Machine Learning Department
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Decision Trees

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Lecture 3
Sep. 3, 2025

Q&A

Q: How do these In-Class Polls work?

A:

- Sign into **Google Form** (**c**lick [Poll] link on Schedule page http://mlcourse.org/schedule.html) using **Andrew Email**
- Answer **during lecture for full credit**, or within 24 hours for half credit
- Avoid the **toxic option** which gives negative points!
- 8 "free poll points" but can't use more than 3 free polls consecutively. All the questions for one lecture are worth 1 point total.

Latest Poll link: http://poll.mlcourse.org

First In-Class Poll

Question: *Which of the following did you bring to class today? Select all that apply.*

A. Smartphone

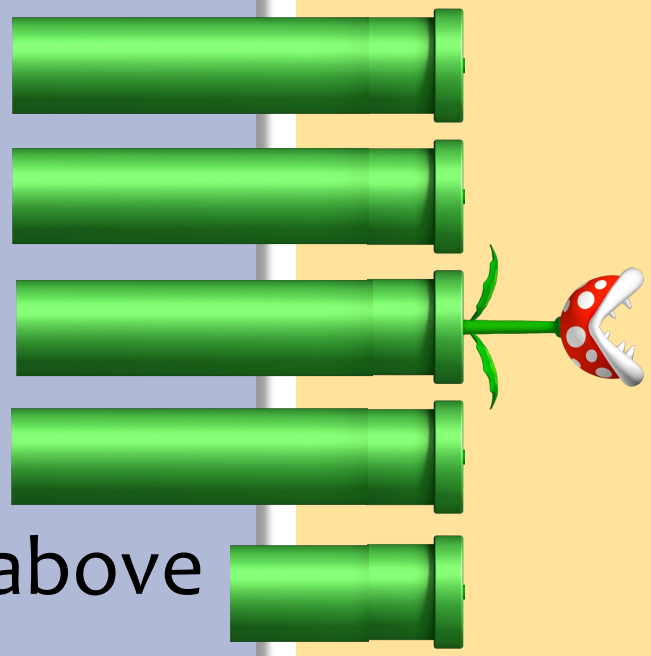
B. Flip phone

C. Pay phone

D. No phone

E. None of the above

Answer:



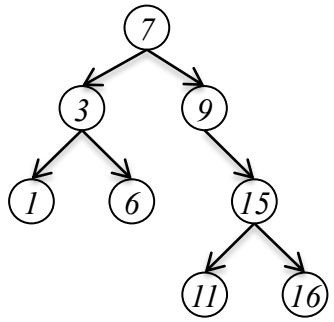
Reminders

- **Homework 1: Background**
  - **Out: Mon, Aug 25**
  - **Due: Wed, Sep 3 at 11:59pm**
  - unique policy for this assignment: we will grant (essentially) any and all extension requests
- **Homework 2: Decision Trees**
  - **Out: Wed, Sep. 3**
  - **Due: Mon, Sep. 15 at 11:59pm**

MAKING PREDICTIONS WITH A DECISION TREES

Background: Recursion

- *Def*: a **binary search tree** (BST) consists of nodes, where each node:
  - has a value, v
  - up to 2 children
  - all its left descendants have values less than v, and its right descendants have values greater than v
- We like BSTs because they permit search in O(log(n)) time, assuming n nodes in the tree



Node Data Structure

```
class Node:
        int value
        Node left
        Node right
```

Iterative Search

```
def contains(node, key):
        cur = node
        while true:
                if key < cur.value & cur.left != null:
                        cur = cur.left
                else if cur.value < key & cur.right != null:
                        cur = cur.right
                else:
                        break
        return key == cur.value
```

Recursive Search

```
def contains(node, key):
        if key < node.value & node.left != null:
                return contains(node.left, key)
        else if node.value < key & node.right != null:
                return contains(node.right, key)
        else:
                return key == node.value
```

Algorithms for Classification

Algorithm 3 **decision stump**: based on a single feature, x_d, predict the most common label in the training dataset among all data points that have the same value for x_d

	y	x_1	x_2	x_3	x_4
predictions	**allergic?**	**hives?**	**sneezing?**	**red eye?**	**has cat?**
-	-	Y	N	N	N
+	-	N	Y	N	N
+	+	Y	Y	N	N
-	-	Y	N	Y	Y
+	+	N	Y	Y	N

But why use just one feature…

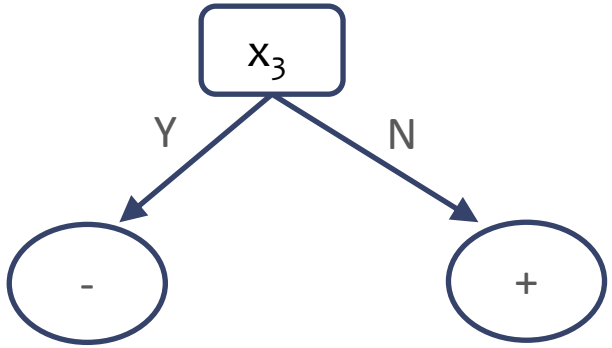
Nonzero training error, but perhaps still better than the memorizer

Example decision stump:

$$h(\mathbf{x}) = \begin{cases} + \text{ if sneezing} = Y \\ - \text{ otherwise} \end{cases}$$

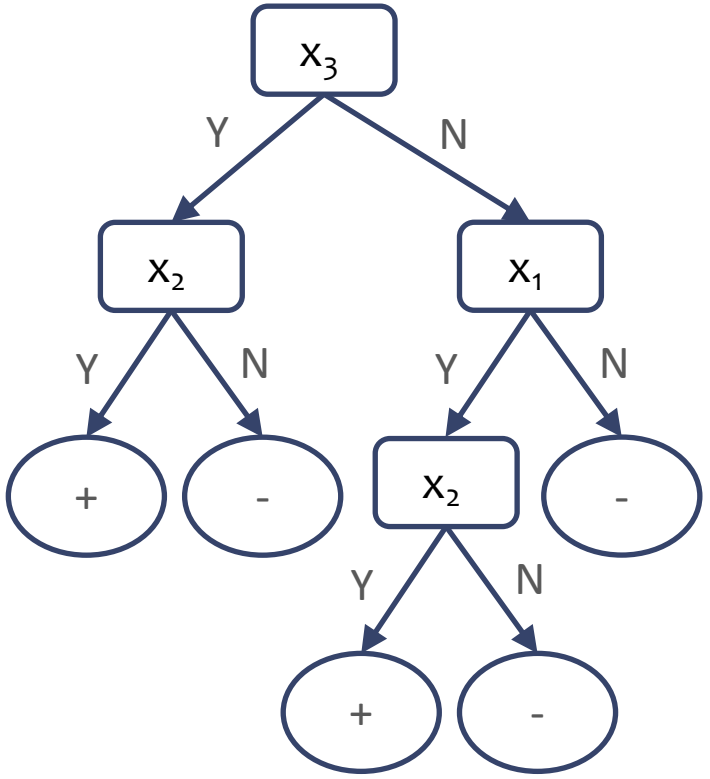
From Decision Stump to Decision Tree

	y	x_1	x_2	x_3	x_4
predic-tions	**allergic?**	**hives?**	**sneezing?**	**red eye?**	**has cat?**
-	-	Y	N	N	N
-	-	N	Y	N	N
+	+	Y	Y	N	N
-	-	Y	N	Y	Y
+	+	N	Y	Y	N



From Decision Stump to Decision Tree

	y	x_1	x_2	x_3	x_4
predic-tions	**allergic?**	**hives?**	**sneezing?**	**red eye?**	**has cat?**
-	-	Y	N	N	N
-	-	N	Y	N	N
+	+	Y	Y	N	N
-	-	Y	N	Y	Y
+	+	N	Y	Y	N



Decision Tree: In-Class Activity

1. Group 1: Answer the questions to determine which leaf node corresponds to your feature values
2. Group 2: (part 1)
   a) Take a blue sticky note if you prefer dogs to cats; otherwise, take a red sticky note
   b) Answer the questions to determine which leaf node corresponds to your feature values and place your sticky note there
3. Group 2: (part 2)
   a) Answer the new question to determine which new leaf node to move your sticky note to

Decision Tree: Prediction

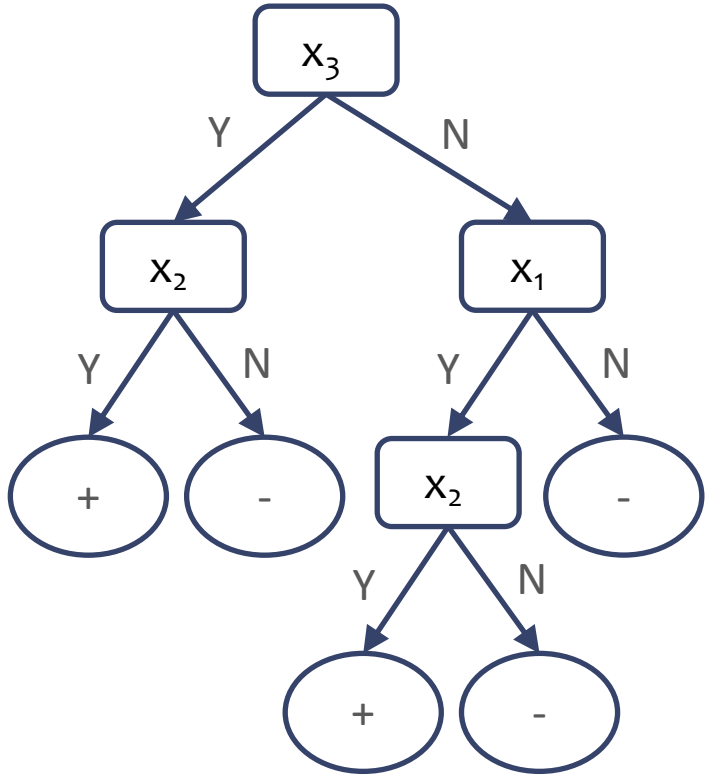
```
def h(x'):
    Let current node = root
    while(true):
        if current node is internal (non-leaf):
            Let m = attribute associated with current node
            Go down branch labeled with value x'_m
        if current node is a leaf:
            return label y stored at that leaf
```

```
class Node:
    str type   // "leaf" or "internal"
    y vote   // label for leaf node
    {} branches // map from feature
                // values to Node objects
    int m // feature for internal node
```

Decision Tree: Prediction

Algorithm 4 **decision tree:** recursively walk from root to a leaf, following the attribute values labeled on the branches, and return the label at the leaf

	y	x_1	x_2	x_3	x_4
predic-tions	allergic?	hives?	sneezing?	red eye?	has cat?
-	-	Y	N	N	N
-	-	N	Y	N	N
+	+	Y	Y	N	N
-	-	Y	N	Y	Y
+	+	N	Y	Y	N



Zero training error!

Decision Tree: Prediction (Iterative)

```
def h(x'):
    Let current node = root
    while(true):
        if current node is internal (non-leaf):
            Let m = attribute associated with current node
            Go down branch labeled with value x'_m
        if current node is a leaf:
            return label y stored at that leaf
```

Question: The original h(x') pseudocode is an iterative implementation. Can you implement h(x') recursively?

```
class Node:
    str type   // "leaf" or "internal"
    y vote   // label for leaf node
    {} branches // map from feature
                // values to Node objects
    int m // feature for internal node
```

Decision Tree: Prediction (Recursive)

def h(**x**'):

Question: The original h(x') pseudocode is an iterative implementation. Can you implement h(x') recursively?

```
class Node:
    str type   // "leaf" or "internal"
    y vote   // label for leaf node
    {} branches // map from feature
                // values to Node objects
    int m // feature for internal node
```

Decision Tree Example

Learned from medical records of 1000 women (Sims et al., 2000)

Negative examples are C-sections

```
[833+,167-] .83+ .17-
Fetal_Presentation = 1: [822+,116-] .88+ .12-
| Previous_Csection = 0: [767+,81-] .90+ .10-
| | Primiparous = 0: [399+,13-] .97+ .03-
| | Primiparous = 1: [368+,68-] .84+ .16-
| | | Fetal_Distress = 0: [334+,47-] .88+ .12-
| | | Fetal_Distress = 1: [34+,21-] .62+ .38-
| Previous_Csection = 1: [55+,35-] .61+ .39-
Fetal_Presentation = 2: [3+,29-] .11+ .89-
Fetal_Presentation = 3: [8+,22-] .27+ .73-
```

Figure from Tom Mitchell

LEARNING A DECISION TREE

Decision Tree Learning

- *Definition*: a **splitting criterion** is a function that measures the effectiveness of splitting on a particular attribute
- Our decision tree learner **selects the "best" attribute** as the one that maximizes the splitting criterion
- Lots of options for a splitting criterion:
  - error rate (minimize)
  - accuracy = 1 - error rate (maximize)
  - Mutual information (maximize)
  - Gini gain (maximize)

Decision Tree Learning

Decision Tree Learning Example

Dataset:

y	x_1	x_2	x_3
-	Y	N	N
-	Y	N	Y
-	Y	N	N
+	N	N	Y
+	Y	Y	N
+	Y	Y	Y
+	Y	Y	N
+	Y	Y	Y

In-Class Exercise: Using **error rate** as the splitting criterion, what decision tree would be learned?

Recursive Training for Decision Trees

- def train(dataset D'):
  - Let p = new Node()
  - ***Base Case:*** If (1) all labels $y^{(i)}$ in D' are identical (2) D' is empty (3) for each attribute, all values are identical
    - p.type = Leaf // The node p is a leaf node
    - p.label = majority_vote(D') // Store the label
    - return p
  - ***Recursive Step:*** Otherwise
    - Make an internal node
      - p.type = Internal // The node p is an internal node
    - Pick the *best* attribute X_m according to splitting criterion
      - p.attr = argmax_m splitting_criterion(D', X_m) // Store the attribute on which to split
    - For each value v of attribute X_m:
      - $D_{X_m = v}$ = {(**x**,y) in D' : $x_m = v$} // Select a partition of the data
      - child_v = train($D_{X_m = v}$) // Recursively build the child
      - p.branches[v] = child_v // Create a branch with label v
      - return p

SPLITTING CRITERION: ERROR RATE

Decision Tree Learning Example

Dataset:

Label Y, Features A and B

Y	A	B
-	1	0
-	1	0
+	1	0
+	1	0
+	1	1
+	1	1
+	1	1
+	1	1

Poll Question 2

Which attribute would **error rate** select for the next split?

1. A
2. B
3. A or B (tie)
4. Neither

Decision Tree Learning Example

Dataset:

Label Y, Features A and B

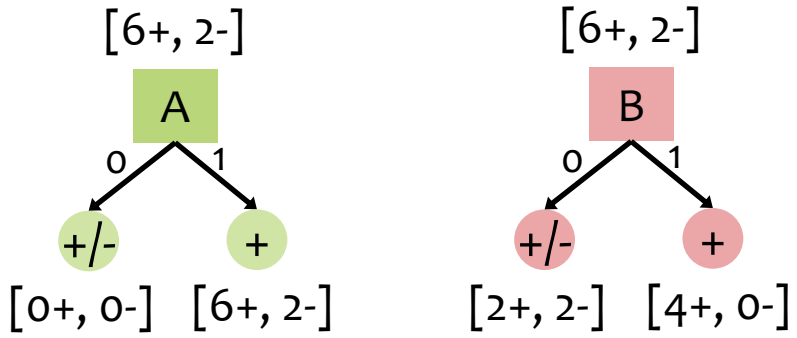
Y	A	B
-	1	0
-	1	0
+	1	0
+	1	0
+	1	1
+	1	1
+	1	1
+	1	1

Decision Tree Learning Example

Dataset:

Label Y, Features A and B

Y	A	B
-	1	0
-	1	0
+	1	0
+	1	0
+	1	1
+	1	1
+	1	1
+	1	1



Error Rate

$error(h_A, D) = 2/8$

$error(h_B, D) = 2/8$

error rate treats attributes A and B as equally good

Decision Tree Learning

- *Definition*: a **splitting criterion** is a function that measures the effectiveness of splitting on a particular attribute
- Our decision tree learner **selects the "best" attribute** as the one that maximizes the splitting criterion
- Lots of options for a splitting criterion:
  - error rate (minimize)
  - accuracy = 1 - error rate (maximize)
  - Mutual information (maximize)
  - Gini gain (maximize)

SPLITTING CRITERION: MUTUAL INFORMATION

Entropy

- The **entropy** of a *random variable* describes the uncertainty of its outcome: the higher the entropy, the less certain we are about what the outcome will be.

$$H(X) = -\sum_{v \in V(X)} P(X = v) \log_2\big(P(X = v)\big)$$

where X is a (discrete) random variable

$V(X)$ is the set of possible values X can take on

Slide from Henry Chai

Entropy

- The **entropy** of a *set* describes how uniform or pure it is: the higher the entropy, the more impure or "mixed-up" the set is

$$H(S) = -\sum_{v \in V(S)} \frac{|S_v|}{|S|} \log_2 \left(\frac{|S_v|}{|S|} \right)$$

where S is a collection of values,

$V(S)$ is the set of unique values in S

S_v is the collection of elements in S with value v

- If all the elements in S are the same, then

Slide from Henry Chai

Entropy

- The **entropy** of a *set* describes how uniform or pure it is: the higher the entropy, the more impure or "mixed-up" the set is

$$H(S) = -\sum_{v \in V(S)} \frac{|S_v|}{|S|} \log_2 \left(\frac{|S_v|}{|S|} \right)$$

where S is a collection of values,

$V(S)$ is the set of unique values in S

S_v is the collection of elements in S with value v

- If S is split fifty-fifty between two values, then

Slide from Henry Chai

Mutual Information

- The **mutual information** between *two random variables* describes how much clarity knowing the value of one random variables provides about the other

$$I(Y;X) = H(Y) - H(Y|X)$$

$$= H(Y) - \sum_{v \in V(X)} P(X = v)H(Y|X = v)$$

where X and Y are random variables

$V(X)$ is the set of possible values X can take on

$H(Y|X = v)$ is the conditional entropy of Y given $X = v$

Slide from Henry Chai

Mutual Information

- The **mutual information** between *a feature and the label* describes how much clarity knowing the feature provides about the label

$$I(y; x_d) = H(y) - H(y|x_d)$$

$$= H(y) - \sum_{v \in V(x_d)} f_v * H\left(Y_{x_d=v}\right)$$

where x_d is a feature and y is the set of all labels

$V(x_d)$ is the set of possible values x_d can take on

f_v is the fraction of data points where $x_d = v$

$Y_{x_d=v}$ is the set of all labels where $x_d = v$

Slide from Henry Chai

Decision Tree Learning Example

Dataset:

Label Y, Features A and B

Y	A	B
-	1	0
-	1	0
+	1	0
+	1	0
+	1	1
+	1	1
+	1	1
+	1	1

Poll Question 3

Which feature would **mutual information** select for the next split?

1. A
2. B
3. A or B (tie)
4. Neither

Decision Tree Learning Example

Dataset:

Label Y, Features A and B

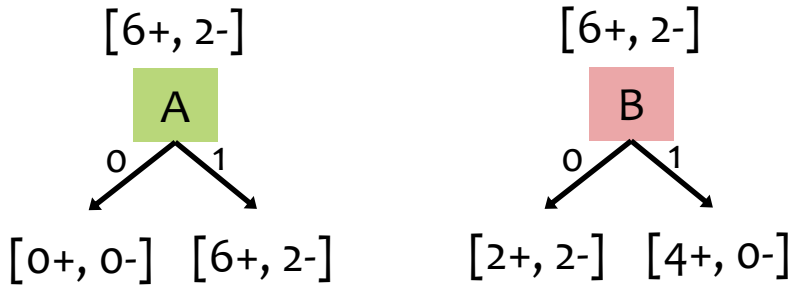
Y	A	B
-	1	0
-	1	0
+	1	0
+	1	0
+	1	1
+	1	1
+	1	1
+	1	1

Decision Tree Learning Example

Dataset:

Label Y, Features A and B

Y	A	B
-	1	0
-	1	0
+	1	0
+	1	0
+	1	1
+	1	1
+	1	1
+	1	1



Mutual Information

$H(Y) = -2/8 \log(2/8) - 6/8 \log(6/8)$

$H(Y|A=0)$ = "undefined"

$H(Y|A=1) = -2/8 \log(2/8) - 6/8 \log(6/8)$
$= H(Y)$

$H(Y|A) = P(A=0)H(Y|A=0) + P(A=1)H(Y|A=1)$
$= 0 + H(Y|A=1) = H(Y)$

$I(Y; A) = H(Y) - H(Y|A=1) = 0$

$H(Y|B=0) = -2/4 \log(2/4) - 2/4 \log(2/4)$

$H(Y|B=1) = -0 \log(0) - 1 \log(1) = 0$

$H(Y|B) = 4/8(0) + 4/8(H(Y|B=0))$

$I(Y; B) = H(Y) - 4/8\ H(Y|B=0) > 0$