10-301/601: Introduction to Machine Learning Lecture 9 – Logistic Regression

Henry Chai & Matt Gormley

9/28/22

Q & A:

Where does the $O(n^3)$ computational complexity for matrix inversion come from?

- And that other number you quoted for that matter?

Matrix inversion	One $n \times n$ matrix	One $n \times n$ matrix	Gauss–Jordan elimination	$O(n^3)$
			Strassen algorithm	$O(n^{2.807})$
			Coppersmith–Winograd algorithm	$O(n^{2.376})$
			Optimized CW-like algorithms	$O(n^{2.373})$

Source: https://en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations

Front Matter

- Announcements:
  - HW3 released 9/21, due 9/28 (today!) at 11:59 PM
    - **Only two grace days allowed on HW3**
    - HW3 exit poll has also been released: you have until one week from the due date to complete it
  - Exam 1 on 10/4 (one week from yesterday) from 6:30 PM - 8:30 PM
    - Exam 1 practice problems released on the course website, under Coursework
  - Lecture instead of recitation this Friday (9/30)

Probabilistic Learning

- Previously:
  - (Unknown) Target function, $c^*: \mathcal{X} \rightarrow \mathcal{Y}$
  - Classifier, $h : \mathcal{X} \rightarrow \mathcal{Y}$
  - Goal: find a classifier, h, that best approximates c^*
- Now:
  - (Unknown) Target *distribution*, $y \sim p^*(Y|\boldsymbol{x})$
  - Distribution, $p(Y|\boldsymbol{x})$
  - Goal: find a distribution, p, that best approximates p^*

Likelihood

- Given N independent, identically distribution (iid) samples $\mathcal{D} = \{x^{(1)}, \ldots, x^{(N)}\}$ of a random variable X
  - If X is discrete with probability mass function (pmf) $p(X|\theta)$, then the *likelihood* of $\mathcal{D}$ is

$$L(\theta) = \prod_{i=1}^{N} p\left(x^{(i)}|\theta\right)$$

  - If X is continuous with probability density function (pdf) $f(X|\theta)$, then the *likelihood* of $\mathcal{D}$ is

$$L(\theta) = \prod_{i=1}^{N} f\left(x^{(i)}|\theta\right)$$

Log-Likelihood

- Given N independent, identically distribution (iid) samples $\mathcal{D} = \{x^{(1)}, \ldots, x^{(N)}\}$ of a random variable X
  - If X is discrete with probability mass function (pmf) $p(X|\theta)$, then the *log-likelihood* of $\mathcal{D}$ is

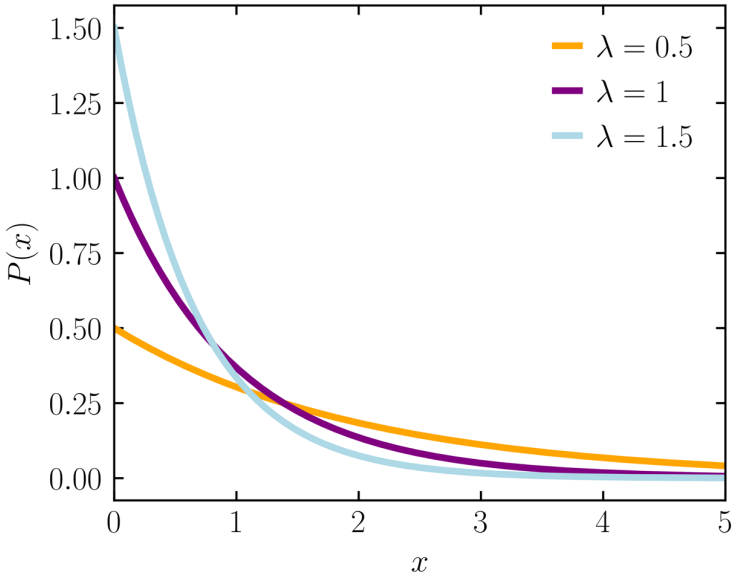
$$\ell(\theta) = \log \prod_{i=1}^{N} p\left(x^{(i)}|\theta\right) = \sum_{i=1}^{N} \log p\left(x^{(i)}|\theta\right)$$

  - If X is continuous with probability density function (pdf) $f(X|\theta)$, then the *log-likelihood* of $\mathcal{D}$ is

$$\ell(\theta) = \log \prod_{i=1}^{N} f\left(x^{(i)}|\theta\right) = \sum_{i=1}^{N} \log f\left(x^{(i)}|\theta\right)$$

Maximum Likelihood Estimation (MLE)

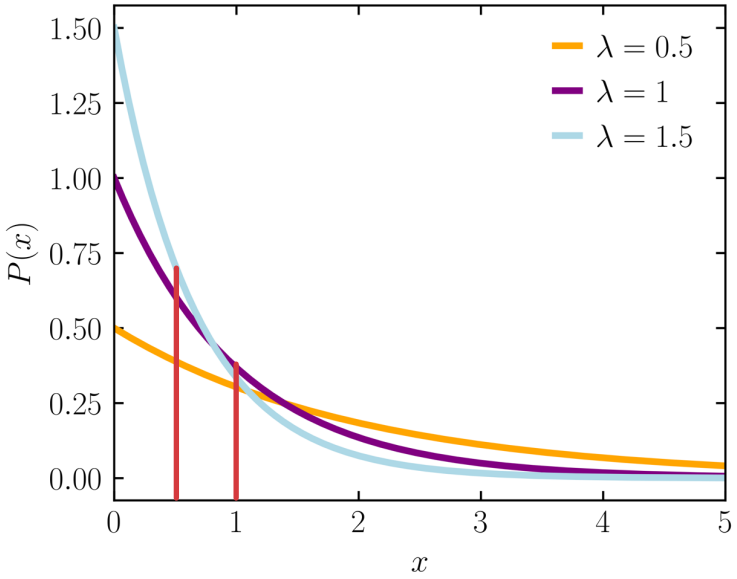
- Insight: every valid probability distribution has a finite amount of probability mass as it must sum/integrate to 1
- Idea: set the parameter(s) so that the likelihood of the samples is maximized
- Intuition: assign as much of the (finite) probability mass to the observed data *at the expense of unobserved data*
- Example: the exponential distribution



Source: https://en.wikipedia.org/wiki/Exponential_distribution#/media/File:Exponential_probability_density.svg

Maximum Likelihood Estimation (MLE)

- Insight: every valid probability distribution has a finite amount of probability mass as it must sum/integrate to 1
- Idea: set the parameter(s) so that the likelihood of the samples is maximized
- Intuition: assign as much of the (finite) probability mass to the observed data *at the expense of unobserved data*
- Example: the exponential distribution

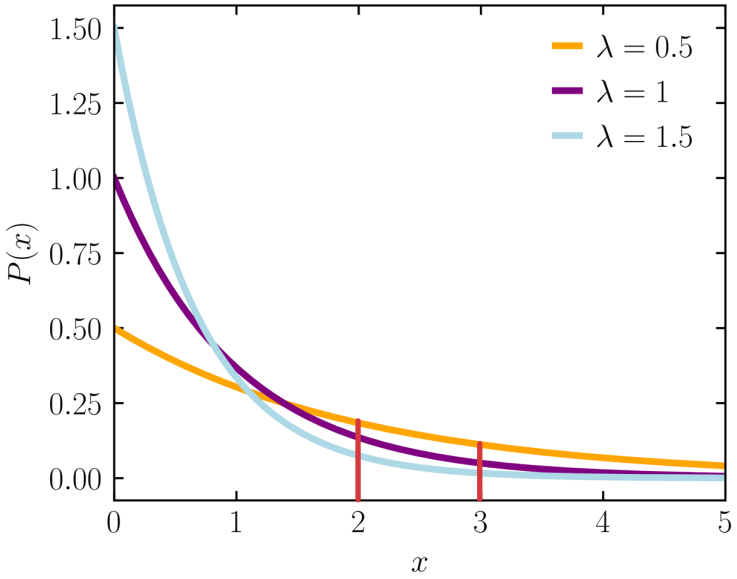


$\{x^{(1)} = 0.5, x^{(2)} = 1\}$

Source: https://en.wikipedia.org/wiki/Exponential_distribution#/media/File:Exponential_probability_density.svg

Maximum Likelihood Estimation (MLE)

- Insight: every valid probability distribution has a finite amount of probability mass as it must sum/integrate to 1
- Idea: set the parameter(s) so that the likelihood of the samples is maximized
- Intuition: assign as much of the (finite) probability mass to the observed data *at the expense of unobserved data*
- Example: the exponential distribution



$\{x^{(1)} = 2, x^{(2)} = 3\}$

Source: https://en.wikipedia.org/wiki/Exponential_distribution#/media/File:Exponential_probability_density.svg

Exponential Distribution MLE

- The pdf of the exponential distribution is

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

- Given N iid samples $\{x^{(1)}, \ldots, x^{(N)}\}$, the likelihood is

$$L(\lambda) = \prod_{i=1}^{N} f\left(x^{(i)}|\lambda\right) = \prod_{i=1}^{N} \lambda e^{-\lambda x^{(i)}}$$

Exponential Distribution MLE

- The pdf of the exponential distribution is

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

- Given N iid samples $\{x^{(1)}, \ldots, x^{(N)}\}$, the log-likelihood is

$$\ell(\lambda) = \sum_{i=1}^{N} \log f\left(x^{(i)}|\lambda\right) = \sum_{i=1}^{N} \log \lambda e^{-\lambda x^{(i)}}$$

$$= \sum_{i=1}^{N} \log \lambda + \log e^{-\lambda x^{(i)}} = N \log \lambda - \lambda \sum_{i=1}^{N} x^{(i)}$$

Okay, but now what?

How do we actually find the best value for the parameter, λ?

- The pdf of the exponential distribution is

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

- Given N iid samples $\{x^{(1)}, \ldots, x^{(N)}\}$, the log-likelihood is

$$\ell(\lambda) = \sum_{i=1}^{N} \log f\left(x^{(i)}|\lambda\right) = \sum_{i=1}^{N} \log \lambda e^{-\lambda x^{(i)}}$$

$$= \sum_{i=1}^{N} \log \lambda + \log e^{-\lambda x^{(i)}} = N \log \lambda - \lambda \sum_{i=1}^{N} x^{(i)}$$

Exponential Distribution MLE

- The pdf of the exponential distribution is

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

- Given N iid samples $\{x^{(1)}, \ldots, x^{(N)}\}$, the log-likelihood is

$$\ell(\lambda) = \sum_{i=1}^{N} \log f\left(x^{(i)}|\lambda\right) = \sum_{i=1}^{N} \log \lambda e^{-\lambda x^{(i)}}$$

$$= \sum_{i=1}^{N} \log \lambda + \log e^{-\lambda x^{(i)}} = N \log \lambda - \lambda \sum_{i=1}^{N} x^{(i)}$$

- Taking the partial derivative and setting it equal to 0 gives

$$\frac{\partial \ell}{\partial \lambda} = \frac{N}{\lambda} - \sum_{i=1}^{N} x^{(i)}$$

Exponential Distribution MLE

- The pdf of the exponential distribution is

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

- Given N iid samples $\{x^{(1)}, \ldots, x^{(N)}\}$, the log-likelihood is

$$\ell(\lambda) = \sum_{i=1}^{N} \log f\left(x^{(i)}|\lambda\right) = \sum_{i=1}^{N} \log \lambda e^{-\lambda x^{(i)}}$$

$$= \sum_{i=1}^{N} \log \lambda + \log e^{-\lambda x^{(i)}} = N \log \lambda - \lambda \sum_{i=1}^{N} x^{(i)}$$

- Taking the partial derivative and setting it equal to 0 gives

$$\frac{N}{\hat{\lambda}} - \sum_{i=1}^{N} x^{(i)} = 0 \rightarrow \frac{N}{\hat{\lambda}} = \sum_{i=1}^{N} x^{(i)} \rightarrow \hat{\lambda} = \frac{N}{\sum_{i=1}^{N} x^{(i)}}$$

Building a Probabilistic Classifier

- Define a decision rule
  - Given a test data point $\boldsymbol{x}'$, predict its label $\hat{y}$ using the posterior distribution $P(Y = y|\boldsymbol{x}')$
  - Common choice: $\hat{y} = \underset{y}{\operatorname{argmax}}\, P(Y = y|\boldsymbol{x}')$
- Idea: model $P(Y|\boldsymbol{x})$ as some parametric function of $\boldsymbol{x}$

Modelling the Posterior

- Suppose we have binary labels $y \in \{0,1\}$ and D-dimensional inputs $\boldsymbol{x} = [1, x_1, \ldots, x_D]^T \in \mathbb{R}^{D+1}$

  1 prepended to $\boldsymbol{x}$

- **Assume**

$$P(Y = 1|\boldsymbol{x}, \boldsymbol{\theta}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x}) = \frac{1}{1 + \exp(-\boldsymbol{\theta}^T \boldsymbol{x})} = \frac{\exp(\boldsymbol{\theta}^T \boldsymbol{x})}{\exp(\boldsymbol{\theta}^T \boldsymbol{x}) + 1}$$

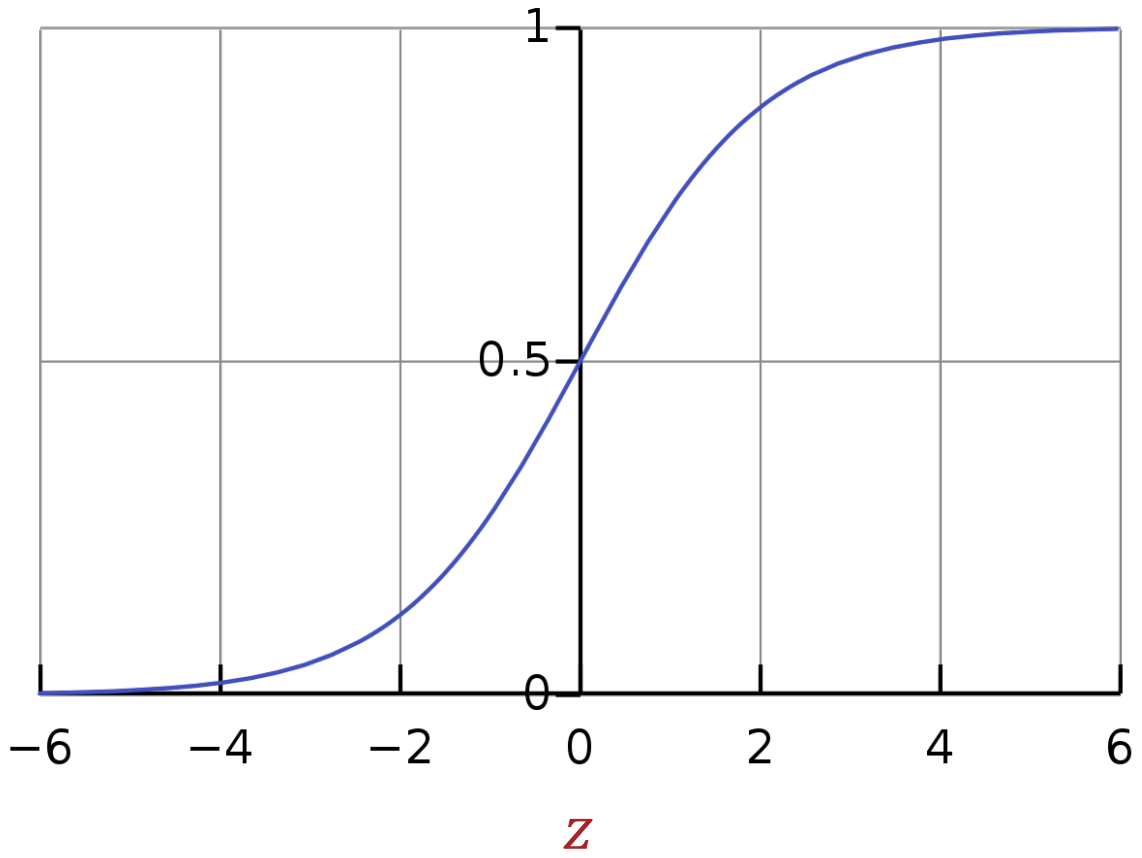
- This implies two useful facts:

1. $P(Y = 0|\boldsymbol{x}, \boldsymbol{\theta}) = 1 - P(Y = 1|\boldsymbol{x}, \boldsymbol{\theta}) = \frac{1}{\exp(\boldsymbol{\theta}^T \boldsymbol{x}) + 1}$

2. $\frac{P(Y = 1|\boldsymbol{x}, \boldsymbol{\theta})}{P(Y = 0|\boldsymbol{x}, \boldsymbol{\theta})} = \exp(\boldsymbol{\theta}^T \boldsymbol{x}) \rightarrow \log \frac{P(Y = 1|\boldsymbol{x}, \boldsymbol{\theta})}{P(Y = 0|\boldsymbol{x}, \boldsymbol{\theta})} = \boldsymbol{\theta}^T \boldsymbol{x}$

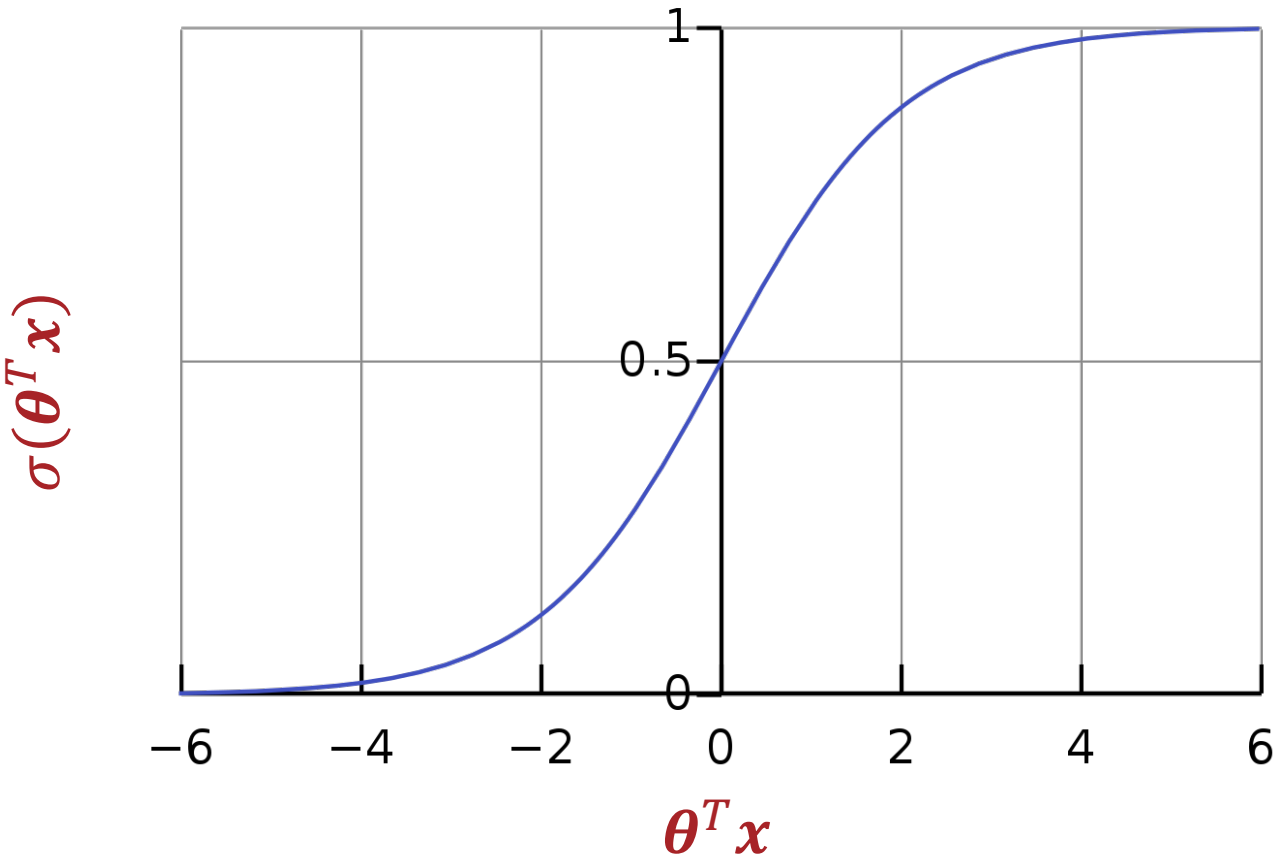
Logistic Function

$$\sigma(z) = \frac{1}{1+e^{-z}}$$



Source: https://en.wikipedia.org/wiki/Logistic_function#/media/File:Logistic-curve.svg

Why use the Logistic Function?



- Differentiable everywhere
- $\sigma: \mathbb{R} \to [0, 1]$
- The decision boundary is linear in $\boldsymbol{x}$!

Source: https://en.wikipedia.org/wiki/Logistic_function#/media/File:Logistic-curve.svg

Logistic Regression Decision Boundary

$$\hat{y} = \begin{cases} 1 \text{ if } P(Y = 1|\boldsymbol{x}, \boldsymbol{\theta}) \geq \frac{1}{2} \\ 0 \text{ otherwise.} \end{cases}$$

$$P(Y = 1|\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x}) = \frac{1}{1 + \exp(-\boldsymbol{\theta}^T \boldsymbol{x})} \geq \frac{1}{2}$$

$$2 \geq 1 + \exp(-\boldsymbol{\theta}^T \boldsymbol{x})$$

$$1 \geq \exp(-\boldsymbol{\theta}^T \boldsymbol{x})$$

$$\log(1) \geq -\boldsymbol{\theta}^T \boldsymbol{x}$$

$$0 \leq \boldsymbol{\theta}^T \boldsymbol{x}$$

Logistic Regression Decision Boundary

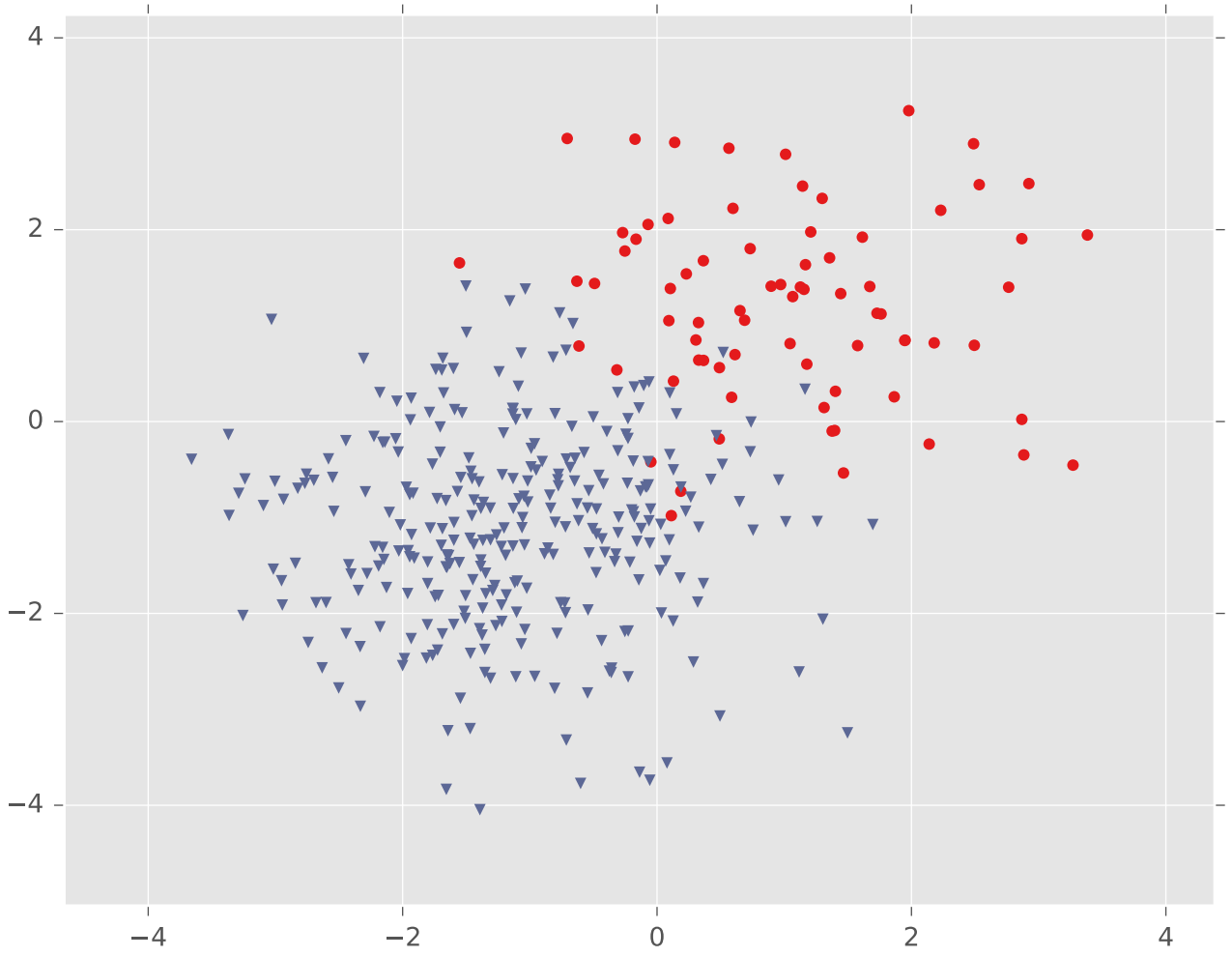


Figure courtesy of Matt Gormley

Logistic Regression Decision Boundary

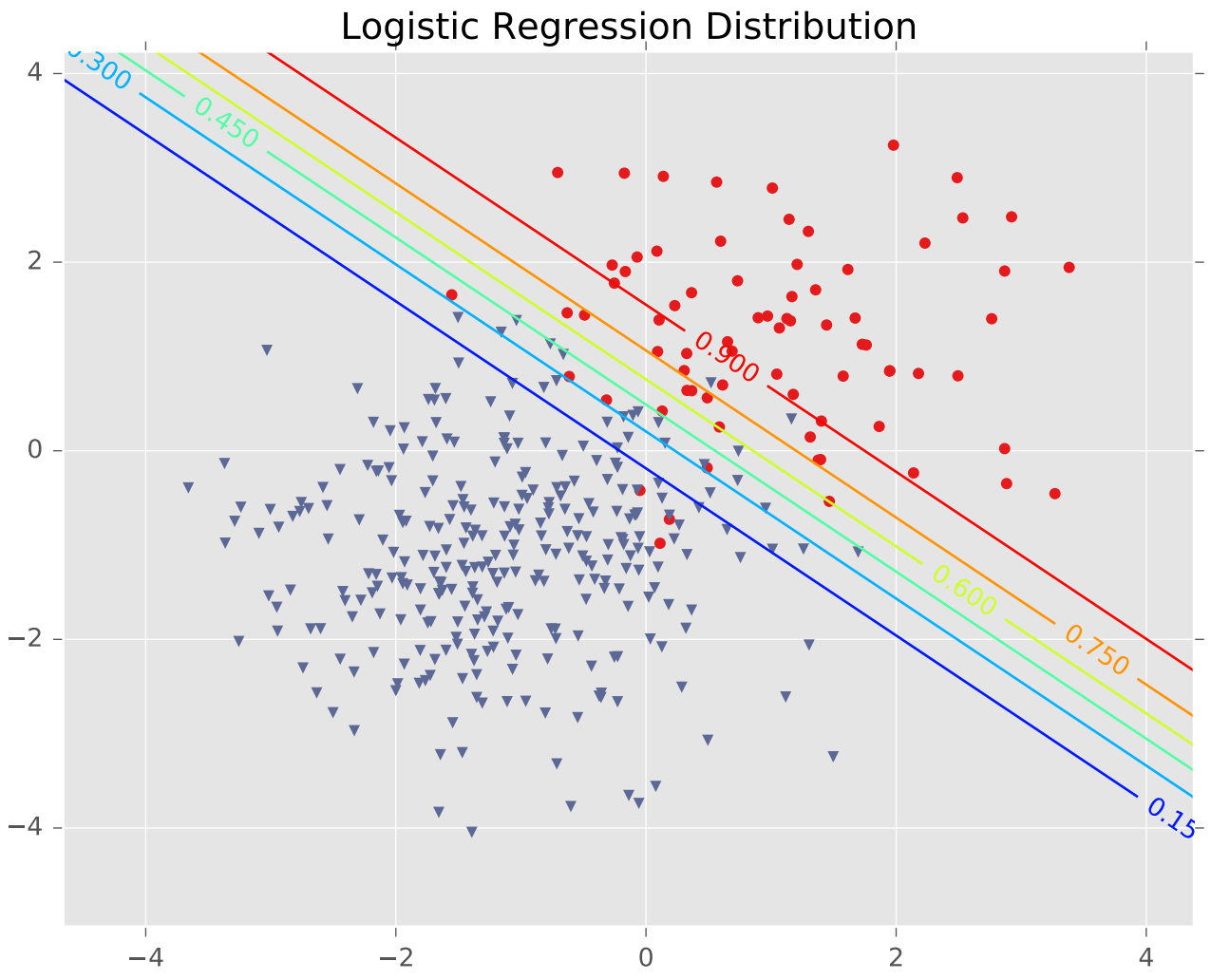


Figure courtesy of Matt Gormley

Logistic Regression Decision Boundary

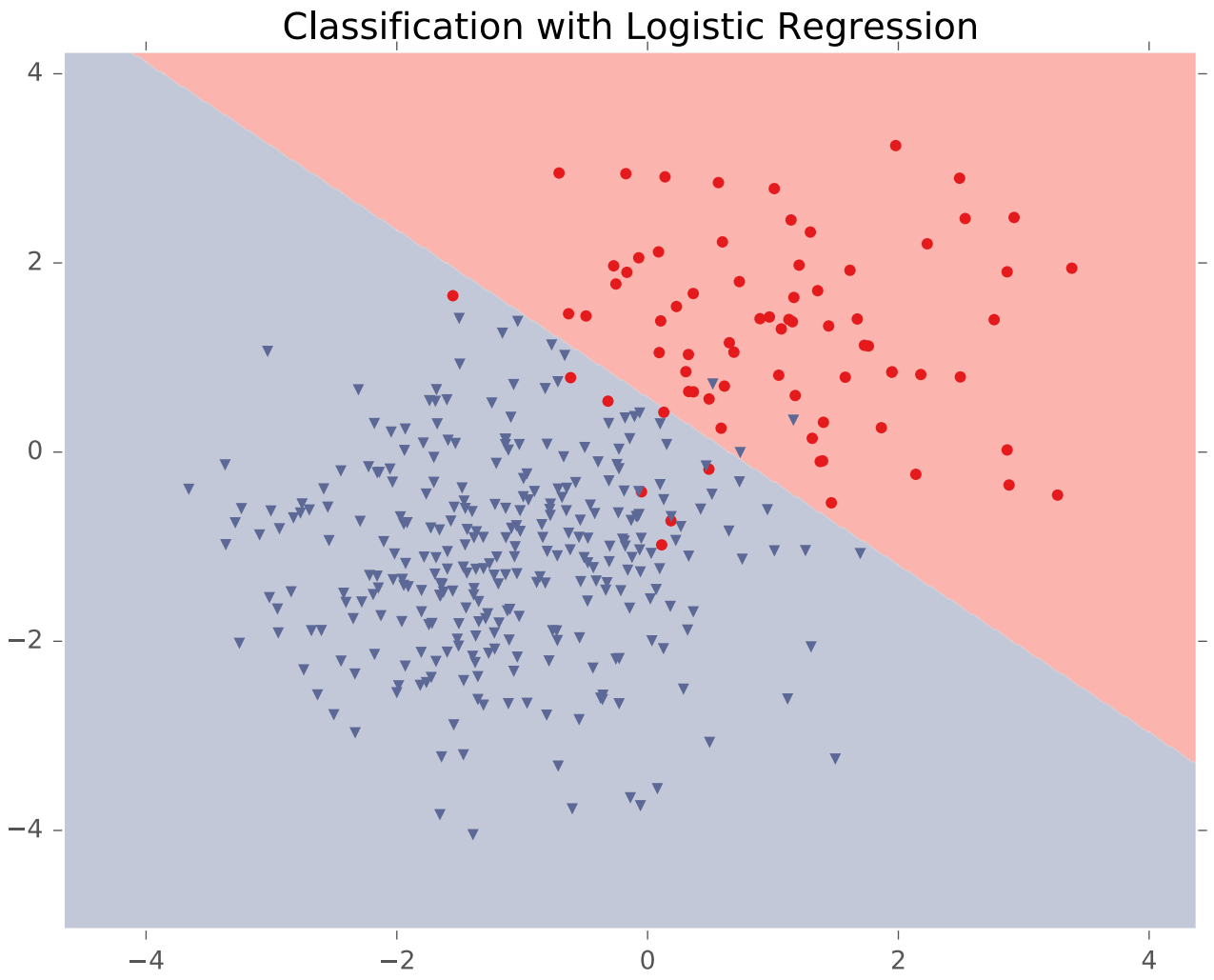


Figure courtesy of Matt Gormley

Setting the Parameters via Minimum Negative Conditional (log-)Likelihood Estimation (MCLE)

- Find $\boldsymbol{\theta}$ that minimizes

$$\ell(\boldsymbol{\theta}) = -\log P\left(y^{(1)}, \ldots, y^{(N)} \middle| \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N)}, \boldsymbol{\theta}\right) = -\log \prod_{i=1}^{N} P\left(y^{(i)} \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right)$$

$$= -\log \prod_{i=1}^{N} P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right)^{y^{(i)}} \left(P\left(Y = 0 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right)\right)^{1-y^{(i)}}$$

$$= -\sum_{i=1}^{N} y^{(i)} \log P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right) + \left(1 - y^{(i)}\right) \log P\left(Y = 0 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right)$$

$$= -\sum_{i=1}^{N} y^{(i)} \log \frac{P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right)}{P\left(Y = 0 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right)} + \log P\left(Y = 0 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right)$$

$$= -\sum_{i=1}^{N} y^{(i)} \boldsymbol{\theta}^T \boldsymbol{x}^{(i)} - \log\left(1 + \exp\left(\boldsymbol{\theta}^T \boldsymbol{x}^{(i)}\right)\right)$$

$$J(\boldsymbol{\theta}) = \frac{1}{N} \ell(\boldsymbol{\theta}) = -\frac{1}{N} \sum_{i=1}^{N} y^{(i)} \boldsymbol{\theta}^T \boldsymbol{x}^{(i)} - \log\left(1 + \exp\left(\boldsymbol{\theta}^T \boldsymbol{x}^{(i)}\right)\right)$$

Minimizing the Negative Conditional (log-)Likelihood

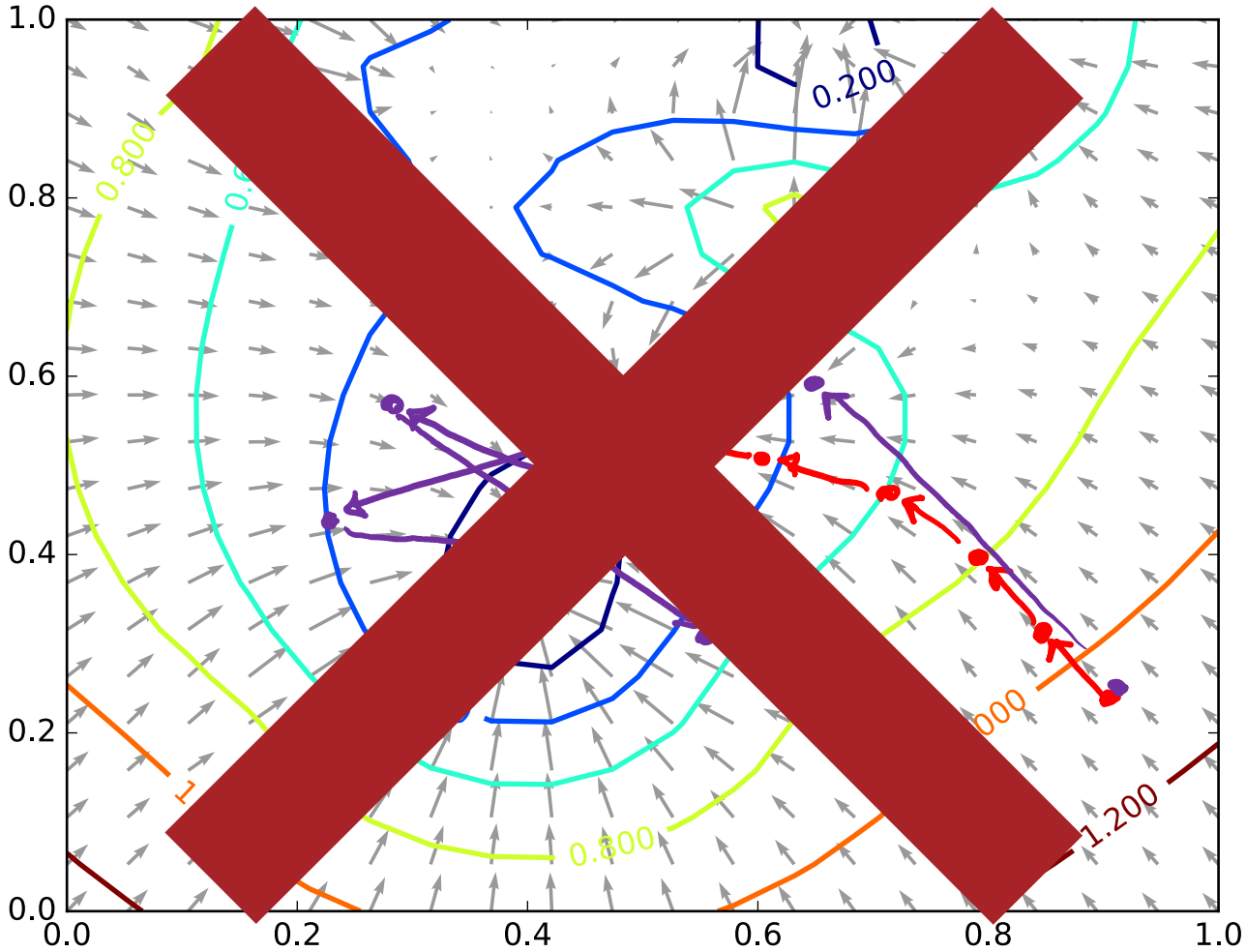
$$J(\boldsymbol{\theta}) = -\frac{1}{N}\sum_{i=1}^{N} y^{(i)}\boldsymbol{\theta}^T\boldsymbol{x}^{(i)} - \log\left(1 + \exp\left(\boldsymbol{\theta}^T\boldsymbol{x}^{(i)}\right)\right)$$

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = -\frac{1}{N}\sum_{i=1}^{N} y^{(i)}\nabla_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}^T\boldsymbol{x}^{(i)}\right) - \nabla_{\boldsymbol{\theta}}\log\left(1 + \exp\left(\boldsymbol{\theta}^T\boldsymbol{x}^{(i)}\right)\right)$$

$$= -\frac{1}{N}\sum_{i=1}^{N} y^{(i)}\boldsymbol{x}^{(i)} - \frac{\exp\left(\boldsymbol{\theta}^T\boldsymbol{x}^{(i)}\right)}{1 + \exp\left(\boldsymbol{\theta}^T\boldsymbol{x}^{(i)}\right)}\boldsymbol{x}^{(i)}$$

$$= \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}^{(i)}\left(P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}\right) - y^{(i)}\right)$$

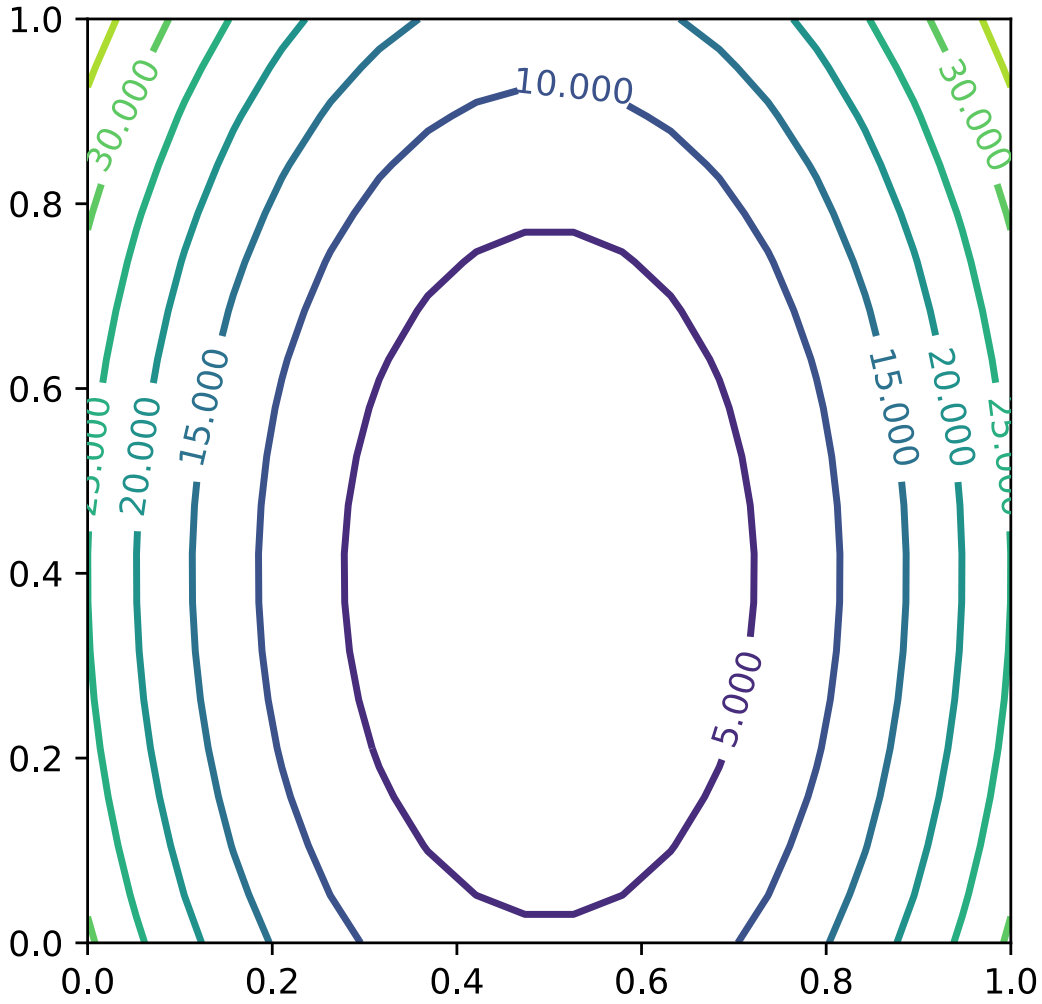
Recall: Gradient Descent



Good news: the negative conditional log-likelihood is convex!

(See HW/recitation)

Recall: Gradient Descent



Good news: the negative conditional log-likelihood is convex!

(See HW/recitation)

Gradient Descent

- Input: training dataset $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$ and step size γ

1. Initialize $\boldsymbol{\theta}^{(0)}$ to all zeros and set $t = 0$
2. While TERMINATION CRITERION is not satisfied
   a. Compute the gradient:

$$\nabla_{\boldsymbol{\theta}} J\left(\boldsymbol{\theta}^{(t)}\right) = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}^{(i)}\left(P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}^{(t)}\right) - y^{(i)}\right)$$

   b. Update $\boldsymbol{\theta}$: $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \gamma \nabla_{\boldsymbol{\theta}} J\left(\boldsymbol{\theta}^{(t)}\right)$
   c. Increment t: $t \leftarrow t + 1$

- Output: $\boldsymbol{\theta}^{(t)}$

Poll Question 1:

What is the computational cost of one iteration of gradient descent for logistic regression?

A. $O(1)$ **(TOXIC)** B. $O(N)$ C. $O(D)$ D. $O(ND)$

- Input: training dataset $\mathcal{D} = \{(\boldsymbol{x}^{(i)}, y^{(i)})\}_{i=1}^{N}$ and step size γ

1. Initialize $\boldsymbol{\theta}^{(0)}$ to all zeros and set $t = 0$
2. While TERMINATION CRITERION is not satisfied
   a. Compute the gradient:

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}^{(t)}) = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{x}^{(i)} \left(P(Y = 1 | \boldsymbol{x}^{(i)}, \boldsymbol{\theta}^{(t)}) - y^{(i)} \right)$$

   b. Update $\boldsymbol{\theta}$: $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \gamma \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}^{(t)})$
   c. Increment t: $t \leftarrow t + 1$

- Output: $\boldsymbol{\theta}^{(t)}$

Gradient Descent

- Input: training dataset $\mathcal{D} = \{(\boldsymbol{x}^{(i)}, y^{(i)})\}_{i=1}^{N}$ and step size γ

1. Initialize $\boldsymbol{\theta}^{(0)}$ to all zeros and set $t = 0$
2. While TERMINATION CRITERION is not satisfied
   a. Compute the gradient:

$$O(ND) \left\{ \quad \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}^{(t)}) = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{x}^{(i)} \left(P(Y = 1 | \boldsymbol{x}^{(i)}, \boldsymbol{\theta}^{(t)}) - y^{(i)} \right) \right.$$

   b. Update $\boldsymbol{\theta}$: $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \gamma \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}^{(t)})$
   c. Increment t: $t \leftarrow t + 1$

- Output: $\boldsymbol{\theta}^{(t)}$

Stochastic Gradient Descent (SGD)

- Input: training dataset $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$ and step size γ

1. Initialize $\boldsymbol{\theta}^{(0)}$ to all zeros and set $t = 0$
2. While TERMINATION CRITERION is not satisfied
   a. Randomly sample a data point from $\mathcal{D}$, $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$
   b. Compute the pointwise gradient:
      $\nabla_{\boldsymbol{\theta}} J^{(i)}\left(\boldsymbol{\theta}^{(t)}\right) = \boldsymbol{x}^{(i)}\left(P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}^{(t)}\right) - y^{(i)}\right)$
   c. Update $\boldsymbol{\theta}$: $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \gamma \nabla_{\boldsymbol{\theta}} J^{(i)}\left(\boldsymbol{\theta}^{(t)}\right)$
   d. Increment t: $t \leftarrow t + 1$

- Output: $\boldsymbol{\theta}^{(t)}$

Stochastic Gradient Descent (SGD)

- If the example is sampled uniformly at random, the expected value of the pointwise gradient is the same as the full gradient!

$$E\left[\nabla_{\boldsymbol{\theta}} J^{(i)}(\boldsymbol{\theta})\right] = \sum_{i=1}^{N} \left(\text{probability of selecting } \boldsymbol{x}^{(i)}, y^{(i)}\right) \nabla_{\boldsymbol{\theta}} J^{(i)}(\boldsymbol{\theta})$$

$$= \sum_{i=1}^{N} \left(\frac{1}{N}\right) \nabla_{\boldsymbol{\theta}} J^{(i)}(\boldsymbol{\theta}) = \frac{1}{N} \sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} J^{(i)}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$$

- In practice, the data set is randomly shuffled then looped through so that each data point is used equally often

Stochastic Gradient Descent (SGD)

- Input: training dataset $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$ and step size γ

1. Initialize $\boldsymbol{\theta}^{(0)}$ to all zeros and set $t = 0$
2. While TERMINATION CRITERION is not satisfied
   a. For $i \in \text{shuffle}(\{1, \ldots, N\})$
      i. Compute the pointwise gradient:
      $$\nabla_{\boldsymbol{\theta}} J^{(i)}\left(\boldsymbol{\theta}^{(t)}\right) = \boldsymbol{x}^{(i)}\left(P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}^{(t)}\right) - y^{(i)}\right)$$
      ii. Update $\boldsymbol{\theta}$: $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \gamma \nabla_{\boldsymbol{\theta}} J^{(i)}\left(\boldsymbol{\theta}^{(t)}\right)$
      iii. Increment t: $t \leftarrow t + 1$

- Output: $\boldsymbol{\theta}^{(t)}$

Wait, have we seen something like this before?

- Input: training dataset $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$ and step size γ

1. Initialize $\boldsymbol{\theta}^{(0)}$ to all zeros and set $t = 0$
2. While TERMINATION CRITERION is not satisfied
   a. For $i \in \text{shuffle}(\{1, \ldots, N\})$
      i. Compute the pointwise gradient:
      $$\nabla_{\boldsymbol{\theta}} J^{(i)}\left(\boldsymbol{\theta}^{(t)}\right) = \boldsymbol{x}^{(i)}\left(P\left(Y = 1 \middle| \boldsymbol{x}^{(i)}, \boldsymbol{\theta}^{(t)}\right) - y^{(i)}\right)$$
      ii. Update $\boldsymbol{\theta}$: $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \gamma \nabla_{\boldsymbol{\theta}} J^{(i)}\left(\boldsymbol{\theta}^{(t)}\right)$
      iii. Increment t: $t \leftarrow t + 1$

- Output: $\boldsymbol{\theta}^{(t)}$

Recall: Perceptron Learning Algorithm

- Input: training dataset $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$

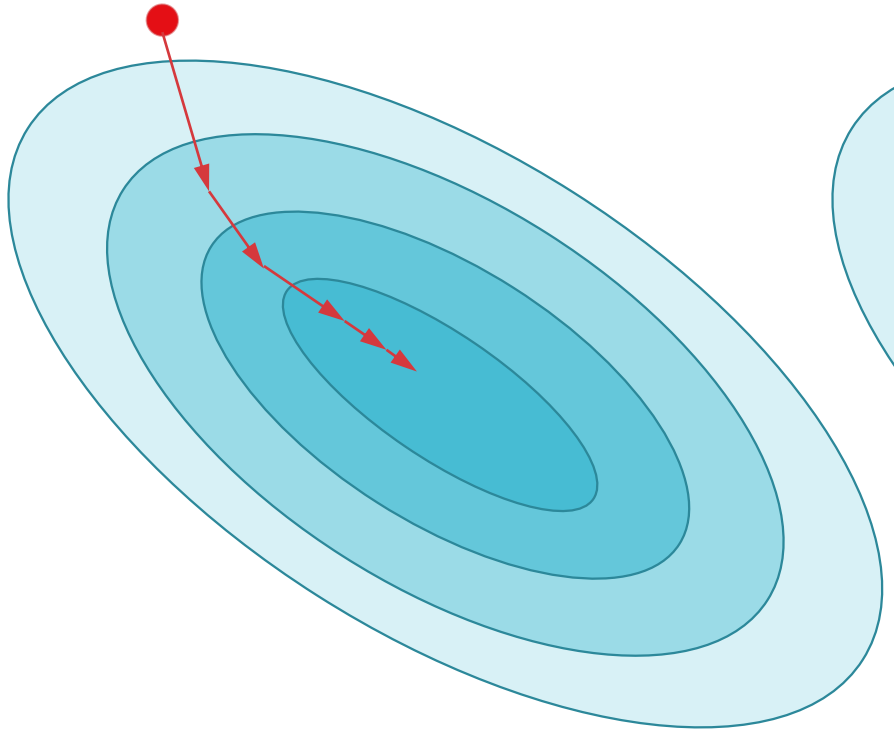
1. Initialize $\boldsymbol{\theta}^{(0)}$ to all zeros and set $t = 0$
2. While NOT CONVERGED
   a. For $t \in \text{shuffle}(\{1, \ldots, N\})$
      i. Predict the label of $\boldsymbol{x}^{(t)}$, $\hat{y} = \text{sign}\left(\boldsymbol{\theta}^T \boldsymbol{x}^{(t)}\right)$
      ii. Observe its true label, $y^{(t)}$
      iii. If we misclassified $\boldsymbol{x}^{(t)}$ ($y^{(t)} \neq \hat{y}$):

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(t)} \boldsymbol{x}^{(t)}$$

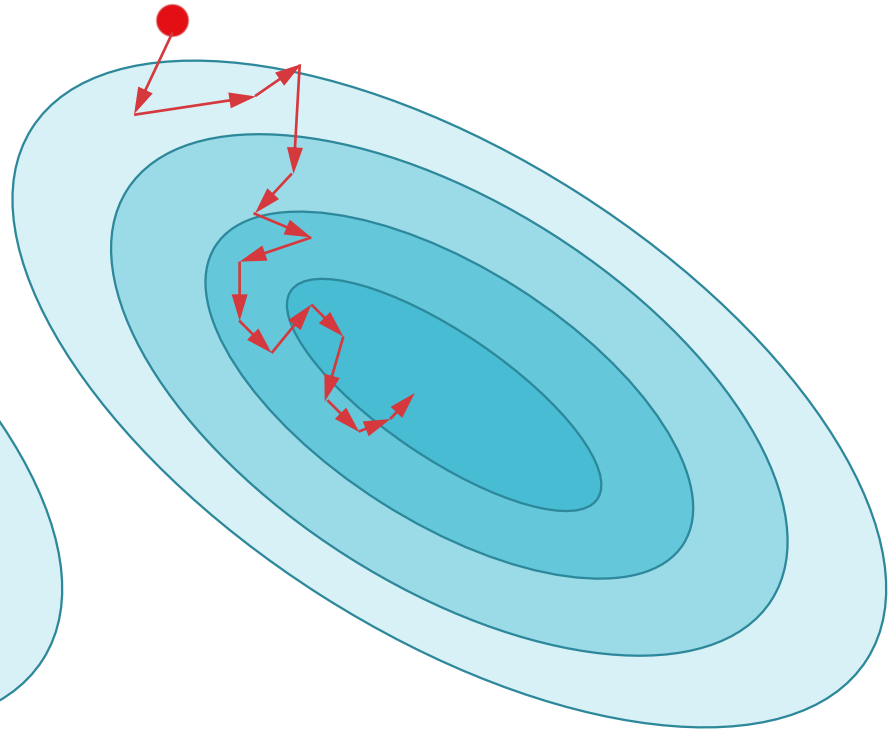
This is the negative gradient of the *hinge loss*

$$J^{(i)}(\boldsymbol{\theta}) = \max\left(0, 1 - y^{(i)} \boldsymbol{\theta}^T \boldsymbol{x}^{(i)}\right)$$

Stochastic Gradient Descent vs. Gradient Descent



Gradient Descent



Stochastic Gradient Descent

Stochastic Gradient Descent vs. Gradient Descent

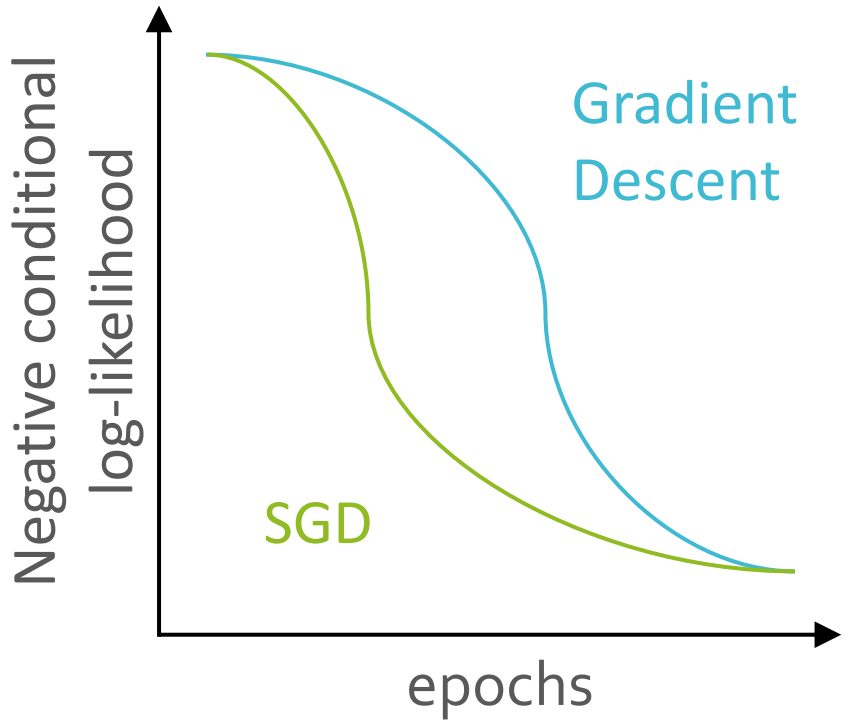
- An *epoch* is a single pass through the entire training dataset
  - Gradient descent updates the parameters once per epoch
  - SGD updates the parameters N times per epoch
- Theoretical comparison:
  - Define convergence to be when $J(\boldsymbol{\theta}^{(t)}) - J(\boldsymbol{\theta}^*) < \epsilon$

Method	Steps to Convergence	Computation per Step
Gradient descent	$O(\log 1/\epsilon)$	$O(ND)$
SGD	$O(1/\epsilon)$	$O(D)$

(with high probability under certain assumptions)

Stochastic Gradient Descent vs. Gradient Descent

- An *epoch* is a single pass through the entire training dataset
  - Gradient descent updates the parameters once per epoch
  - SGD updates the parameters N times per epoch



Empirically, SGD reduces the negative conditional log-likelihood much faster than gradient descent

Optimization for ML Learning Objectives

You should be able to…

- Apply gradient descent to optimize a function
- Apply stochastic gradient descent (SGD) to optimize a function
- Apply knowledge of zero derivatives to identify a closed-form solution (if one exists) to an optimization problem
- Distinguish between convex, concave, and nonconvex functions
- Obtain the gradient (and Hessian) of a (twice) differentiable function

Logistic Regression Learning Objectives

You should be able to…

- Apply the principle of maximum likelihood estimation (MLE) to learn the parameters of a probabilistic model
- Given a discriminative probabilistic model, derive the conditional log-likelihood, its gradient, and the corresponding Bayes Classifier
- Explain the practical reasons why we work with the log of the likelihood
- Implement logistic regression for binary (and multiclass) classification
- Prove that the decision boundary of binary logistic regression is linear