10-301/601: Introduction to Machine Learning Lecture 15 — Learning Theory (Infinite Case)

Henry Chai & Matt Gormley & Hoda Heidari 10/23/23

Front Matter

- Announcements
 - HW5 released 10/9, due 10/27 (Friday) at 11:59 PM
 - Exam 3 scheduled
 - Tuesday, December 12th from 5:30 PM to 8:30 PM
 - Sign up for peer tutoring! See Piazza for more details

Recall Theorem 1: Finite, Realizable Case

• For a *finite* hypothesis set \mathcal{H} such that $c^* \in \mathcal{H}$ (realizable) and arbitrary distribution p^* , if the number of labelled training data points satisfies

$$\frac{1}{\epsilon} \left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right) \right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ with

$$\widehat{R}(h) = 0$$
 have $R(h) \le \epsilon$

Recall Theorem 1: Finite, Realizable Case

• For a *finite* hypothesis set \mathcal{H} such that $c^* \in \mathcal{H}$ (realizable) and arbitrary distribution p^* , if the number of labelled training data points satisfies

$$N = \frac{1}{\epsilon} \left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right) \right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ with $\widehat{R}(h) = 0$ have $R(h) \le \epsilon$

• Making the bound tight and solving for ϵ gives...

Statistical Learning Theory Corollary

• For a *finite* hypothesis set \mathcal{H} such that $\underline{c}^* \in \mathcal{H}$ (*realizable*) and arbitrary distribution p^* , given a training dataset S where |S| = N, all $h \in \mathcal{H}$ with $\widehat{R}(h) = 0$ have

$$R(h) \le \frac{1}{N} \left(\ln(|\mathcal{H}|) + \ln\left(\frac{1}{\delta}\right) \right)$$

with probability at least $1 - \delta$.

Recall Theorem 2: Finite, Agnostic Case

• For a *finite* hypothesis set ${\mathcal H}$ and arbitrary distribution p^* , if the number of labelled training data points satisfies

$$N \ge \frac{1}{2\epsilon^2} \left(\ln(|\mathcal{H}|) + \ln\left(\frac{2}{\delta}\right) \right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ satisfy

$$\left| R(h) - \hat{R}(h) \right| \le \epsilon$$

• Bound is inversely quadratic in ϵ , e.g., halving ϵ means we need four times as many labelled training data points

Statistical Learning Theory Corollary

• For a *finite* hypothesis set $\mathcal H$ and arbitrary distribution p^* , given a training dataset S where |S|=N, all $h\in\mathcal H$ have

$$R(h) \leq \hat{R}(h) + \sqrt{\frac{1}{2N} \left(\ln(|\mathcal{H}|) + \ln\left(\frac{2}{\delta}\right) \right)}$$
 with probability at least $1 - \delta$.

What happens when $|\mathcal{H}| = \infty$?

• For a *finite* hypothesis set $\mathcal H$ and arbitrary distribution p^* , given a training data set S where |S|=N, all $h\in\mathcal H$ have

$$R(h) \le \hat{R}(h) + \sqrt{\frac{1}{2N}} \left(\ln(|\mathcal{H}|) + \ln\left(\frac{2}{\delta}\right) \right)$$

with probability at least $1 - \delta$.

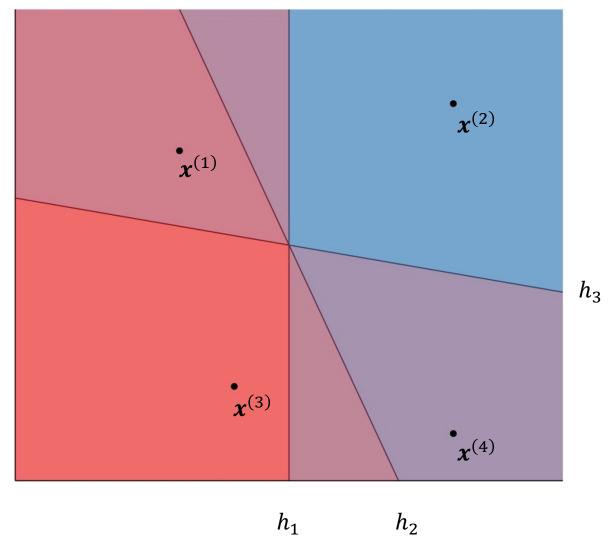
Labellings

- Given some finite set of data points $S = \{x^{(1)}, ..., x^{(N)}\}$ and some hypothesis $h \in \mathcal{H}$, applying h to each point in S results in a labelling (binary dessitication)
 - $[h(x^{(1)}), ..., h(x^{(N)})]$ is a vector of N +1's and -1's (recall: our discussion of PAC learning assumes binary classification)
- Given $S = \{x^{(1)}, ..., x^{(N)}\}$, each hypothesis in \mathcal{H} induces a labelling but not necessarily a unique labelling
 - The set of labellings induced by \mathcal{H} on S is

$$\mathcal{H}(S) = \{ [h(\mathbf{x}^{(1)}), \dots, h(\mathbf{x}^{(N)})] \mid h \in \mathcal{H} \} \quad |\mathcal{H}(S)| \in 2^{|S|}$$

$$\mathcal{H} = \{h_1, h_2, h_3\} \qquad \mathcal{J} + \{(S) = ?$$

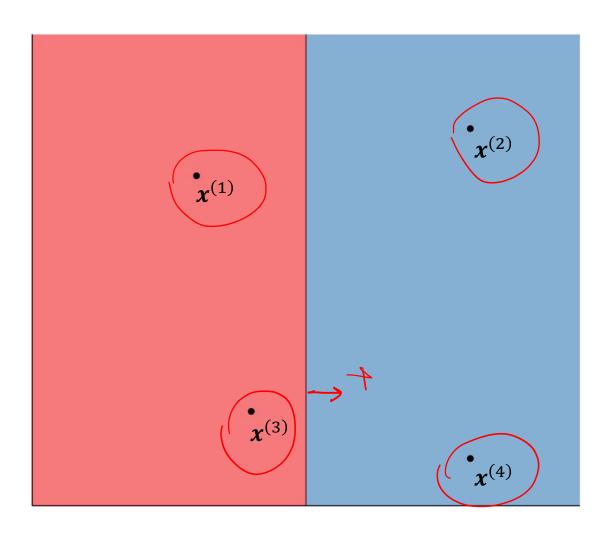
$$S = \{X^{(1)}, \dots, A^{(4)}\}$$



$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$[h_1(x^{(1)}), h_1(x^{(2)}), h_1(x^{(3)}), h_1(x^{(4)})]$$

$$= (-1, +1, -1, +1)$$

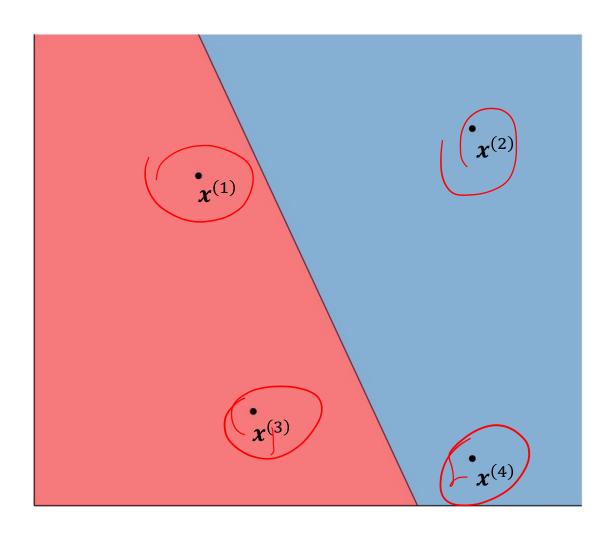


 h_1

$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$[h_{2}(x^{(1)}), h_{2}(x^{(2)}), h_{2}(x^{(3)}), h_{2}(x^{(4)})]$$

$$= (-1, +1) -1, +1)$$

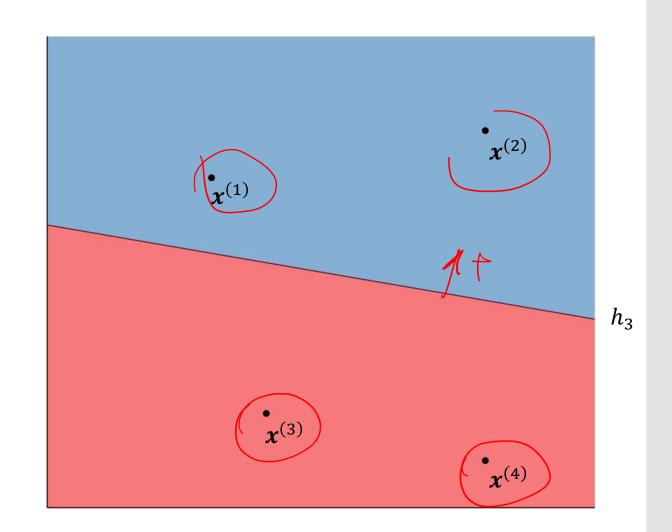


 h_2

$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$[h_1(x^{(1)}), h_1(x^{(2)}), h_1(x^{(3)}), h_1(x^{(4)})]$$

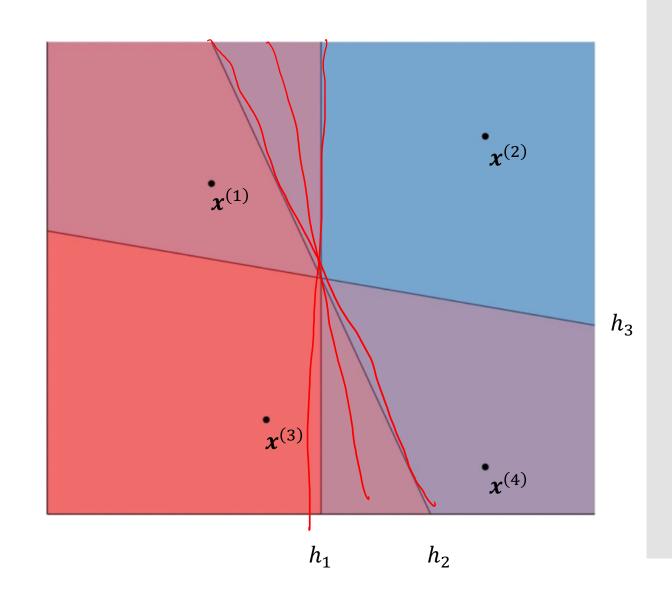
$$= (+1, +1, -1, -1)$$



$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$\mathcal{H}(S)$$
= {[+1, +1, -1, -1], [-1, +1, -1, +1]}

$$|\mathcal{H}(S)| = 2$$

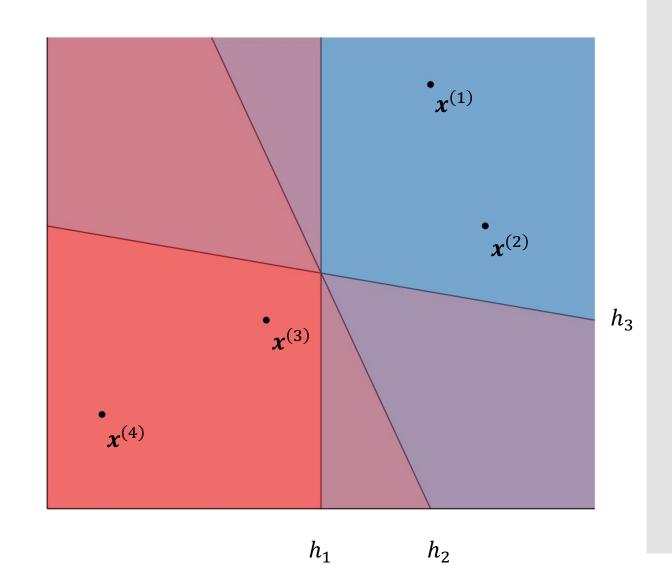


$$\mathcal{H} = \{h_1, h_2, h_3\}$$

$$\mathcal{H}(S)$$

= {[+1, +1, -1, -1]}

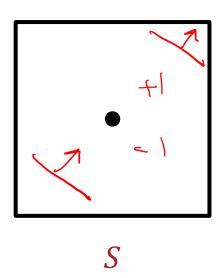
$$|\mathcal{H}(S)| = 1$$



VC-Dimension

- $\mathcal{H}(S)$ is the set of all labellings induced by \mathcal{H} on S
 - If |S| = N, then $|\mathcal{H}(S)| \le 2^N$
 - \mathcal{H} shatters S if $|\mathcal{H}(S)| = 2^N$
- The VC-dimension of \mathcal{H} , $VC(\mathcal{H})$, is the size of the largest set S that can be shattered by \mathcal{H} .
 - If \mathcal{H} can shatter arbitrarily large finite sets, then $VC(\mathcal{H}) = \infty$
- To prove that $VC(\mathcal{H}) = d$, you need to show
- VC(H) $\gg \lambda$ 1. \exists some set of d data points that \mathcal{H} can shatter and VC(H) $\not \sim \lambda$ 2. $\not \equiv$ a set of d+1 data points that \mathcal{H} can shatter

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
- VC(H) > 1 Can \mathcal{H} shatter some set of 1 point?



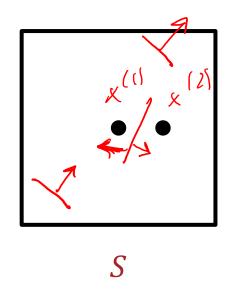
• $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators

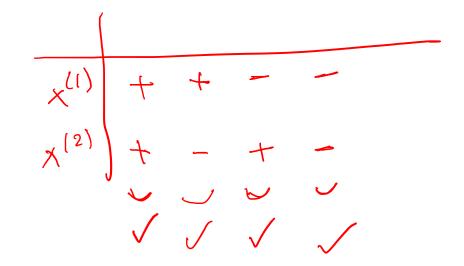
• What is $VC(\mathcal{H})$?

• Can \mathcal{H} shatter some set of 1 point?

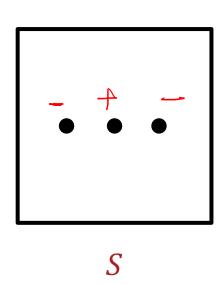
VC(H) > 2 Can \mathcal{H} shatter some set of 2 points?

VC-Dimension: Example

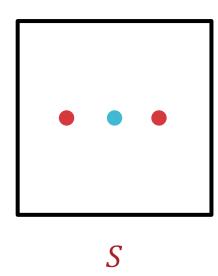




- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
- VC(H) 73 Can \mathcal{H} shatter some set of 3 points?

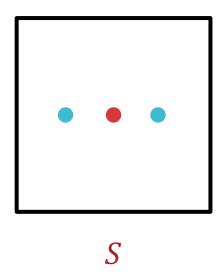


- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?



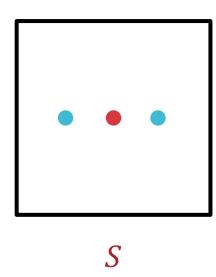
20

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?



21

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter **some** set of 3 points?



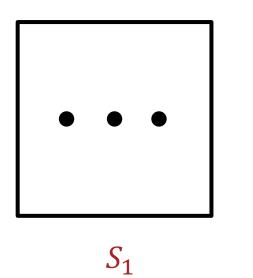
• What is $VC(\mathcal{H})$?

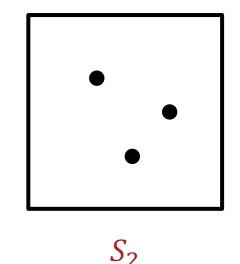
• Can ${\mathcal H}$ shatter some set of 1 point?

• $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators

- Can \mathcal{H} shatter some set of 2 points?
- Can \mathcal{H} shatter some set of 3 points?

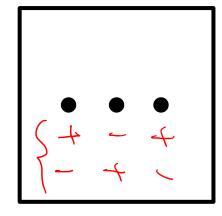
VC-Dimension: Example



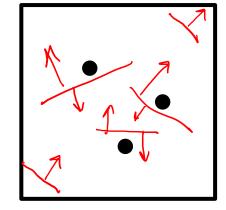


• $x \in \mathbb{R}^2$ and $\mathcal{H}=$ all 2-dimensional linear separators • What is $VC(\mathcal{H})$?

- Can \mathcal{H} shatter some set of 1 point?
- Can \mathcal{H} shatter some set of 2 points?
- Can \mathcal{H} shatter some set of 3 points?

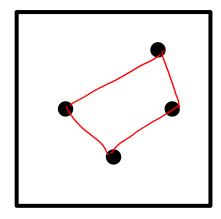


$$|\mathcal{H}(S_1)| = 6$$

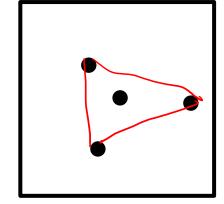


$$|\mathcal{H}(S_2)| = 8$$

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?



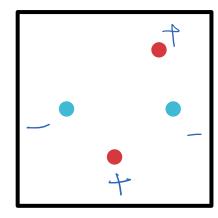
 S_1 All points on the convex hull



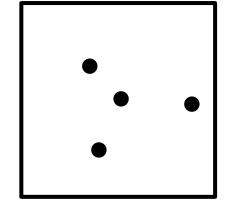
At least one point nside the convex hul

10/23/23 **25**

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?

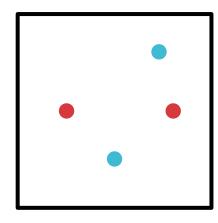


All points on the convex hull

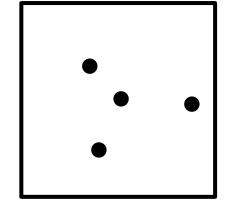


 S_2 At least one point inside the convex hull

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?

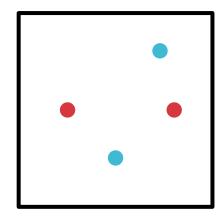


All points on the convex hull

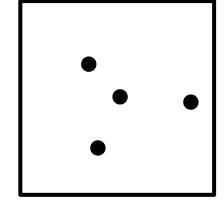


 S_2 At least one point inside the convex hull

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?

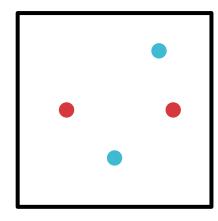


 $|\mathcal{H}(S_1)| = 14$ All points on the convex hull



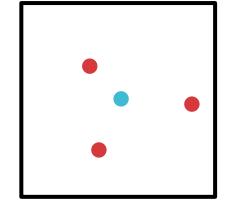
 S_2 At least one point inside the convex hull

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?



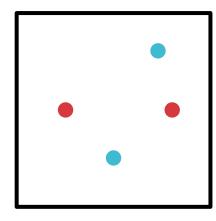
$$|\mathcal{H}(S_1)| = 14$$

All points on the convex hull



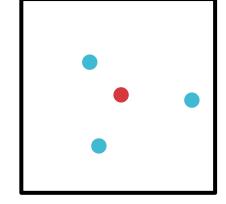
 S_2 At least one point inside the convex hull

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?



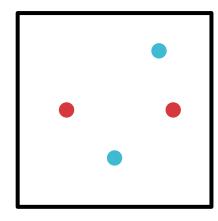
$$|\mathcal{H}(S_1)| = 14$$

All points on the convex hull



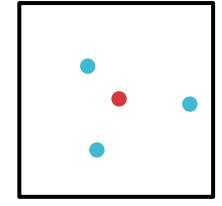
 S_2 At least one point inside the convex hull

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- What is $VC(\mathcal{H})$?
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?



$$|\mathcal{H}(S_1)| = 14$$

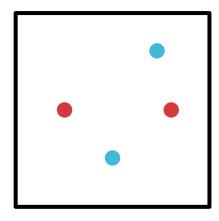
All points on the convex hull



$$|\mathcal{H}(S_2)| = 14$$

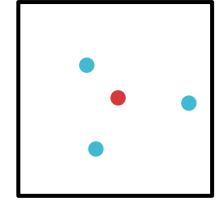
At least one point
inside the convex hull

- $x \in \mathbb{R}^2$ and $\mathcal{H} =$ all 2-dimensional linear separators
- $VC(\mathcal{H}) = 3$
 - Can \mathcal{H} shatter some set of 1 point?
 - Can \mathcal{H} shatter some set of 2 points?
 - Can \mathcal{H} shatter some set of 3 points?
 - Can \mathcal{H} shatter some set of 4 points?



$$|\mathcal{H}(S_1)| = 14$$

All points on the convex hull



$$|\mathcal{H}(S_2)| = 14$$

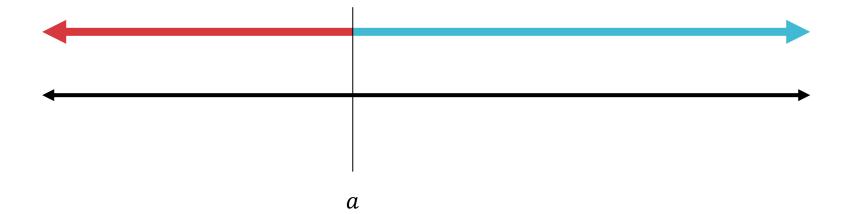
At least one point
inside the convex hull

• $x \in \mathbb{R}^d$ and $\mathcal{H} = \text{all } d\text{-dimensional linear separators}$

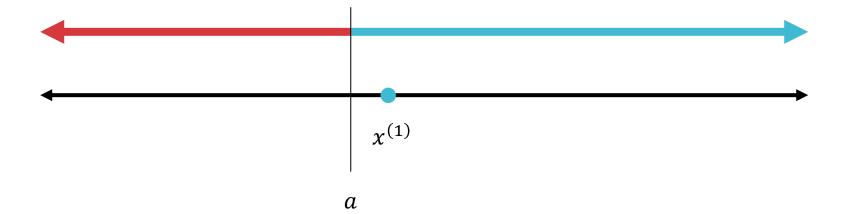
• $VC(\mathcal{H}) = d + 1$

VC-Dimension: Example

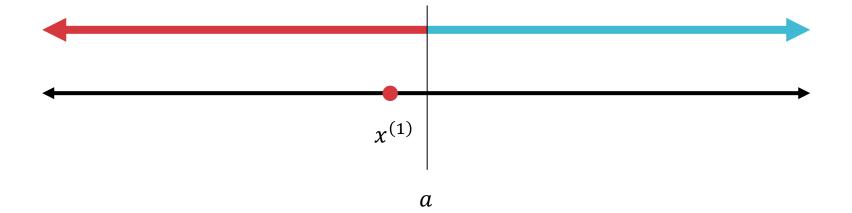
• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x; a) = \operatorname{sign}(x - a)$



• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x; a) = \operatorname{sign}(x - a)$

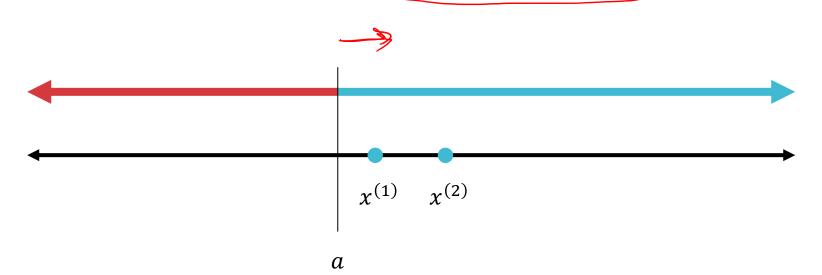


• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x; a) = \operatorname{sign}(x - a)$



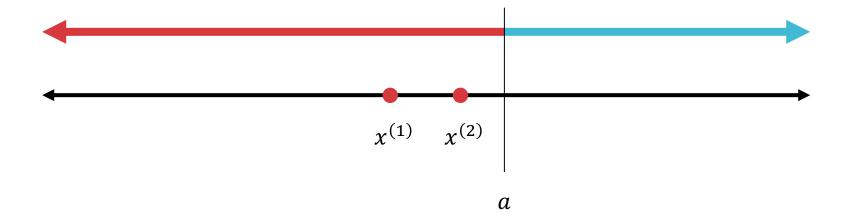
VC-Dimension: Example

• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x; a) = \operatorname{sign}(x - a)$



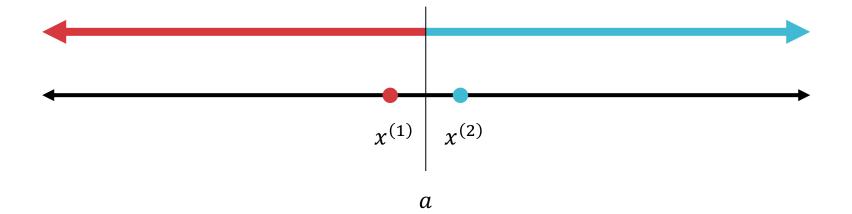
• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x; a) = \operatorname{sign}(x - a)$

VC-Dimension: Example



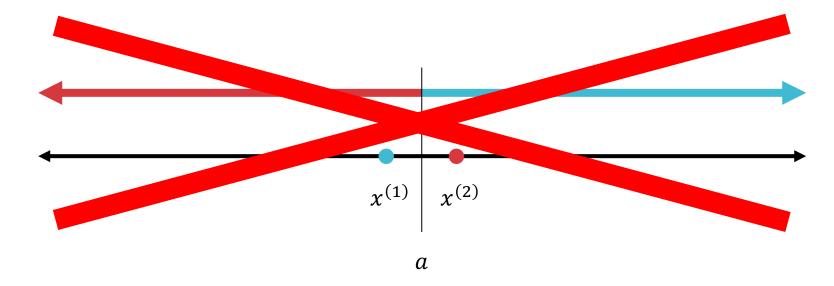
• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x; a) = \operatorname{sign}(x - a)$

VC-Dimension: Example



VC-Dimension: Example

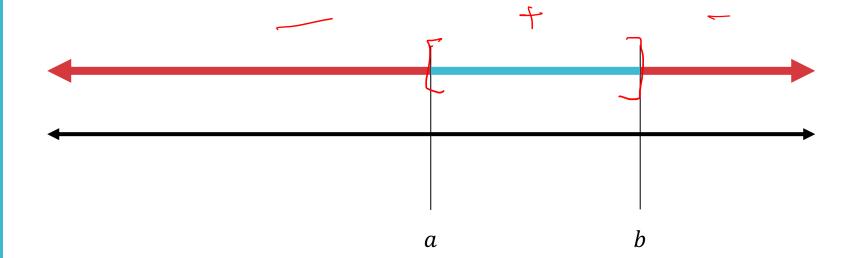
• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive rays, i.e., all hypotheses of the form $h(x; a) = \operatorname{sign}(x - a)$



• $VC(\mathcal{H}) = 1$

• $x \in \mathbb{R}$ and $\mathcal{H} =$ all 1-dimensional positive intervals

VC-Dimension: Example

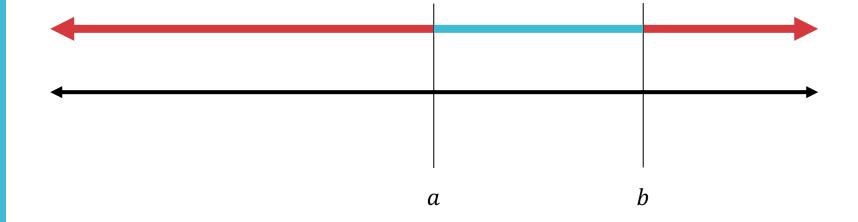


Poll Question 1:

What is $VC(\mathcal{H})$?

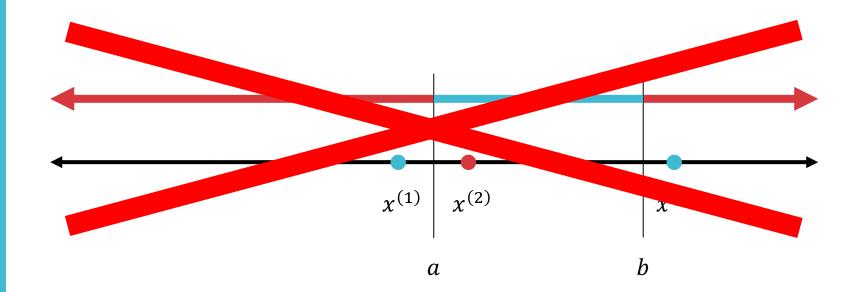
A. 0
B. 1
C. 1.5 (TOXIC)
D. 2
F. 3

• $x \in \mathbb{R}$ and $\mathcal{H} = \text{all 1-dimensional positive intervals}$



• $x \in \mathbb{R}$ and $\mathcal{H} = \text{all 1-dimensional positive intervals}$

VC-Dimension: Example



• $VC(\mathcal{H}) = 2$

Theorem 3: Vapnik-Chervonenkis (VC)-Bound

• Infinite, realizable case: for any hypothesis set \mathcal{H} such that $c^* \in \mathcal{H}$ and arbitrary distribution p^* , if the number of labelled training data points satisfies

$$\mathbf{N} = \mathbf{O}\left(\frac{1}{\epsilon}\left(\mathbf{VC}(\mathcal{H})\log\left(\frac{1}{\epsilon}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ with

$$\widehat{R}(h) = 0$$
 have $R(h) \leq \epsilon$

Statistical Learning Theory Corollary 3

• Infinite, realizable case: for any hypothesis set \mathcal{H} such that $c^* \in \mathcal{H}$ and arbitrary distribution p^* , given a training dataset S where |S| = N, all $h \in \mathcal{H}$ with $\hat{R}(h) = 0$ have

$$R(h) \le O\left(\frac{1}{N}\left(VC(\mathcal{H})\log\left(\frac{N}{VC(\mathcal{H})}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$$

with probability at least $1 - \delta$.

Theorem 4: Vapnik-Chervonenkis (VC)-Bound

• Infinite, agnostic case: for any hypothesis set ${\cal H}$ and arbitrary distribution p^* , if the number of labelled training data points satisfies

$$N = O\left(\frac{1}{\epsilon^2} \left(VC(\mathcal{H}) + \log\left(\frac{1}{\delta}\right)\right)\right)$$

then with probability at least $1 - \delta$, all $h \in \mathcal{H}$ have

$$\left| R(h) - \hat{R}(h) \right| \le \epsilon$$

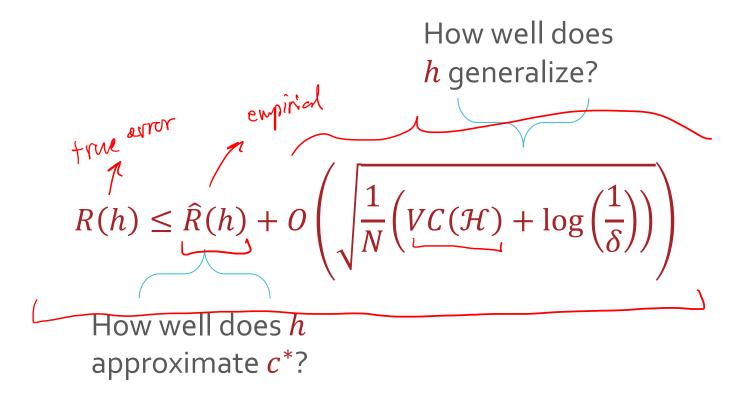
Statistical Learning Theory Corollary 4

• Infinite, agnostic case: for any hypothesis set \mathcal{H} and arbitrary distribution p^* , given a training dataset S where |S| = N, all $h \in \mathcal{H}$ have

$$R(h) \le \widehat{R}(h) + O\left(\sqrt{\frac{1}{N}\left(VC(\mathcal{H}) + \log\left(\frac{1}{\delta}\right)\right)}\right)$$

with probability at least $1 - \delta$.

Approximation Generalization Tradeoff



Approximation Generalization Tradeoff

 $Increases as \\ VC(\mathcal{H}) \ increases \\ R(h) \leq \widehat{R}(h) + O\left(\sqrt{\frac{1}{N}\Big(VC(\mathcal{H}) + \log\Big(\frac{1}{\delta}\Big)\Big)}\right) \\ Decreases as \\ VC(\mathcal{H}) \ increases \\$

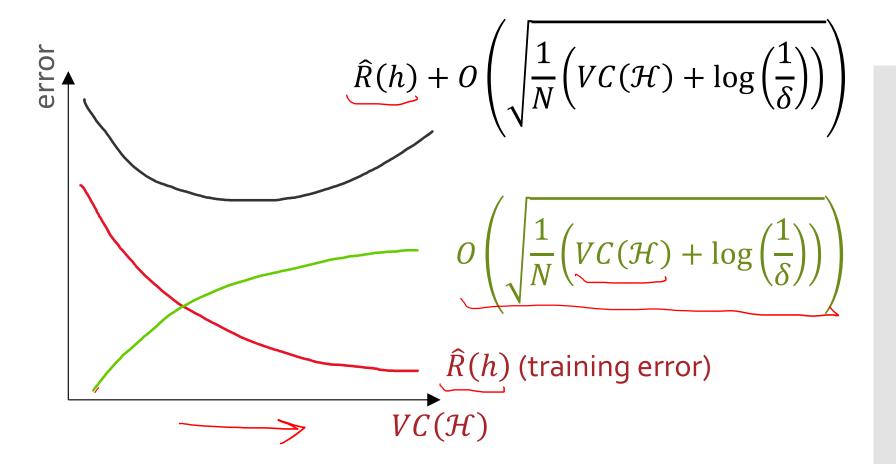
Can we use this corollary to guide model selection?

• Infinite, agnostic case: for any hypothesis set \mathcal{H} and arbitrary distribution p^* , given a training dataset S where |S| = N, all $h \in \mathcal{H}$ have

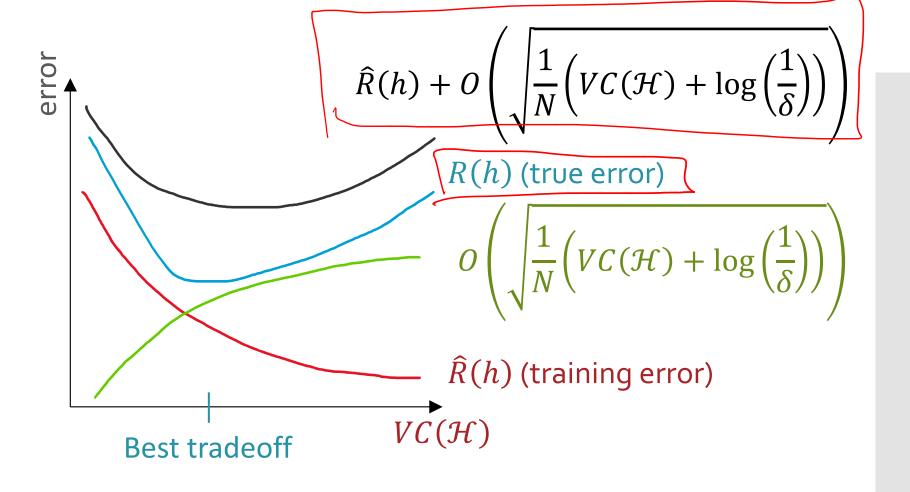
$$R(h) \le \widehat{R}(h) + O\left(\sqrt{\frac{1}{N}}\left(VC(\mathcal{H}) + \log\left(\frac{1}{\delta}\right)\right)\right)$$

with probability at least $1 - \delta$.

Learning
Theory and
Model
Selection



Learning Theory and Model Selection



- How can we find this "best tradeoff" for linear separators?
- Use a regularizer! By (effectively) reducing the number of features our model considers, we reduce its VC-dimension.

Learning Theory Learning Objectives

You should be able to...

- Identify the properties of a learning setting and assumptions required to ensure low generalization error
- Distinguish true error, train error, test error
- Define PAC and explain what it means to be approximately correct and what occurs with high probability
- Apply sample complexity bounds to real-world machine learning examples
- Theoretically motivate regularization

10-301/601: Introduction to Machine Learning Lecture 15 — Societal Impacts of ML

Henry Chai & Matt Gormley & Hoda Heidari 10/23/23

ML in Societal Applications

8 WAYS MACHINE LEARNING WILL IMPROVE EDUCATION

BY MATTHEW LYNCH / ② JUNE 12, 2018 / ○ 5





Features Technology Innovation Partner Zone the techies

Home > Features > Emerging tech & innovation Features

Researcher explains how algorithms can create a fairer legal system

Deep learning is being used to predict critical COVID-19 cases

Artificial Intelligence and Accessibility: Examples of a Technology that Serves People with Disabilities

Medicine.



From mental health apps to robot surgeons, artificial intelligence is already changing the practice of medicine.

TheUpshot

ROBO RECRUITING

Can an Algorithm Hire Better Than a Human?

By Claire Cain Miller



By DAVID KLEPPER July 29, 2020

Artificial intelligence is slated to disrupt 4.5 million jobs for African Americans, who have a 10% greater likelihood of automation-based job loss than other workers

Allana Akhtar Oct 7, 2019, 12:57 PM

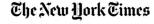
 $(f) (\square) (r)$

The algorithms that power fintech may discriminate can be difficult to anticipate—and financial institut accountable even when alleged discrimination is o unintentional.

Wanted: The 'perfect babysitter.' Must pass AI scan for respect and attitude.

Misinformation on coronavirus is proving highly contagious





I.R.S. Changes Audit Practice That Discriminated Against Black **Taxpayers**

The agency will overhaul how it scrutinizes returns that claim the earned-income tax credit, which is aimed at alleviating poverty.





