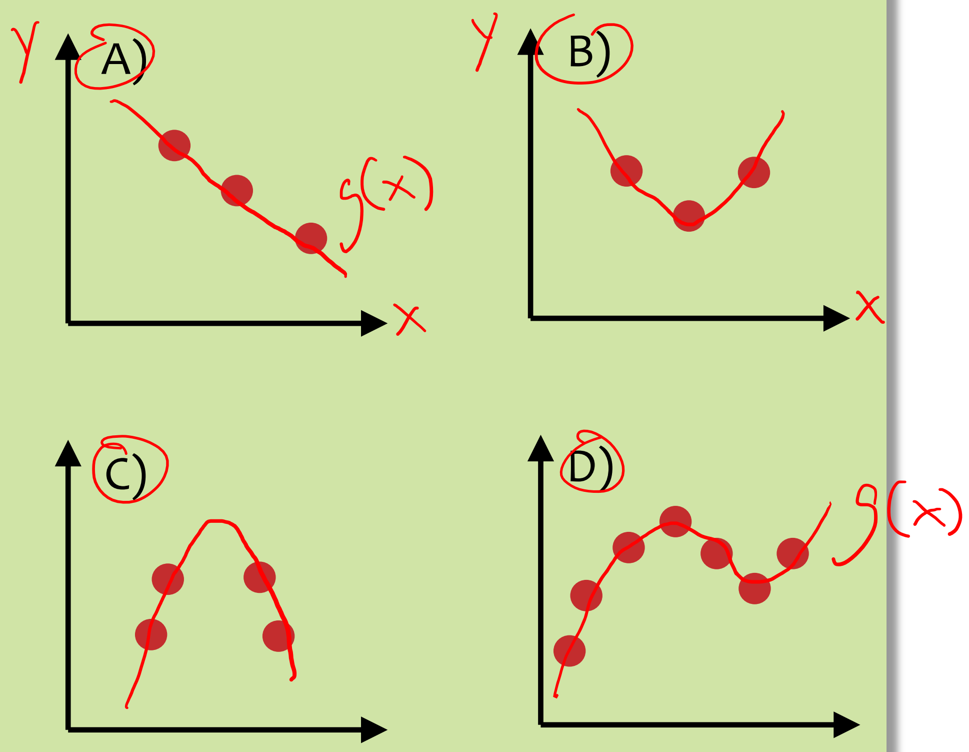
Neural Network Errors

Recall…

Poll Question 2: For which of the datasets below does there exist a one-hidden layer neural network that achieves zero *classification* error? **Select all that apply.**



Poll Question 3: For which of the datasets below does there exist a one-hidden layer neural network for *regression* that achieves *nearly* zero MSE? **Select all that apply.**



Neural Network Architectures

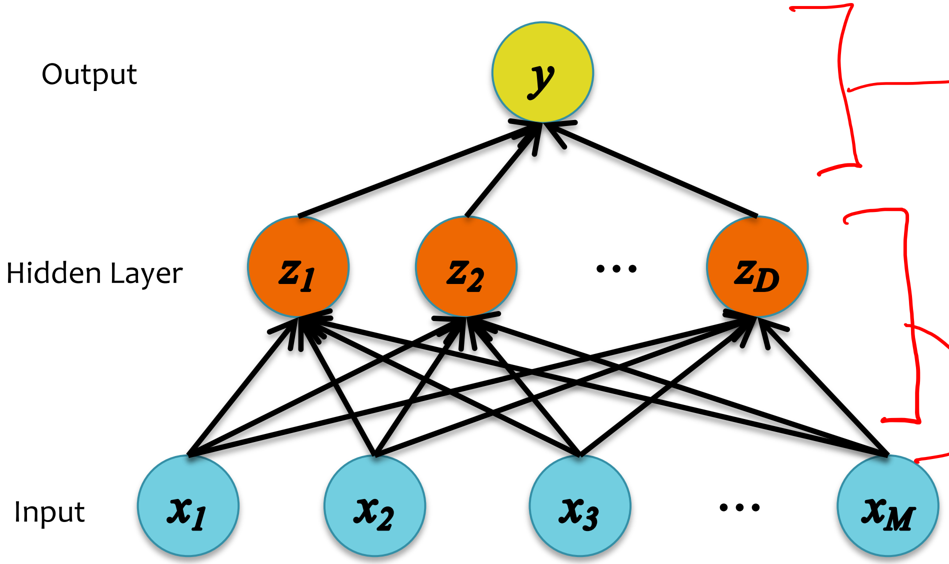
Even for a basic Neural Network, there are many design decisions to make:

1. # of hidden layers (depth)
2. # of units per hidden layer (width)
3. Type of activation function (nonlinearity)
4. Form of objective function
5. How to initialize the parameters

ACTIVATION FUNCTIONS

Activation Functions

Neural Network with sigmoid activation functions



(F) Loss
$J = \frac{1}{2}(y - y^*)^2$

(E) Output (sigmoid)
$y = \frac{1}{1+\exp(-b)}$

(D) Output (linear)
$b = \sum_{j=0}^{D} \beta_j z_j$

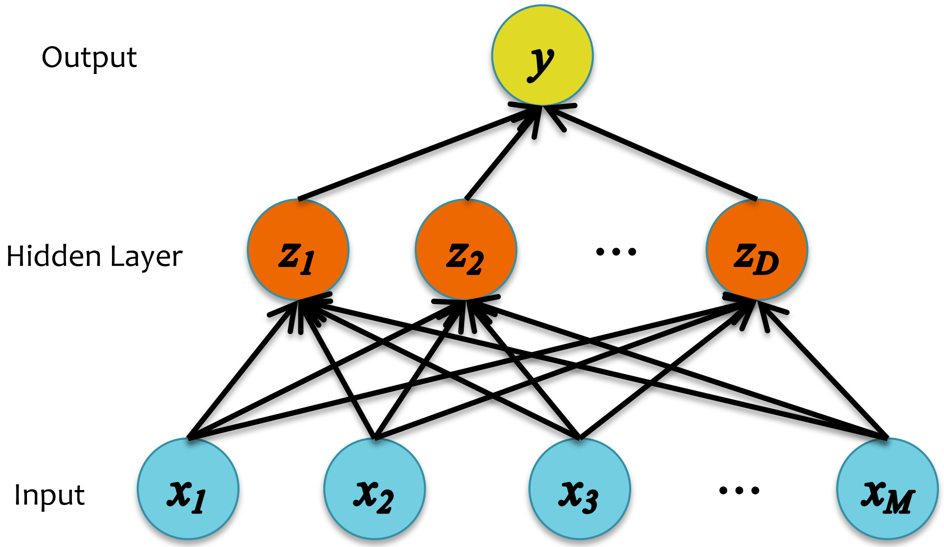
(C) Hidden (sigmoid)
$z_j = \frac{1}{1+\exp(-a_j)}, \ \forall j$

(B) Hidden (linear)
$a_j = \sum_{i=0}^{M} \alpha_{ji} x_i, \ \forall j$

(A) Input
Given $x_i, \ \forall i$

Activation Functions

Neural Network with arbitrary nonlinear activation functions



(F) **Loss**
$J = \frac{1}{2}(y - y^*)^2$

(E) **Output (nonlinear)**
$y = \sigma(b)$

(D) **Output (linear)**
$b = \sum_{j=0}^{D} \beta_j z_j$

(C) **Hidden (nonlinear)**
$z_j = \sigma(a_j), \ \forall j$

(B) **Hidden (linear)**
$a_j = \sum_{i=0}^{M} \alpha_{ji} x_i, \ \forall j$

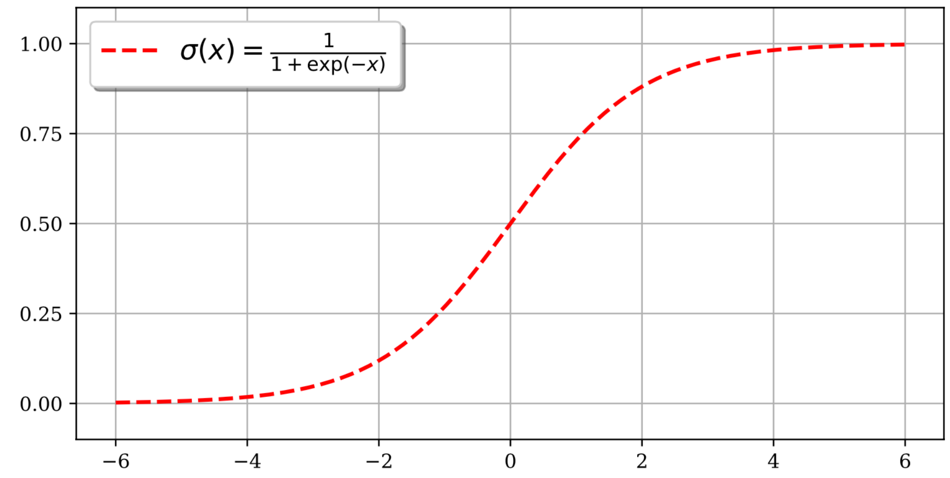
(A) **Input**
Given $x_i, \ \forall i$

Activation Functions

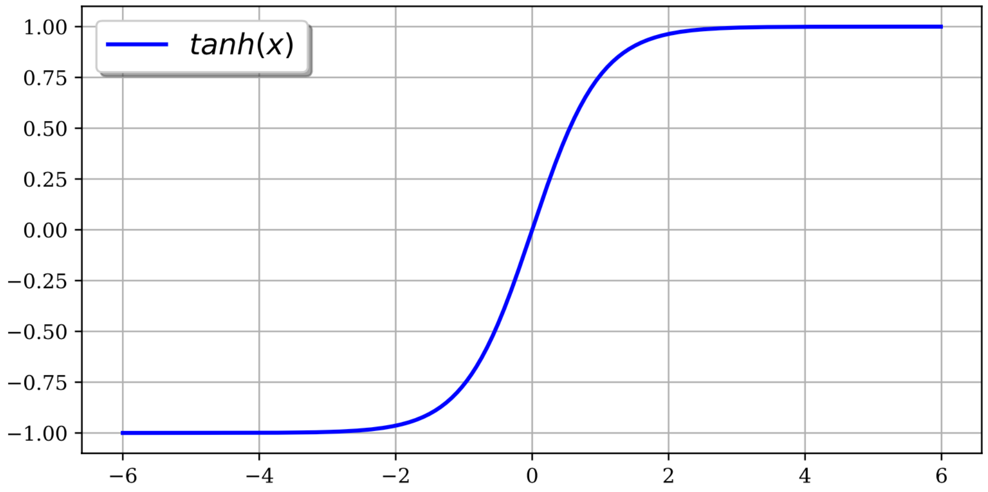
So far, we've assumed that the activation function (nonlinearity) is always the sigmoid function…

…but the sigmoid is not widely used in modern neural networks

Sigmoid (aka. logistic) function



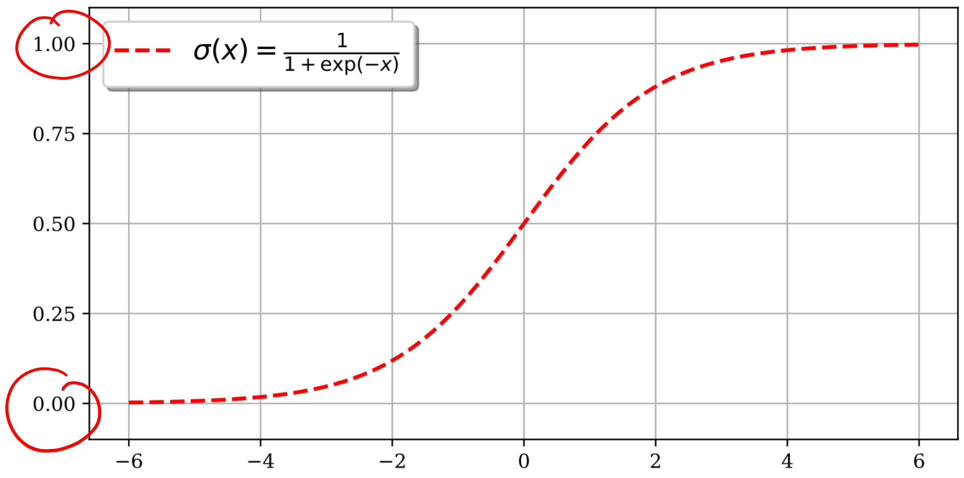
Hyperbolic tangent function



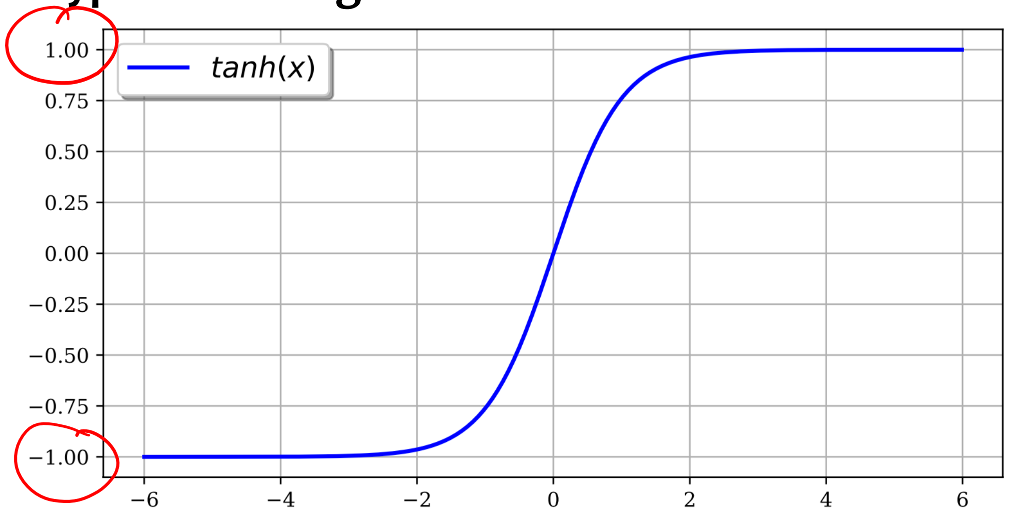
Activation Functions

- sigmoid, σ(x)
  - output in range (0,1)
  - good for probabilistic outputs
- hyperbolic tangent, tanh(x)
  - similar shape to sigmoid, but output in range (-1,+1)

Sigmoid (aka. logistic) function

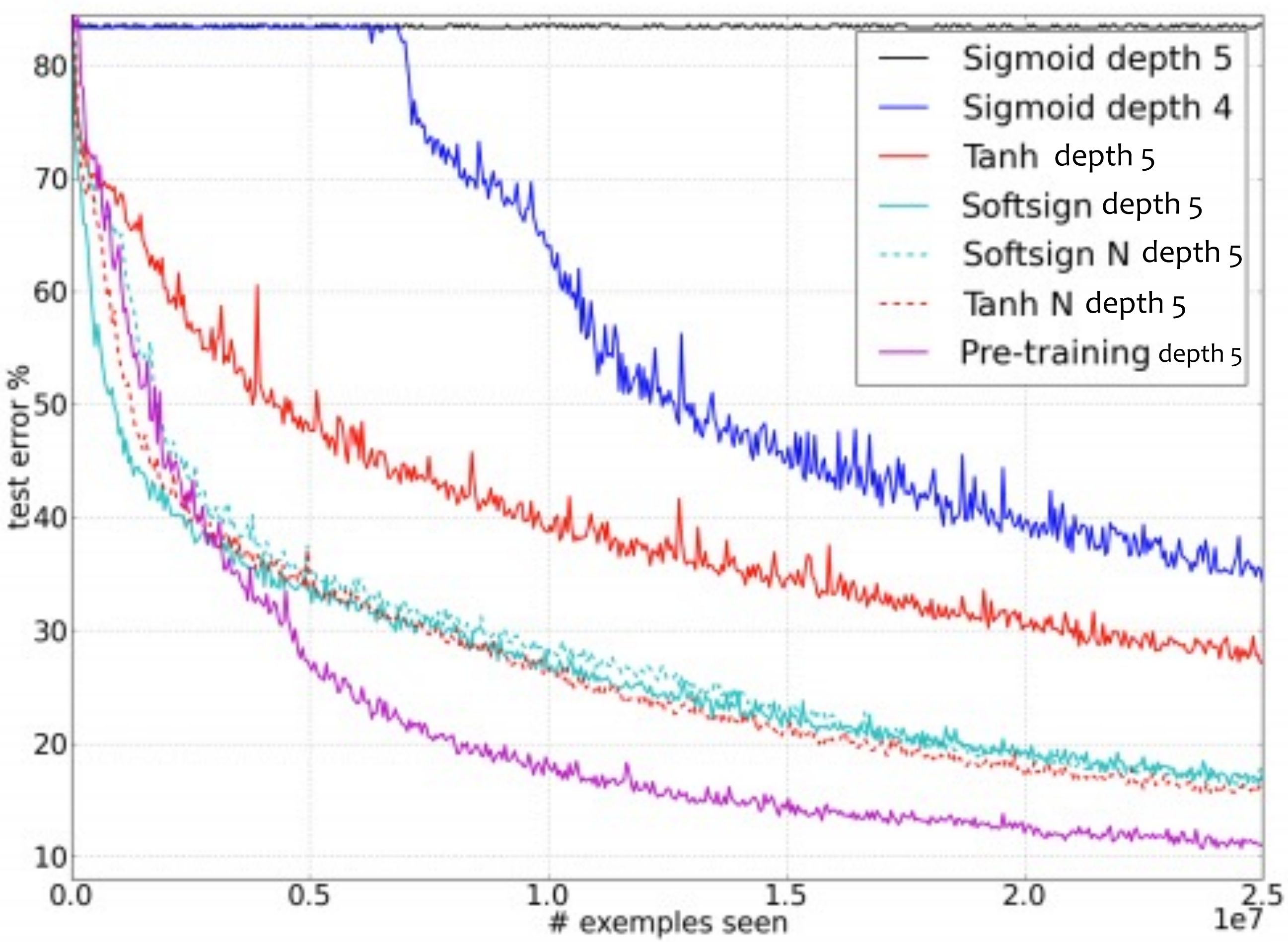


Hyperbolic tangent function



Understanding the difficulty of training deep feedforward neural networks

AI Stats 2010



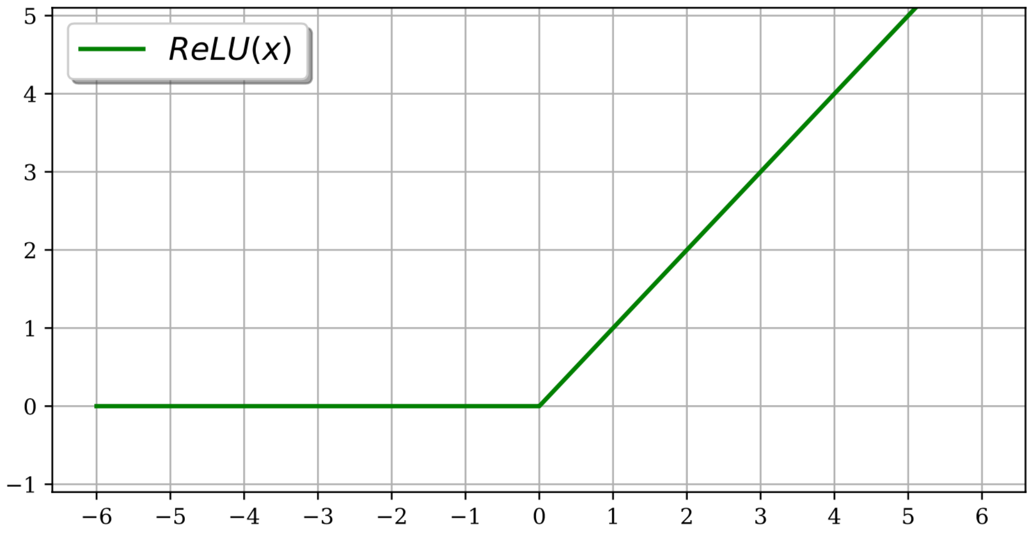
sigmoid
vs.
tanh

Figure from Glorot & Bentio (2010)

Activation Functions

- Rectified Linear Unit (ReLU)
  - avoids the vanishing gradient problem
  - derivative is fast to compute

$$\text{ReLU}(x) = max(0, x)$$



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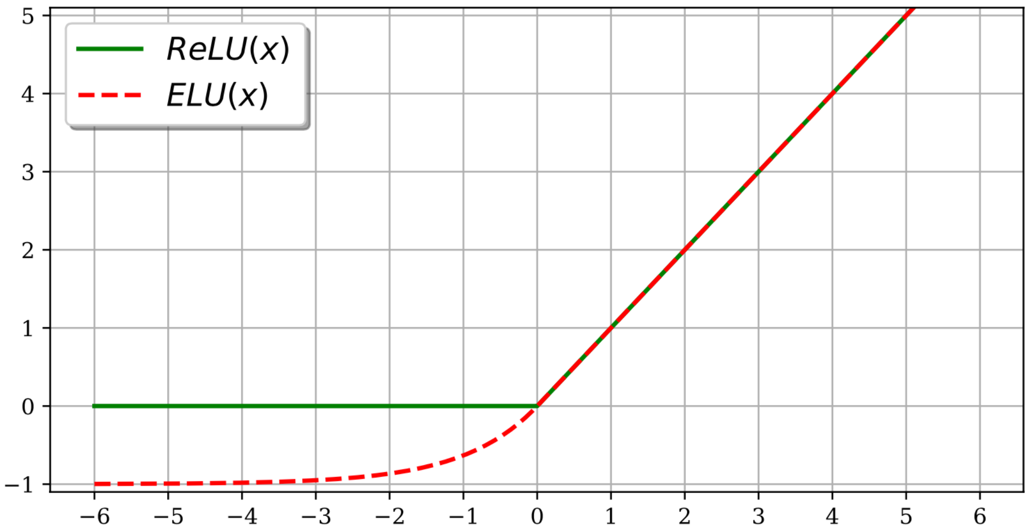

- Exponential Linear Unit (ELU)
  - same as ReLU on positive inputs
  - unlike ReLU, allows negative outputs and smoothly transitions for x < 0

$$\text{ELU}(x) = \begin{cases} x, & \text{if } x > 0 \\ \alpha(\exp(x) - 1), & \text{if } x \leq 0 \end{cases}$$

Activation Functions

Image Classification Benchmark (CIFAR-10)

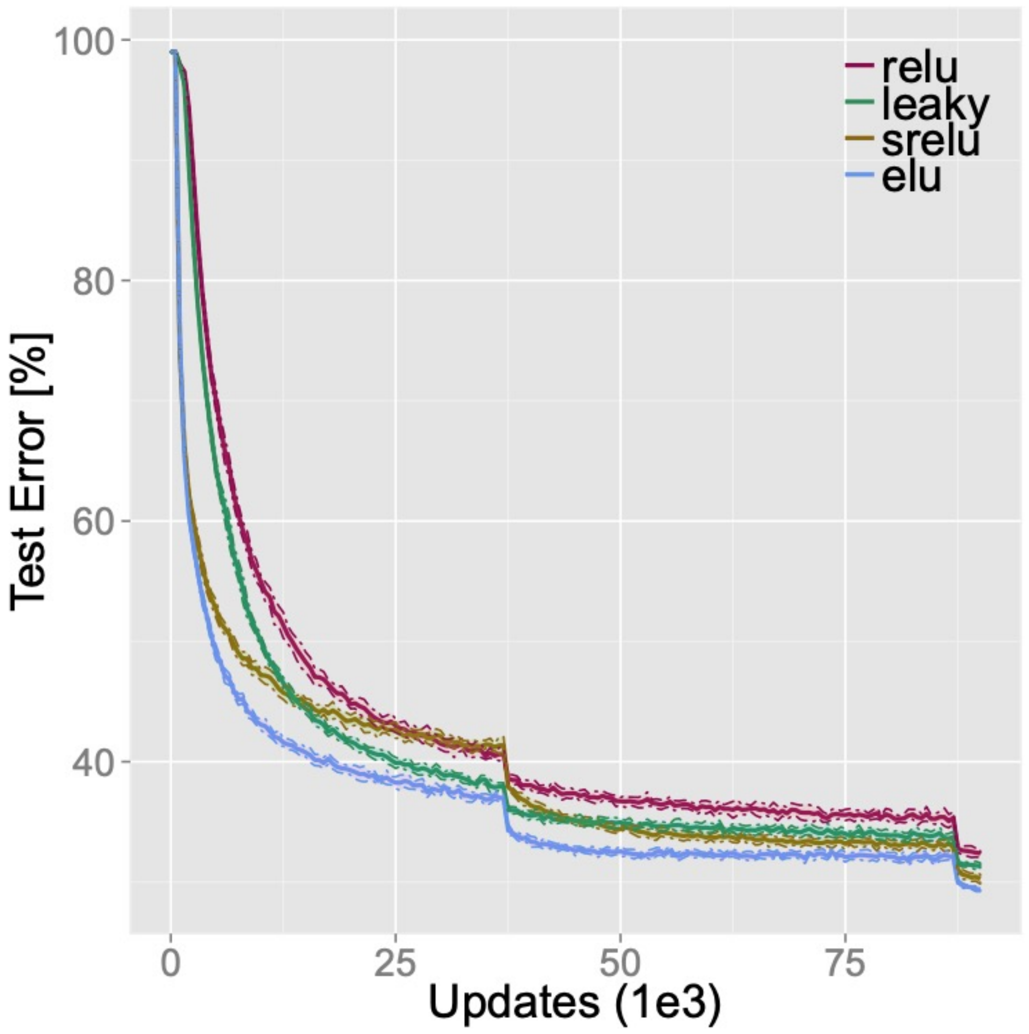

Figure from Clevert et al. (2016)

1. Training loss converges fastest with ELU
2. ELU(x) yields lower test error than ReLU(x) on CIFAR-10

Neural Networks Objectives

You should be able to…

- Explain the biological motivations for a neural network
- Combine simpler models (e.g. linear regression, binary logistic regression, multinomial logistic regression) as components to build up feed-forward neural network architectures
- Explain the reasons why a neural network can model nonlinear decision boundaries for classification
- Compare and contrast feature engineering with learning features
- Identify (some of) the options available when designing the architecture of a neural network
- Implement a feed-forward neural network

LOSS FUNCTIONS & OUTPUT LAYERS

Background

A Recipe for Machine Learning

1. Given training data:

$$\mathcal{D} = \{\mathbf{x}^{(i)}, y^{(i)}\}_{i=1}^{N}$$

2. Choose each of these:
   - Decision function

$$\hat{y} = h_{\boldsymbol{\theta}}(\mathbf{x})$$

   - Loss function

$$\ell(\hat{y}, y) \in \mathbb{R}$$

3. Define goal:

$$\hat{\boldsymbol{\theta}} \approx \underset{\boldsymbol{\theta}}{\text{argmin}} \sum_{i=1}^{N} \ell(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}), y^{(i)})$$

4. Train with SGD:

(take small steps opposite the gradient)

$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)} - \eta_t \nabla \ell(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}), y^{(i)})$$

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3. Define goal:

opposite the gradient)

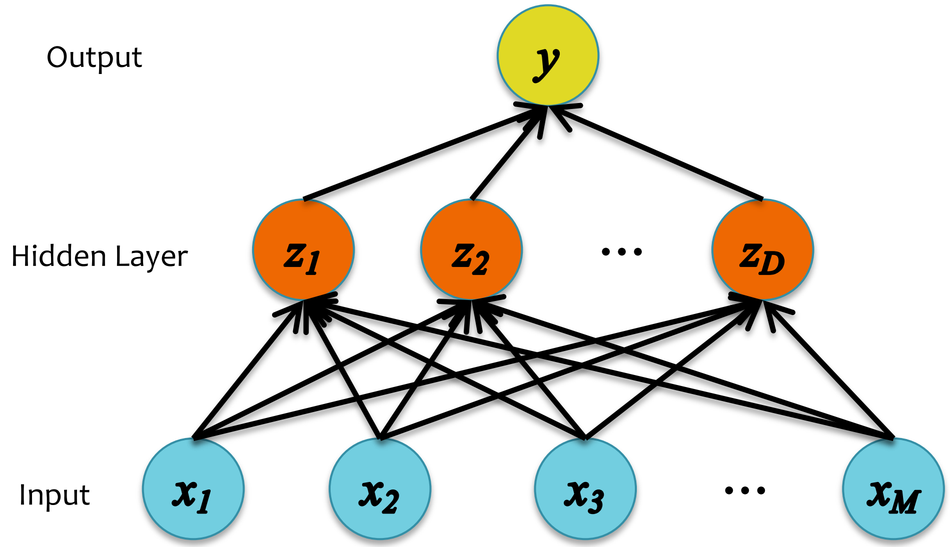
$$\ldots^{+1)} = \boldsymbol{\theta}^{(t)} - \eta_t \nabla \ell(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}), y^{(i)})$$

The Loss Function

Which loss function you choose depends on the problem you are trying to solve

Neural Network for Classification

Binary



(E) Output (sigmoid)
$y = \frac{1}{1+\exp(-b)}$ $\in (0,1)$

(D) Output (linear)
$b = \sum_{j=0}^{D} \beta_j z_j$

(C) Hidden (sigmoid)
$z_j = \frac{1}{1+\exp(-a_j)}, \ \forall j$

(B) Hidden (linear)
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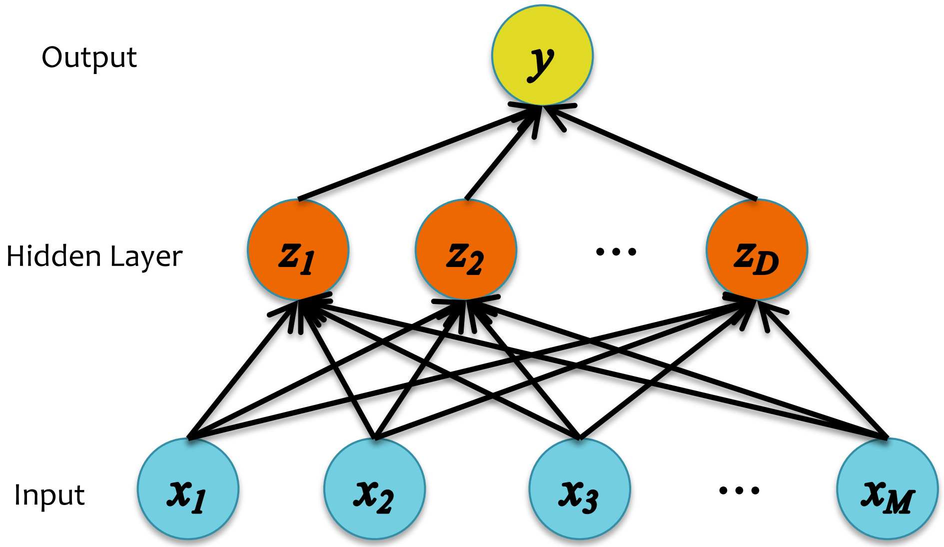
(A) Input
Given $x_i, \ \forall i$

Neural Network for Regression



(D) Output (linear)
$y = \sum_{j=0}^{D} \beta_j z_j$ $\in \mathbb{R}$

(C) Hidden (sigmoid)
$z_j = \frac{1}{1+\exp(-a_j)}, \ \forall j$

(B) Hidden (linear)
$a_j = \sum_{i=0}^{M} \alpha_{ji} x_i, \ \forall j$

(A) Input
Given $x_i, \ \forall i$

Objective Functions for NNs

1. Binary Cross-Entropy:
   - the same objective as Binary Logistic Regression
   - i.e. negative log likelihood
   - This requires our output y to be a probability in [0,1]

$$J = \ell_{CE}(y, y^{(i)}) = -(y^{(i)} \log(y) + (1 - y^{(i)}) \log(1 - y))$$

$$\frac{dJ}{dy} = -\left(y^{(i)} \frac{1}{y} + (1 - y^{(i)}) \frac{1}{y - 1}\right)$$

2. Quadratic Loss:
   - the same objective as Linear Regression
   - i.e. mean squared error

nueral net output

$$J = \ell_Q(y, y^{(i)}) = \frac{1}{2}(y - y^{(i)})^2$$

true output

$$\frac{dJ}{dy} = y - y^{(i)}$$

Cross-entropy vs. Quadratic loss

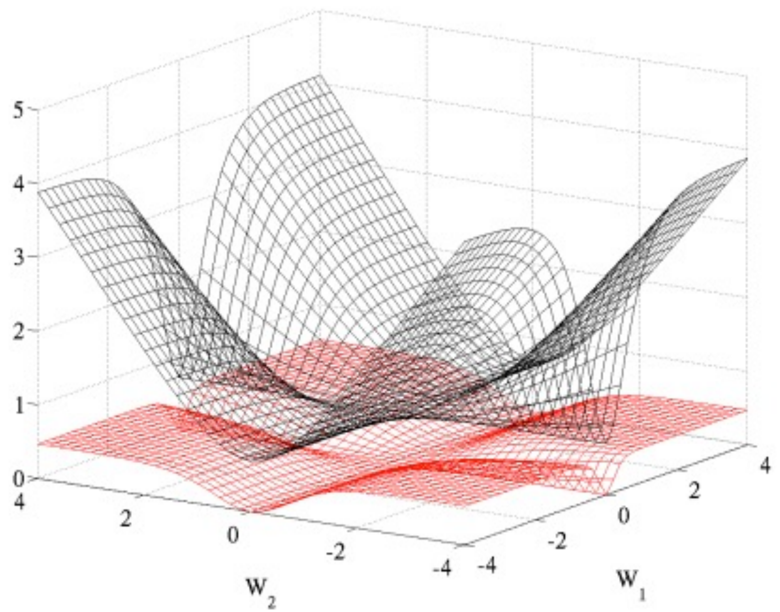
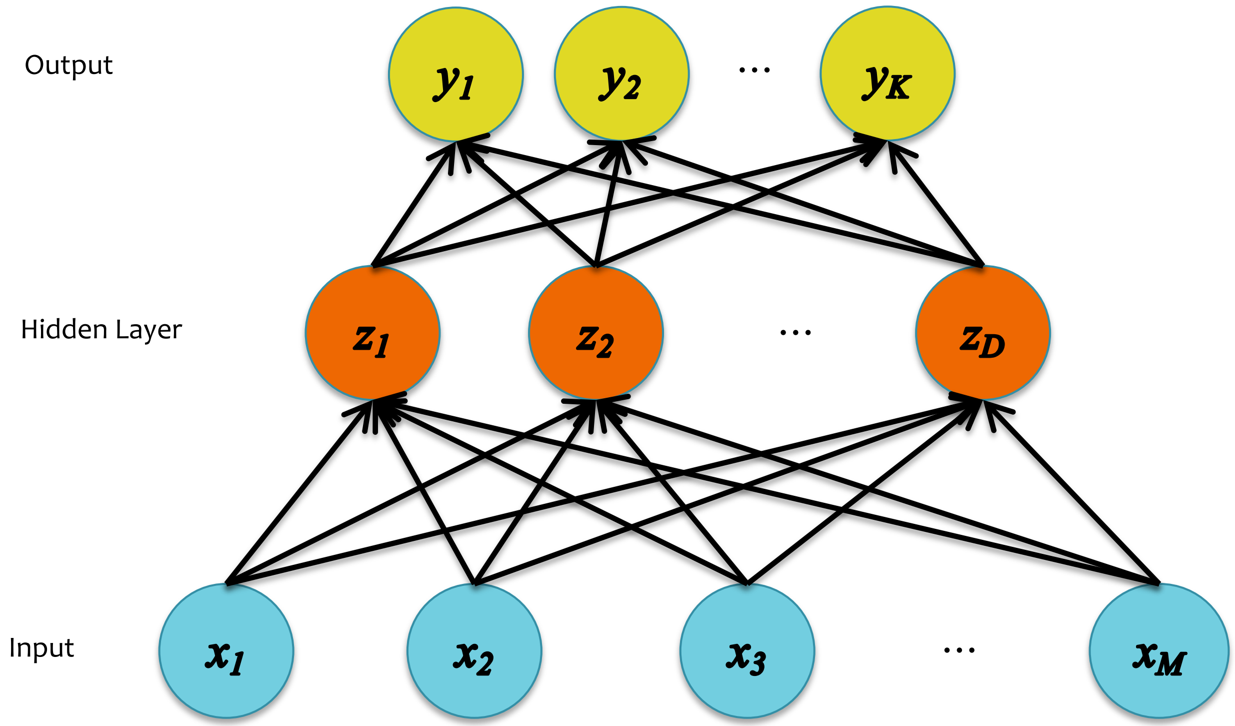


Figure 5: *Cross entropy (black, surface on top) and quadratic (red, bottom surface) cost as a function of two weights (one at each layer) of a network with two layers, W_1 respectively on the first layer and W_2 on the second, output layer.*

Figure from Glorot & Bentio (2010)

Multiclass Output

Softmax: $$y_k = \frac{\exp(b_k)}{\sum_{l=1}^{K} \exp(b_l)}$$



(F) **Loss**
$J = \sum_{k=1}^{K} y_k^* \log(y_k)$

(E) **Output (softmax)**
$y_k = \frac{\exp(b_k)}{\sum_{l=1}^{K} \exp(b_l)}$ $\in (0,1)$

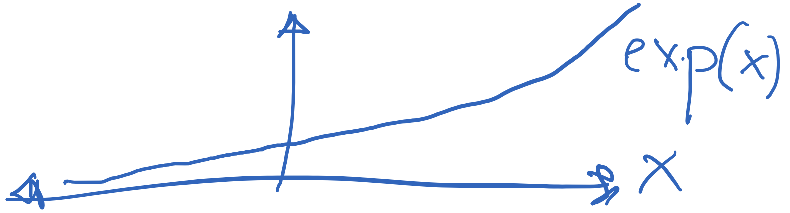
(D) **Output (linear)**
$b_k = \sum_{j=0}^{D} \beta_{kj} z_j \ \forall k$ $\in \mathbb{R}$

(C) **Hidden (nonlinear)**
$z_j = \sigma(a_j), \ \forall j$

(B) **Hidden (linear)**
$a_j = \sum_{i=0}^{M} \alpha_{ji} x_i, \ \forall j$

(A) **Input**
Given $x_i, \ \forall i$

Multiclass Output



Softmax: $$y_k = \frac{\exp(b_k)}{\sum_{l=1}^{K} \exp(b_l)}$$

$\vec{b} = [-1.0,\ 2.7,\ 0.5]$

$\exp(\vec{b}) = [0.3,\ 14.8,\ 1.6]$

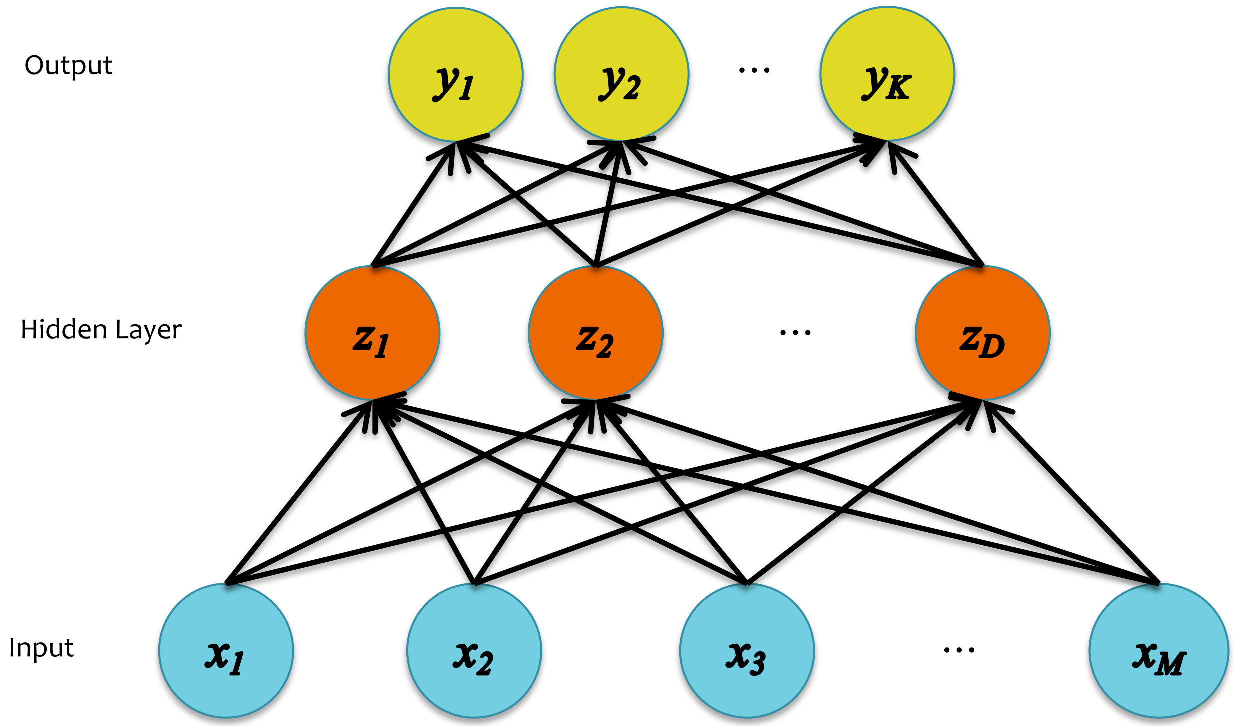
$\text{sum}(\exp(\vec{b})) = 16.7$

$\vec{y} = [0.01,\ 0.90,\ 0.09]$

prob. dist.

① $\left(\sum_{k=1}^{K} y_k\right) = 1.0$

② $y_k \in [0,1]\ \forall k$



Objective Functions for NNs

3. Cross-Entropy for Multiclass Outputs:
   - i.e. negative log likelihood for multiclass outputs
   - Suppose output is a random variable Y that takes one of K values
   - Let $\mathbf{y}^{(i)}$ represent our true label as a one-hot vector:

$$\mathbf{y}^{(i)} = \begin{array}{cccccccc} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{\ldots} & \boxed{0} \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots & K \end{array} \quad \Leftarrow \quad \tilde{y}^{(i)} = 4$$

   - Assume our model outputs a length K vector of probabilities:

$$\mathbf{y} = \text{softmax}(f_{\text{scores}}(\mathbf{x}, \boldsymbol{\theta}))$$

$$= [y_1, y_2, \ldots, y_K]$$

   - Then we can write the log-likelihood of a single training example $(\mathbf{x}^{(i)}, \mathbf{y}^{(i)})$ as:

$$J = \ell_{CE}(\mathbf{y}, \mathbf{y}^{(i)}) = -\sum_{k=1}^{K} y_k^{(i)} \log(y_k) = -\log(y[\tilde{y}^{(i)}])$$

$$= -(0 \cdot \log(y_1) + 0 \cdot \log(y_2) + 0 \cdot \log(y_3) + 1 \cdot \log(y_4) + 0 \cdot \log(y_5) + \cdots + 0 \cdot \log(y_K))$$

$$= -\log(y_4) = -\log \Pr[Y=4 \mid \vec{x}]$$

Binary Classification with Neural Nets

Poll Question 1:

True or False. Suppose we are given a binary classification dataset D. Ignoring issues of nonconvexity, if we train a neural network with a single sigmoid output using cross-entropy loss:

$$-(y^{(i)} \log(y) + (1 - y^{(i)}) \log(1 - y))$$

then this will most likely learn the same classifier function as if we trained a neural network with two softmax-ed output units using a multi-class cross entropy loss:

$$-\sum_{k=1}^{K} y_k^{(i)} \log(y_k)$$

Justify your answer. (no toxic option)

Answer:

Computing Gradients

APPROACHES TO DIFFERENTIATION

A Recipe for Machine Learning

1. Given training data:

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2. Choose each of these:
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$$\ell(\hat{y}, y) \in \mathbb{R}$$

3. Define goal:

$$\hat{\boldsymbol{\theta}} \approx \operatorname*{argmin}_{\boldsymbol{\theta}} \sum_{i=1}^{N} \ell(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}), y^{(i)})$$

4. Train with SGD:

(take small steps opposite the gradient)

$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)} - \eta_t \nabla \ell(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}), y^{(i)})$$

Background

A Recipe for

Gradients

Backpropagation can compute this gradient!

And it's a **special case of a more general algorithm** called reverse-mode automatic differentiation that can compute the gradient of any differentiable function efficiently!

1. Given training data:

$$\mathcal{D} = \{\mathbf{x}^{(i)}, y^{(i)}\}_{i=1}^{N}$$

2. Choose each of these:

– Decision function

$$\hat{y} = h_{\boldsymbol{\theta}}(\mathbf{x})$$

– Loss function

$$\ell(\hat{y}, y) \in \mathbb{R}$$

opposite the gradient)

$$\boldsymbol{\theta}^{(t} \quad {}^{(t)} - \eta_t \nabla \ell(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}), y^{(i)})$$

Approaches to Differentiation

- **Question 1:**
  When can we compute the gradients for an arbitrary neural network?

- **Question 2:**
  When can we make the gradient computation efficient?

Given $f : \mathbb{R}^A \rightarrow \mathbb{R}^B, f(\mathbf{x})$

Compute $\frac{\partial f(\mathbf{x})_i}{\partial x_j} \forall i, j$

Approaches to Differentiation

1. Finite Difference Method

- Pro: Great for testing implementations of backpropagation
- Con: Slow for high dimensional inputs / outputs
- Required: Ability to call the function f(**x**) on any input **x**

2. Symbolic Differentiation

- Note: The method you learned in high-school
- Note: Used by Mathematica / Wolfram Alpha / Maple
- Pro: Yields easily interpretable derivatives
- Con: Leads to exponential computation time if not carefully implemented
- Required: Mathematical expression that defines f(**x**)

Given $f : \mathbb{R}^A \rightarrow \mathbb{R}^B, f(\mathbf{x})$

Compute $\frac{\partial f(\mathbf{x})_i}{\partial x_j} \forall i, j$

Approaches to Differentiation

3. Automatic Differentiation – Reverse Mode

- Note: Called *Backpropagation* when applied to Neural Nets
- Pro: Computes partial derivatives of one output $f(\mathbf{x})_i$ with respect to all inputs x_j in time polynomial in the computation time of $f(\mathbf{x})$
- Con: Slow for high dimensional outputs (e.g. vector-valued functions)
- Required: Algorithm for computing $f(\mathbf{x})$

4. Automatic Differentiation - Forward Mode

- Note: Easy to implement. Uses dual numbers.
- Pro: Computes partial derivatives of all outputs $f(\mathbf{x})_i$ with respect to one input x_j in time polynomial in the computation time of $f(\mathbf{x})$
- Con: Slow for high dimensional inputs (e.g. vector-valued $\mathbf{x}$)
- Required: Algorithm for computing $f(\mathbf{x})$

THE FINITE DIFFERENCE METHOD

Training

Finite Difference Method

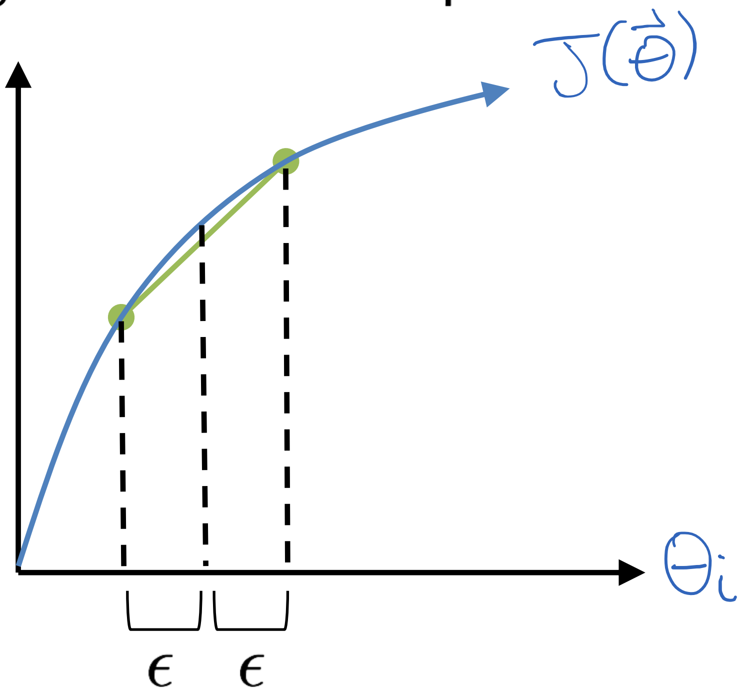
The *centered* finite difference approximation is:

$$\frac{\partial}{\partial \theta_i} J(\boldsymbol{\theta}) \approx \frac{(J(\boldsymbol{\theta} + \epsilon \cdot \boldsymbol{d}_i) - J(\boldsymbol{\theta} - \epsilon \cdot \boldsymbol{d}_i))}{2\epsilon} \quad (1)$$

where $\boldsymbol{d}_i$ is a 1-hot vector consisting of all zeros except for the ith entry of $\boldsymbol{d}_i$, which has value 1.

Notes:

- Suffers from issues of floating point precision, in practice
- Typically only appropriate to use on small examples with an appropriately chosen epsilon



Differentiation Quiz

Speed Quiz:
2 minute time limit.

Poll Question 2: Differentiation Quiz #1

Suppose x = 2 and z = 3, what are dy/dx and dy/dz for the function below? **Round your answer to the nearest integer**.

$$y = \exp(xz) + \frac{xz}{\log(x)} + \frac{\sin(\log(x))}{xz}$$

Answer: *Answers below are in the form [dy/dx, dy/dz]*

A. [42, -72]
B. [72, -42]
C. [100, 127]
D. [127, 100]
E. [1208, 810]
F. [810, 1208]
G. [1505, 94]
H. [94, 1505]

Differentiation Quiz

Differentiation Quiz #2:

A neural network with 2 hidden layers can be written as:

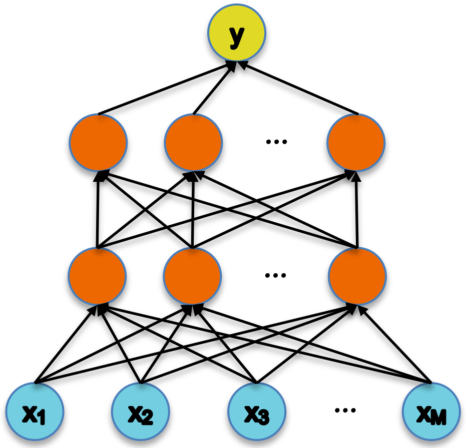
$$y = \sigma(\boldsymbol{\beta}^T \sigma((\boldsymbol{\alpha}^{(2)})^T \sigma((\boldsymbol{\alpha}^{(1)})^T \mathbf{x}))$$

where $y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^{D^{(0)}}$, $\boldsymbol{\beta} \in \mathbb{R}^{D^{(2)}}$ and $\boldsymbol{\alpha}^{(i)}$ is a $D^{(i)} \times D^{(i-1)}$ matrix. Nonlinear functions are applied elementwise:

$$\sigma(\mathbf{a}) = [\sigma(a_1), \ldots, \sigma(a_K)]^T$$

Let σ be sigmoid: $\sigma(a) = \frac{1}{1+exp{-a}}$

What is $\frac{\partial y}{\partial \beta_j}$ and $\frac{\partial y}{\partial \alpha_j^{(i)}}$ for all i, j.



THE CHAIN RULE OF CALCULUS

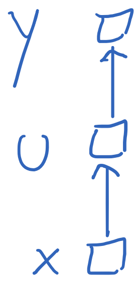
Chain Rule

Definition 1:

Given

$$y = f(u)$$
$$u = g(x)$$

Computation Graph



Chain Rule

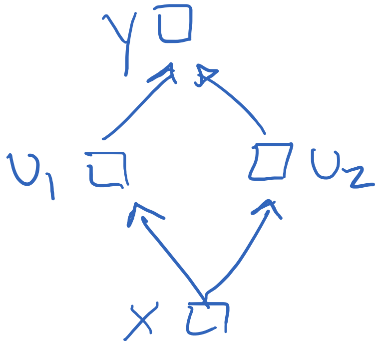
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Definition 2:

Given

$$y = f(u_1, u_2)$$
$$u_2 = g_2(x)$$
$$u_1 = g_1(x)$$

Computation Graph



Chain Rule

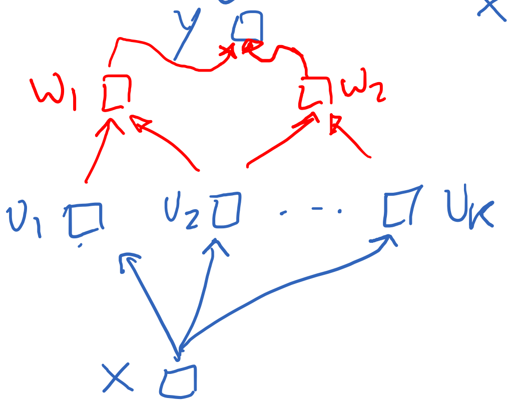
$$\frac{dy}{dx} = \frac{dy}{du_1}\frac{du_1}{dx} + \frac{dy}{du_2}\frac{du_2}{dx}$$

Definition 3:

Given

$$y = f(\vec{u}) \qquad y \in \mathbb{R}$$
$$\vec{u} = g(x) \qquad \vec{u} \in \mathbb{R}^K$$
$$x \in \mathbb{R}$$

Computation Graph



Chain Rule

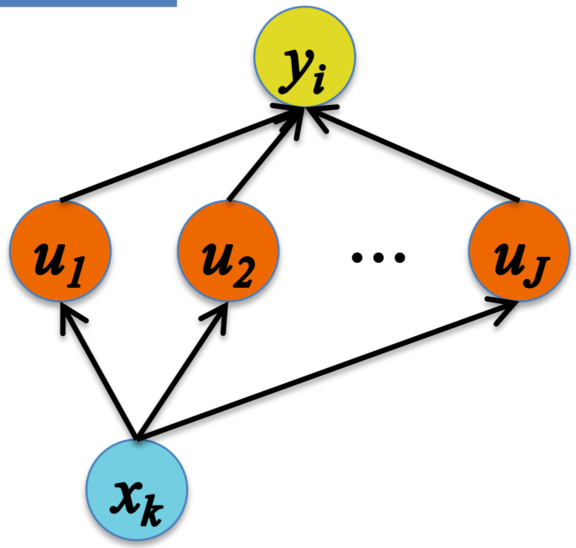
$$\frac{dy}{dx} = \sum_{k=1}^{K} \frac{dy}{du_k}\frac{du_k}{dx}$$

Chain Rule

Given: $\boldsymbol{y} = g(\boldsymbol{u})$ and $\boldsymbol{u} = h(\boldsymbol{x})$.

Chain Rule:

$$\frac{dy_i}{dx_k} = \sum_{j=1}^{J} \frac{dy_i}{du_j} \frac{du_j}{dx_k}, \quad \forall i, k$$



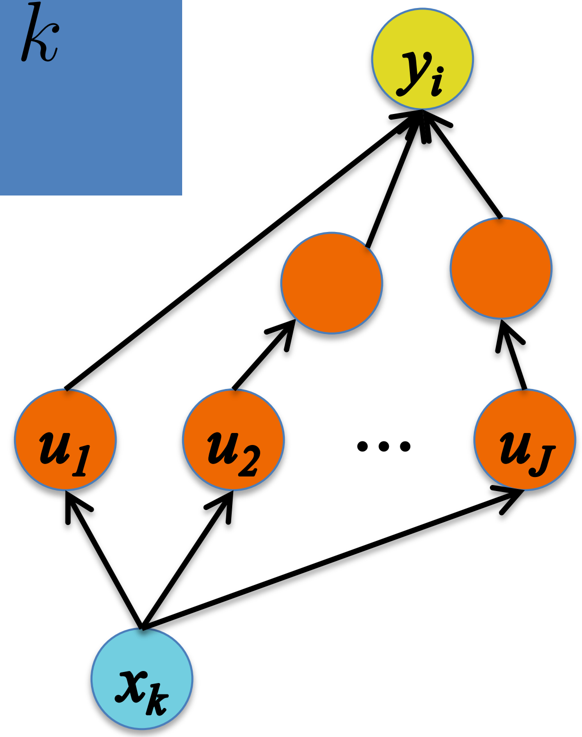
Chain Rule

Given: $\boldsymbol{y} = g(\boldsymbol{u})$ and $\boldsymbol{u} = h(\boldsymbol{x})$.

Chain Rule:

$$\frac{dy_i}{dx_k} = \sum_{j=1}^{J} \frac{dy_i}{du_j}\frac{du_j}{dx_k}, \quad \forall i, k$$

Backpropagation is just repeated application of the **chain rule** from Calculus 101.



Algorithm

FORWARD COMPUTATION FOR A COMPUTATION GRAPH

Backpropagation

Speed Quiz:
2 minute time limit.

Differentiation Quiz #1:

Suppose x = 2 and z = 3, what are dy/dx and dy/dz for the function below? **Round your answer to the nearest integer**.

$$y = \exp(xz) + \frac{xz}{\log(x)} + \frac{\sin(\log(x))}{xz}$$

Now let's solve this using backpropagation!

Given: $y = \exp(xz) + \frac{xz}{\log(x)} + \frac{\sin(\log(x))}{xz} = f(x,z)$

Forward Computation:

Given $x = 2$, $z = 3$

$a = xz$

$b = \log(x)$

$c = \sin(b)$

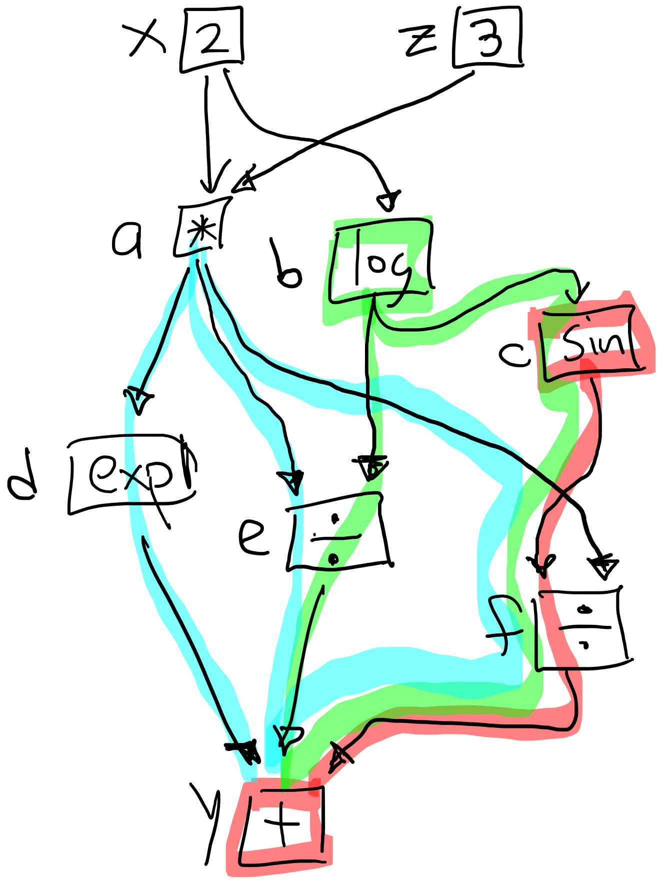
$d = \exp(a)$

$e = a/b$

$f = c/a$

$y = d + e + f$

Computation Graph:



Backward Computation

$g_y = \frac{\partial y}{\partial y} = 1$

$g_f = \frac{\partial y}{\partial f} = 1, \quad g_e = \frac{\partial y}{\partial e} = 1, \quad g_d = \frac{\partial y}{\partial d} = 1$

$g_c = \frac{\partial y}{\partial c} = \frac{\partial y}{\partial f}\frac{\partial f}{\partial c} = (g_f)\left(\frac{1}{a}\right)$

$g_b = \frac{\partial y}{\partial b} = \frac{\partial y}{\partial c}\frac{\partial c}{\partial b} + \frac{\partial y}{\partial e}\frac{\partial e}{\partial b}$

$= (g_c)(\cos(b)) + (g_e)\left(-\frac{a}{b^2}\right)$

$g_a = \frac{\partial y}{\partial a} = \frac{\partial y}{\partial d}\frac{\partial d}{\partial a} + \frac{\partial y}{\partial e}\frac{\partial e}{\partial a} + \frac{\partial y}{\partial f}\frac{\partial f}{\partial a}$

$= (g_d)(\exp(a)) + (g_e)\left(\frac{1}{b}\right) + (g_f)\left(-\frac{c}{a^2}\right)$

$g_x = \frac{\partial y}{\partial x} = (g_a)(z) + (g_b)\left(\frac{1}{x}\right)$

$g_z = \frac{\partial y}{\partial z} = (g_a)(x)$

Backpropagation

Updates for Backpropagation:

$$g_x = \frac{\partial y}{\partial x} = \sum_{k=1}^{K} \frac{\partial y}{\partial u_k} \frac{\partial u_k}{\partial x}$$
$$= \sum_{k=1}^{K} g_{u_k} \frac{\partial u_k}{\partial x}$$

Backprop is efficient b/c of reuse in the forward pass and the backward pass.