## 10-301/601: Introduction to Machine Learning Lecture 15 - Learning Theory (Infinite Case)

Henry Chai \& Matt Gormley \& Hoda Heidari

- Announcements
- HW5 released 10/9, due 10/27 (Friday) at 11:59 PM


## Front Matter

- Exam 3 scheduled
- Tuesday, December $12^{\text {th }}$ from 5:30 PM to 8:30 PM
- Sign up for peer tutoring! See Piazza for more details

Recall Theorem 1:
Finite,
Realizable Case

- For a finite hypothesis set $\mathcal{H}$ such that $c^{*} \in \mathcal{H}$ (realizable) and arbitrary distribution $p^{*}$, if the number of labelled training data points satisfies

$$
N=\frac{1}{\epsilon}\left(\ln (|\mathcal{H}|)+\ln \left(\frac{1}{\delta}\right)\right)
$$

then with probability at least $1-\delta$, all $h \in \mathcal{H}$ with

$$
\underbrace{\hat{R}(h)=0 \text { have } R(h) \leq \epsilon, ~}
$$

Recall Theorem 1:
Finite,
Realizable Case

- For a finite hypothesis set $\mathcal{H}$ such that $c^{*} \in \mathcal{H}$ (realizable) and arbitrary distribution $p^{*}$, if the number of labelled training data points satisfies

$$
\underbrace{}_{S}=\frac{1}{\operatorname{C}}(\ln (\underbrace{|\mathcal{H}|})+\ln \left(\frac{1}{\delta}\right))
$$

then with probability at least $1-\delta$, all $h \in \mathcal{H}$ with $\hat{R}(h)=0$ have $R(h) \leq \epsilon$

- Making the bound tight and solving for $\epsilon$ gives...


## Statistical Learning Theory Corollary

- For a finite hypothesis set $\mathcal{H}$ such that $\underbrace{c^{*} \in \mathcal{H}}$
(realizable) and arbitrary distribution $p^{*}$, given a training dataset $S$ where $|S|=N$, all $h \in \mathcal{H}$ with $\hat{R}(h)=0$ have

$$
R(h) \leq \underbrace{\frac{1}{N}\left(\ln (|\mathcal{H}|)+\ln \left(\frac{1}{\delta}\right)\right)}
$$

with probability at least $1-\delta$.

$$
\text { Agnostic: }\left\{\begin{array}{lll}
c^{*} & \notin & \mathbb{L} \\
c^{*} \in \mathbb{H}
\end{array}\right.
$$

- For a finite hypothesis set $\mathcal{H}$ and arbitrary distribution $p^{*}$, if the number of labelled training data points satisfies

Recall -
Theorem 2:
Finite,
Agnostic Case

$$
N \geq \frac{1}{2 \epsilon^{2}}\left(\ln (|\mathcal{H}|)+\ln \left(\frac{2}{\delta}\right)\right)
$$

then with probability at least $1-\delta$, all $h \in \mathcal{H}$ satisfy $|\underbrace{R(h)}-\underbrace{\hat{R}(h)}| \leq \epsilon$

- Bound is inversely quadratic in $\epsilon$, egg., halving $\epsilon$ means we need four times as many labelled training data points


## Statistical Learning Theory Corollary

- For a finite hypothesis set $\mathcal{H}$ and arbitrary distribution $p^{*}$, given a training dataset $S$ where $|S|=N$, all $h \in \mathcal{H}$ have

$$
\underbrace{R(h)} \leq \underbrace{\hat{R}(h)}+\sqrt{\frac{1}{2 N}\left(\ln (|\mathcal{H}|)+\ln \left(\frac{2}{\delta}\right)\right)}
$$

with probability at least $1-\delta$. $\mathcal{E}$

## What happens when $|\mathcal{H}|=\infty$ ?

- For a finite hypothesis set $\mathcal{H}$ and arbitrary distribution $p^{*}$, given a training data set $S$ where $|S|=N$, all $h \in \mathcal{H}$ have

$$
R(h) \leq \hat{R}(h)+\sqrt{\frac{1}{2 N}\left(\ln (|\mathcal{H}|)+\ln \left(\frac{2}{\delta}\right)\right)}
$$

with probability at least $1-\delta$.

## Labellings

- Given some finite set of data points $S=\left\{x^{(1)}, \ldots, x^{(\mathbb{M})}\right\}$ and some hypothesis $h \in \mathcal{H}$, applying $h$ to each point in $S$ results in a labelling
(bivary dassitication)
- $\left[h\left(\boldsymbol{x}^{(1)}\right), \ldots, h\left(\boldsymbol{x}^{(N)}\right)\right]$ is a vector of $N+1$ 's and -1's (recall: our discussion of PAC learning assumes binary classification)
- Given $S=\left\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N)}\right\}$, each hypothesis in $\mathcal{H}$ induces a labelling but not necessarily a unique labelling
- The set of labellings induced by $\mathcal{H}$ on $S$ is

$$
\mathcal{H}(S)=\{\left[h\left(\boldsymbol{x}^{(1)}\right), \ldots, h\left(\boldsymbol{x}^{(N)}\right)\right]|\underbrace{h \in \mathcal{H}\}} \quad| \mathcal{F}(S) \mid \leqslant 2^{|S|}
$$

## Example: Labellings

$$
\begin{aligned}
& \mathcal{H}=\left\{h_{1}, h_{2}, h_{3}\right\} \quad \mathcal{H}(S)=? \\
& S=\left\{x^{(1)}, \ldots, x^{(4)}\right\}
\end{aligned}
$$



## Example: Labellings

$$
\begin{aligned}
& \mathcal{H}=\left\{h_{1}, h_{2}, h_{3}\right\} \\
& {\left[h_{1}\left(x^{(1)}\right), h_{1}\left(x^{(2)}\right), h_{1}\left(x^{(3)}\right), h_{1}\left(x^{(4)}\right)\right]} \\
& =(\underbrace{-1,}+\underbrace{+1},+1)
\end{aligned}
$$



## Example: Labellings

$$
\begin{aligned}
& \mathcal{H} \mathcal{E}=\left\{h_{1}, h_{2}, h_{3}\right\} \\
& {\left[h_{2}\left(x^{(1)}\right), h_{2}\left(x^{(2)}\right), h_{2}\left(x^{(3)}\right), h_{2}\left(x^{(4)}\right)\right]} \\
& =(-1,-1,-1)
\end{aligned}
$$



## Example: Labellings

$$
\begin{aligned}
& \mathcal{H}=\left\{h_{1}, h_{2}, h_{3}\right\} \\
& {\left[h_{1}\left(x^{(1)}\right), h_{1}\left(x^{(2)}\right), h_{1}\left(x^{(3)}\right), h_{1}\left(x^{(4)}\right)\right]} \\
& =(+1,+1,-1,-1)
\end{aligned}
$$



## Example: Labellings

$$
\begin{aligned}
& \mathcal{H}=\left\{h_{1}, h_{2}, h_{3}\right\} \\
& =\underbrace{\underbrace{\mathcal{H}(S)}} \\
& |\mathcal{H}(S)|=\underbrace{2}
\end{aligned}
$$



## Example: Labellings

$$
\begin{aligned}
& \mathcal{H}=\left\{h_{1}, h_{2}, h_{3}\right\} \\
& \mathcal{H}(S) \\
& =\{[+1,+1,-1,-1]\} \\
& |\mathcal{H}(S)|=1
\end{aligned}
$$



- $\underbrace{\mathcal{H}(S)}$ is the set of all labellings induced by $\mathcal{H}$ on $S$
- If $|S|=N$, then $|\mathcal{H}(S)| \leq 2^{N}$
- $\mathcal{H}$ shatters $S$ if $|\mathcal{H}(S)|=2^{N}$
- The VC-dimension of $\mathcal{H}, V C(\mathcal{H})$, is the size of the largest


## VC-Dimension

 set $S$ that can be shattered by $\mathcal{H}$.- If $\mathcal{H}$ can shatter arbitrarily large finite sets, then

$$
V C(\mathcal{H})=\infty
$$

- To prove that $V C(\mathcal{H})=d$, you need to show
$v \subset(H) \geqslant d \quad 1 . \exists$ some set of $d$ data points that $\mathcal{H}$ can shatter and
VC(H) $\& d 2$. $\nexists$ a set of $d+1$ data points that $\mathcal{H}$ can shatter
- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
$V C(H) \geqslant 1$
- Can $\mathcal{H}$ shatter some set of 1 point?


## VC-Dimension:

 Example
$S$

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
$\operatorname{VC}(H) \geqslant 2^{\circ}$ Can $\mathcal{H}$ shatter some set of 2 points?


## VC-Dimension:

## Example


$S$


- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
$\operatorname{vc}(H) \geqslant 3 \cdot C a n \mathcal{H}$ shatter some set of 3 points?


## VC-Dimension:

 Example
$S$

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?


## VC-Dimension:

 Example
$S$

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?


## VC-Dimension:

 Example
$S$

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?


## VC-Dimension:

 Example
$S$

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?


## VC-Dimension:

 Example
$S_{1}$

$S_{2}$

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?


## VC-Dimension:

## Example



$$
\left|\mathcal{H}\left(S_{1}\right)\right|=6
$$


$\left|\mathcal{H}\left(S_{2}\right)\right|=8$

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?

$S_{1}$
All points on the convex hull

$S_{2}$
At least one point inside the convex hull
- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?


## VC-Dimension: Example


$S_{1}$
All points on the convex hull

$S_{2}$
At least one point inside the convex hull

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?


## VC-Dimension: Example


$S_{1}$
All points on the convex hull

$S_{2}$
At least one point inside the convex hull

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?


## VC-Dimension: Example


$\left|\mathcal{H}\left(S_{1}\right)\right|=14$
All points on the convex hull

$S_{2}$
At least one point inside the convex hull

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?


## VC-Dimension: Example


$\left|\mathcal{H}\left(S_{1}\right)\right|=14$
All points on the convex hull

$S_{2}$
At least one point inside the convex hull

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?


## VC-Dimension: Example


$\left|\mathcal{H}\left(S_{1}\right)\right|=14$
All points on the convex hull

$S_{2}$
At least one point inside the convex hull

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- What is $V C(\mathcal{H})$ ?
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?


## VC-Dimension: Example


$\left|\mathcal{H}\left(S_{1}\right)\right|=14$
All points on the convex hull


$$
\left|\mathcal{H}\left(S_{2}\right)\right|=14
$$

At least one point inside the convex hull

- $\boldsymbol{x} \in \mathbb{R}^{2}$ and $\mathcal{H}=$ all 2-dimensional linear separators
- $V C(\mathcal{H})=3$
- Can $\mathcal{H}$ shatter some set of 1 point?
- Can $\mathcal{H}$ shatter some set of 2 points?
- Can $\mathcal{H}$ shatter some set of 3 points?
- Can $\mathcal{H}$ shatter some set of 4 points?


## VC-Dimension: Example


$\left|\mathcal{H}\left(S_{1}\right)\right|=14$
All points on the convex hull


$$
\left|\mathcal{H}\left(S_{2}\right)\right|=14
$$

At least one point inside the convex hull

- $\boldsymbol{x} \in \mathbb{R}^{d}$ and $\mathcal{H}=$ all $d$-dimensional linear separators
- $V C(\mathcal{H})=d+1$


## VC-Dimension: Example

# VC-Dimension: Example 

- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x ; a)=\operatorname{sign}(x-a)$

- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x ; a)=\operatorname{sign}(x-a)$


## VC-Dimension: Example



- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x ; a)=\operatorname{sign}(x-a)$


## VC-Dimension: Example



## VC-Dimension:

## Example

- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x ; a)=\operatorname{sign}(x-a)$

- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x ; a)=\operatorname{sign}(x-a)$


## VC-Dimension: Example



- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x ; a)=\operatorname{sign}(x-a)$


## VC-Dimension: Example



# VC-Dimension: 

## Example

- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x ; a)=\operatorname{sign}(x-a)$

- $V C(\mathcal{H})=1$
- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive intervals


# VC-Dimension: Example 



- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive intervals


## Poll Question 1:

What is $V C(\mathcal{H})$ ?
A. 0
B. 1
C. 1.5 (TOXIC)
D. 2
E. 3

- $x \in \mathbb{R}$ and $\mathcal{H}=$ all 1-dimensional positive intervals


# VC-Dimension: 

## Example



- $V C(\mathcal{H})=2$

$$
|\mu|=\infty \text { but } \operatorname{Vc}(x)<\infty
$$

- Infinite, realizable case: for any hypothesis set $\mathcal{H}$ such that $c^{*} \in \mathcal{H}$ and arbitrary distribution $p^{*}$, if the number of labelled training data points satisfies

$$
\left.N=0\left(\frac{1}{(\sqrt{\ln |\mathcal{L}|}}\left(\frac{\downarrow}{V C(\mathcal{H})}\right] \log \left(\frac{1}{\epsilon}\right)+\log \left(\frac{1}{\delta}\right)\right)\right)
$$

then with probability at least $1-\delta$, all $h \in \mathcal{H}$ with

$$
\begin{aligned}
& \hat{R}(h)=0 \text { have } R(h) \leq \epsilon \\
& \text { consistent }
\end{aligned}
$$

## Statistical Learning Theory Corollary 3

- Infinite, realizable case: for any hypothesis set $\mathcal{H}$ such that $c^{*} \in \mathcal{H}$ and arbitrary distribution $p^{*}$, given a training dataset $S$ where $|S|=N$, all $h \in \mathcal{H}$ with $\hat{R}(h)=0$ have

$$
R(h) \leq O\left(\frac{1}{N}\left(V C(\mathcal{H}) \log \left(\frac{N}{V C(\mathcal{H})}\right)+\log \left(\frac{1}{\delta}\right)\right)\right)
$$

with probability at least $1-\delta$.

$$
|\mathscr{H}|=\infty \text { bat } \operatorname{vc}(\mathbb{H})<\infty
$$

- Infinite, agnostic case: for any hypothesis set $\mathcal{H}$ and

Theorem 4:
Vapnik-
Chervonenkis (VC)-Bound
arbitrary distribution $p^{*}$, if the number of labelled training data points satisfies
then with probability at least $1-\delta$, all $h \in \mathcal{H}$ have $|R(h)-\hat{R}(h)| \leq \epsilon$

# Statistical <br> Learning Theory Corollary 4 

- Infinite, agnostic case: for any hypothesis set $\mathcal{H}$ and arbitrary distribution $p^{*}$, given a training dataset $S$ where $|S|=N$, all $h \in \mathcal{H}$ have

$$
R(h) \leq \hat{R}(h)+O\left(\sqrt{\frac{1}{N}\left(V C(\mathcal{H})+\log \left(\frac{1}{\delta}\right)\right)}\right)
$$

with probability at least $1-\delta$.

## Approximation Generalization Tradeoff

How well does $h$ generalize?


How well does $h$ approximate $c^{*}$ ?

Increases as
$V C(\mathcal{H})$ increases
Approximation Generalization Tradeoff
$R(h) \leq \underbrace{\hat{R}}(h)+O\left(\sqrt{\frac{1}{N}\left(V C(\mathcal{H})+\log \left(\frac{1}{\delta}\right)\right)}\right)$
Decreases as
$V C(\mathcal{H})$ increases

## Can we use this corollary to guide model selection?

- Infinite, agnostic case: for any hypothesis set $\mathcal{H}$ and arbitrary distribution $p^{*}$, given a training dataset $S$ where $|S|=N$, all $h \in \mathcal{H}$ have

$$
R(h) \leq \hat{R}(h)+O\left(\sqrt{\frac{1}{N}\left(V C(\mathcal{H})+\log \left(\frac{1}{\delta}\right)\right)}\right)
$$

with probability at least $1-\delta$.

# Learning Theory and Model <br> Selection 



# Learning Theory and Model <br> Selection 



- How can we find this "best tradeoff" for linear separators?
- Use a regularizer! By (effectively) reducing the number of features our model considers, we reduce its VC-dimension.

You should be able to...

- Identify the properties of a learning setting and assumptions required to ensure low generalization error
- Distinguish true error, train error, test error
- Define PAC and explain what it means to be approximately correct and what occurs with high probability
- Apply sample complexity bounds to real-world machine learning examples
- Theoretically motivate regularization


## 10-301/601: Introduction

 to Machine Learning Lecture 15 Societal Impacts of MLHenry Chai \& Matt Gormley \& Hoda Heidari 10/23/23

## ML in Societal Applications

## 8 WAYS MACHINE LEARNING WILL IMPROVE EDUCATION

sY Matruewlycch । © June 12, 2018 ।


: TheUpshot
ROBO RECRUITING

Artificial Intelligence and Accessibility: Examples of a Technology that Serves People with Disabilities


Elbe Atcu? Your Future Doctor May Not be Human. This Is the Rise of AI in Medicine.
From mental health apps to robot surgeons, artificial intelligence is already changing the practice of medicine.

## Can an Algorithm Hire

 Better Than a Human?[^0]
## Algorithms and bias: What lenders need to know

Misinformation on coronavirus is proving highly contagious
The algorithms that power fintech may discrimin can be difficult to anticipate-and financial institut
accountable even when alleged discrimination is
unintentional.

Wanted: The 'perfect babysitter.' Must


## I.R.S. Changes Audit Practice That Discriminated Against Black Taxpayers

The agency will overhaul how it scrutinizes returns that claim the earned-income tax credit, which is aimed at alleviating poverty.
$\qquad$

Artificial intelligence is slated to disrupt 4.5 million jobs for African Americans, who have a 10\% greater likelihood of automation-based job loss than other workers

Allana Akhtar Oct7,2019, 12:57 PM

If you're not a white male, artificial intelligence's use in healthcare could be dangerous



How Facebook Is Giving Sex Discrimination in Employment Ads a New Life


By AVVID KLEPPER Juy 29,2020




[^0]:    By Claire Cain Miller

