10-301/601 Introduction to Machine Learning

Machine Learning Department
School of Computer Science
Carnegie Mellon University

Linear Regression

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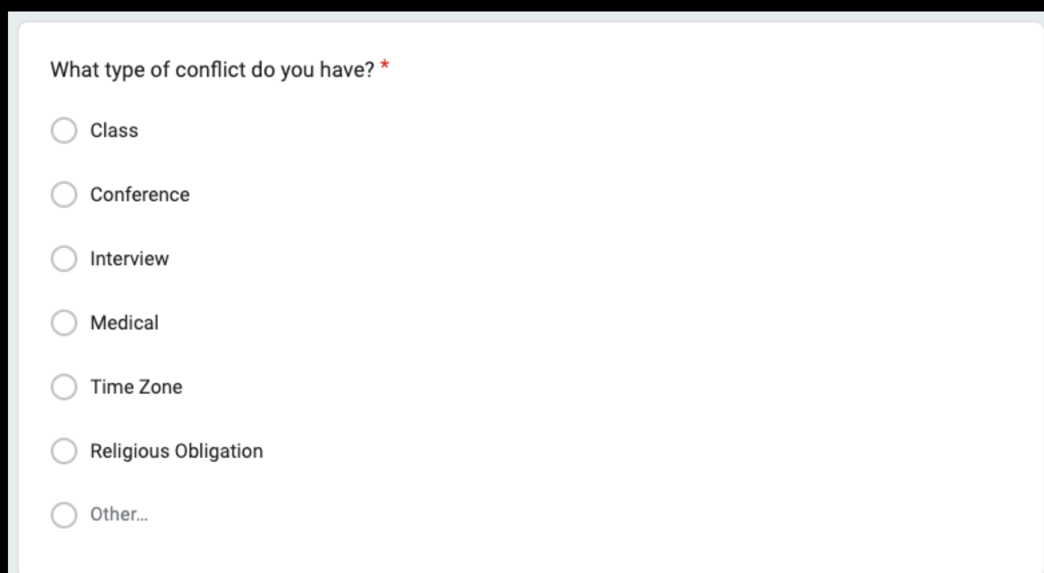
Lecture 7
Sep. 20, 2022

Reminders

- Homework 3: KNN, Perceptron, Lin.Reg.
  - Out: Wed, Sep. 21
  - Due: Wed, Sep. 28 at 11:59pm
  - (only two grace/late days permitted)
- Exam conflicts form



What type of conflict do you have? *

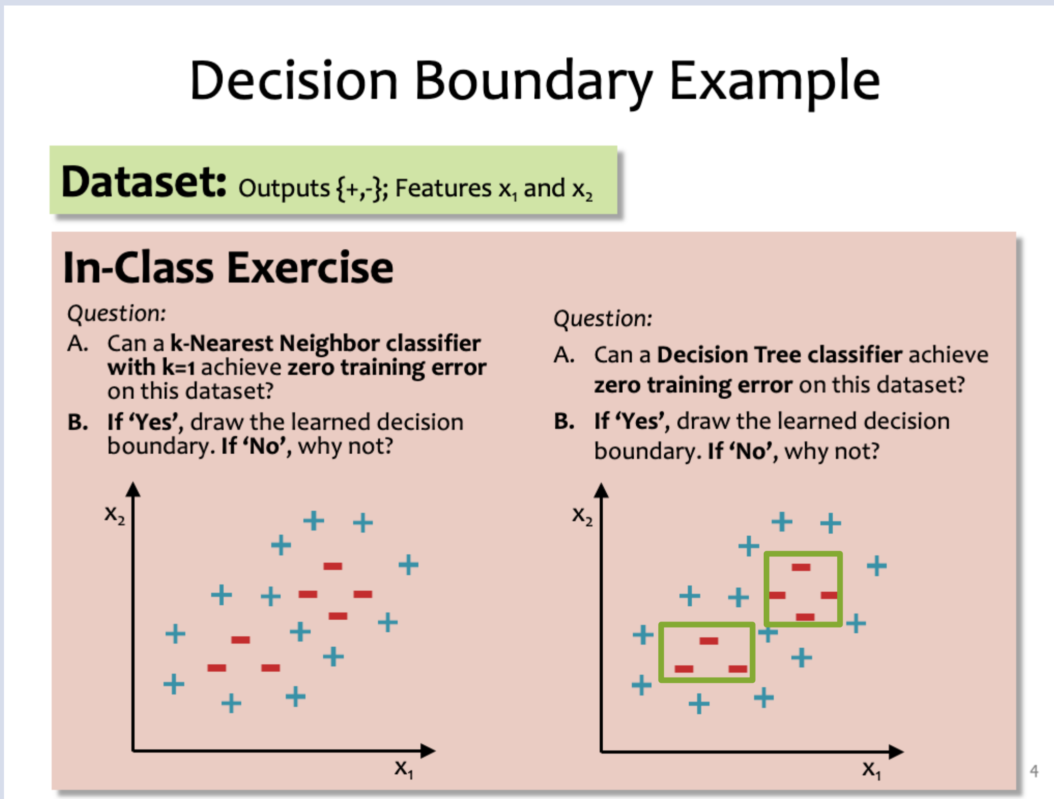
- Class
- Conference
- Interview
- Medical
- Time Zone
- Religious Obligation
- Other...

DECISION TREES WITH REAL-VALUED FEATURES

Q&A

Q: How do we learn a Decision Tree with real-valued features?

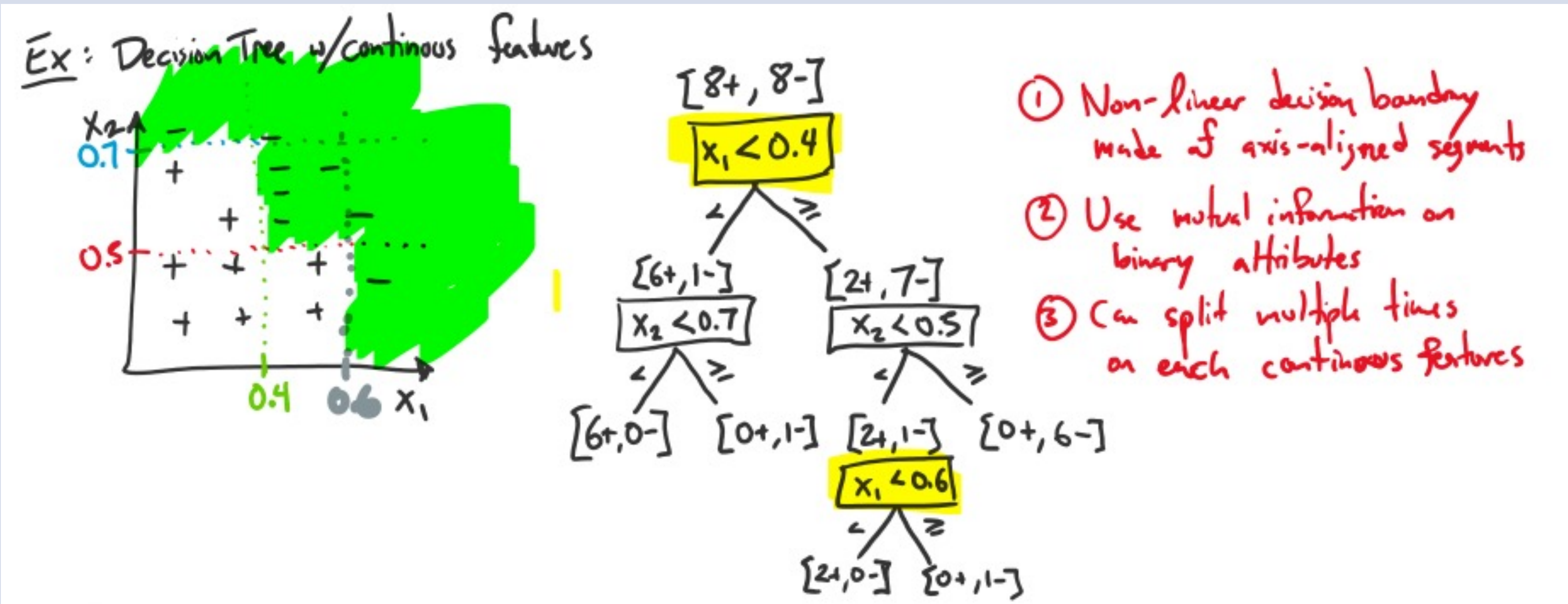
A:



Q&A

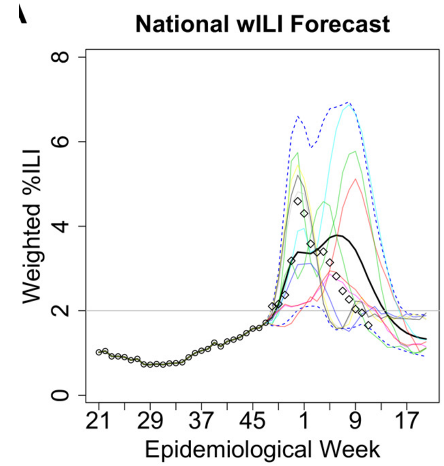
Q: How do we learn a Decision Tree with real-valued features?

A: Make new discrete features out of the real-valued features and then learn the Decision Tree as normal! Here's an example…



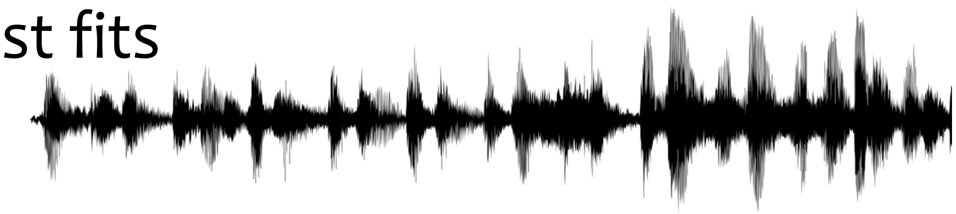
REGRESSION

Regression



Goal:

- Given a training dataset of pairs (**x**,y) where
  - **x** is a vector
  - y is a scalar
- Learn a function (aka. curve or line) y' = h(x) that best fits the training data

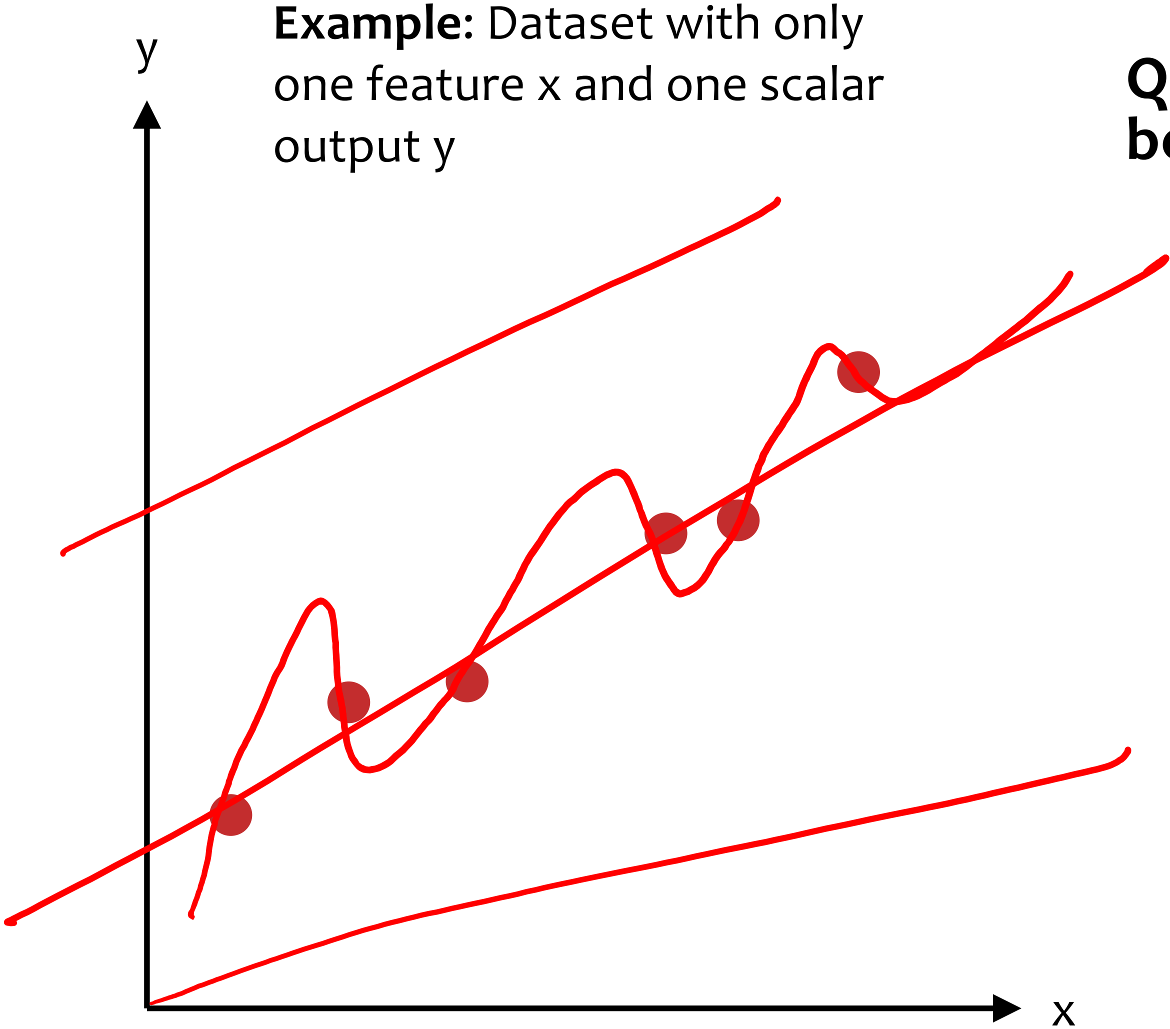


Example Applications:

- Stock price prediction
- Forecasting epidemics
- Speech synthesis
- Generation of images (e.g. *Deep Dream*)

Regression

Example: Dataset with only one feature x and one scalar output y

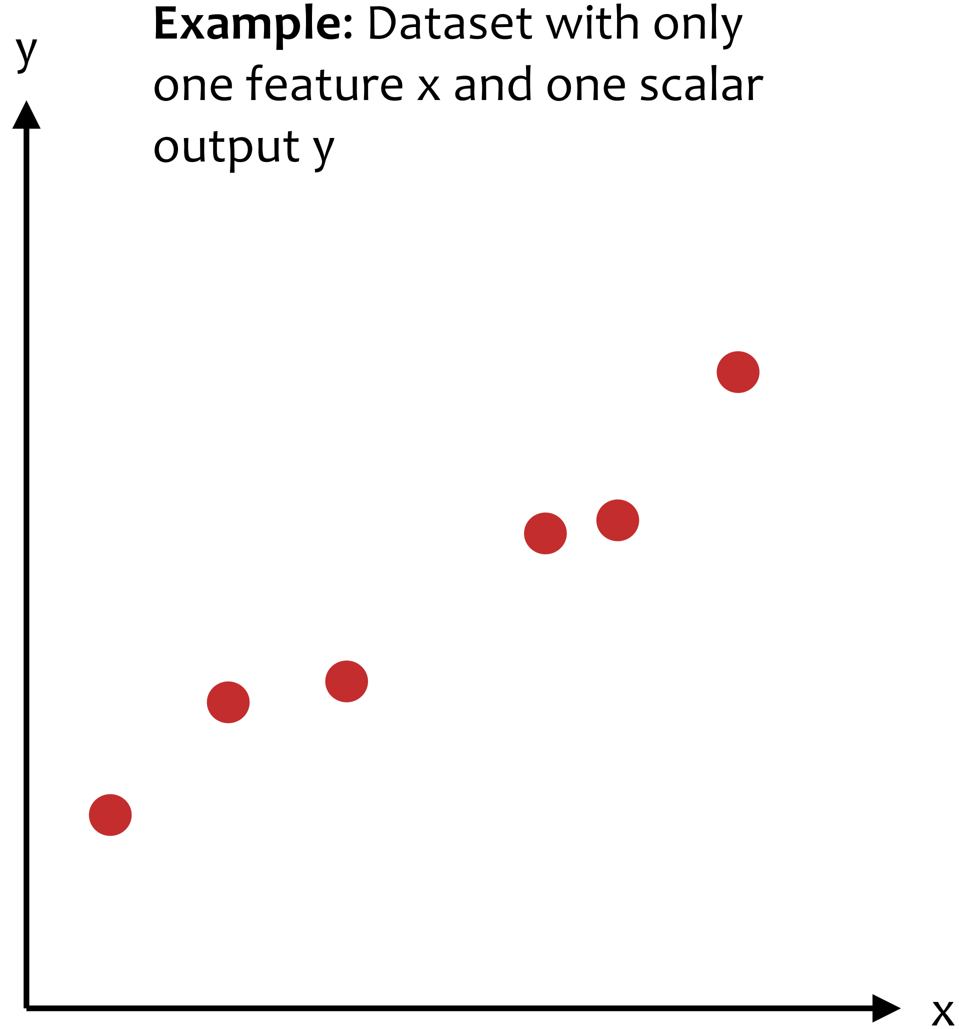


Q: What is the function that best fits these points?

K-NEAREST NEIGHBOR REGRESSION

k-NN Regression

Example: Dataset with only one feature x and one scalar output y



Algorithm 1: k=1 Nearest Neighbor Regression

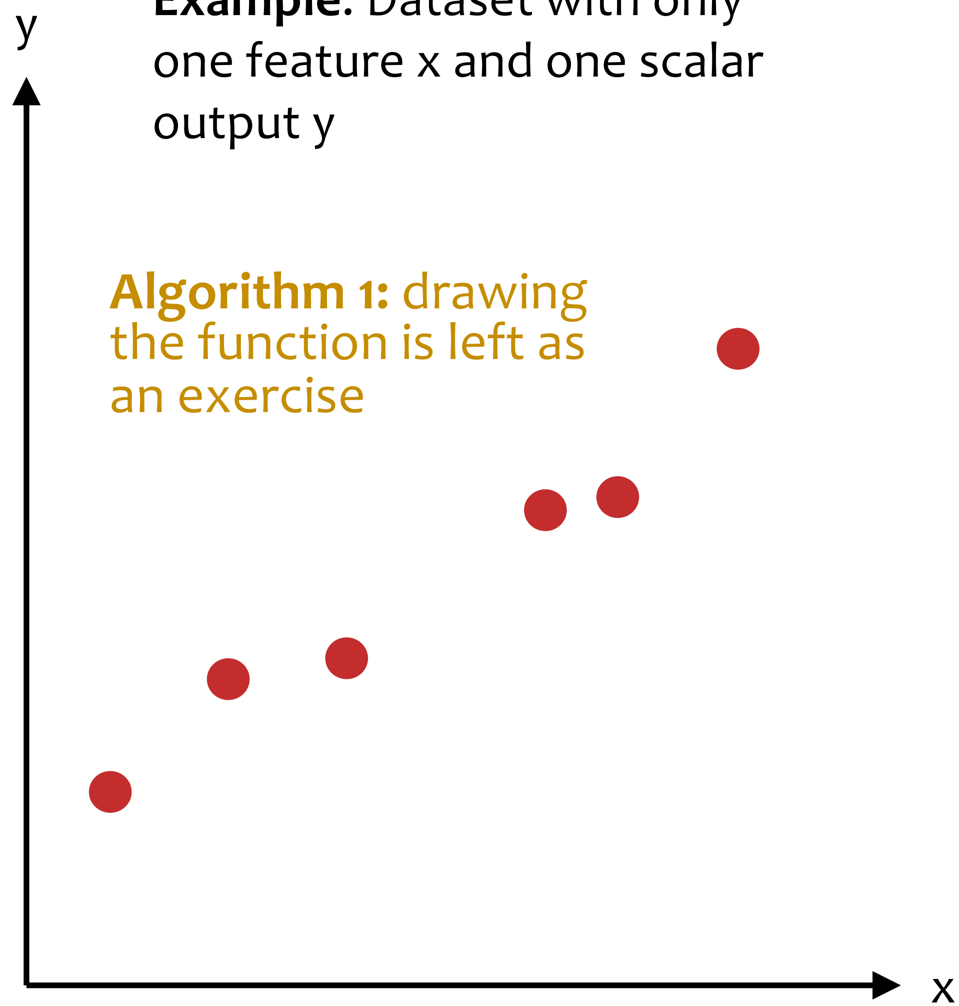
- *Train*: store all (x, y) pairs
- *Predict*: pick the nearest x in training data and return its y

Algorithm 2: k=2 Nearest Neighbors Distance Weighted Regression

- *Train*: store all (x, y) pairs
- *Predict*: pick the nearest two instances $x^{(n1)}$ and $x^{(n2)}$ in training data and return the weighted average of their y values

k-NN Regression

Example: Dataset with only one feature x and one scalar output y



Algorithm 1: drawing the function is left as an exercise

Algorithm 1: k=1 Nearest Neighbor Regression

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k-NN Regression

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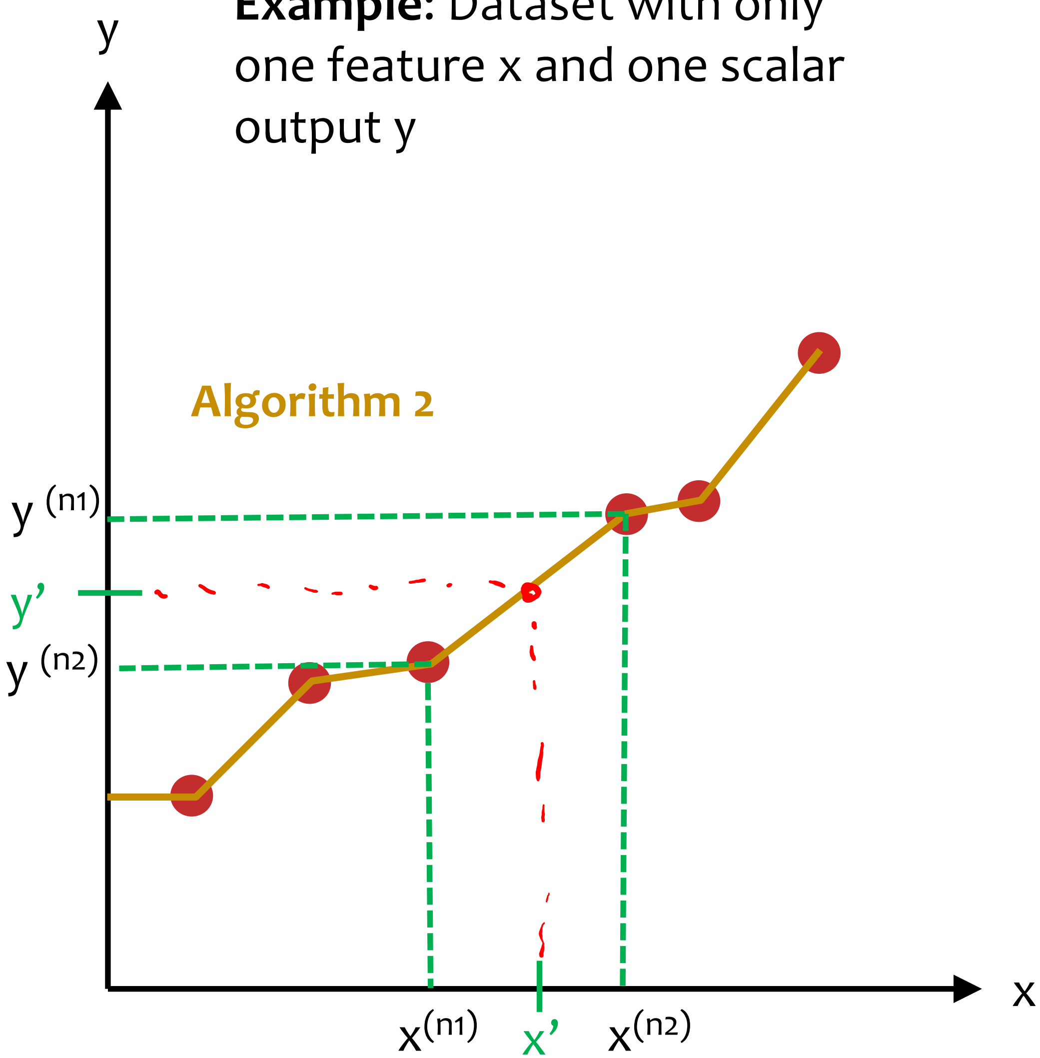

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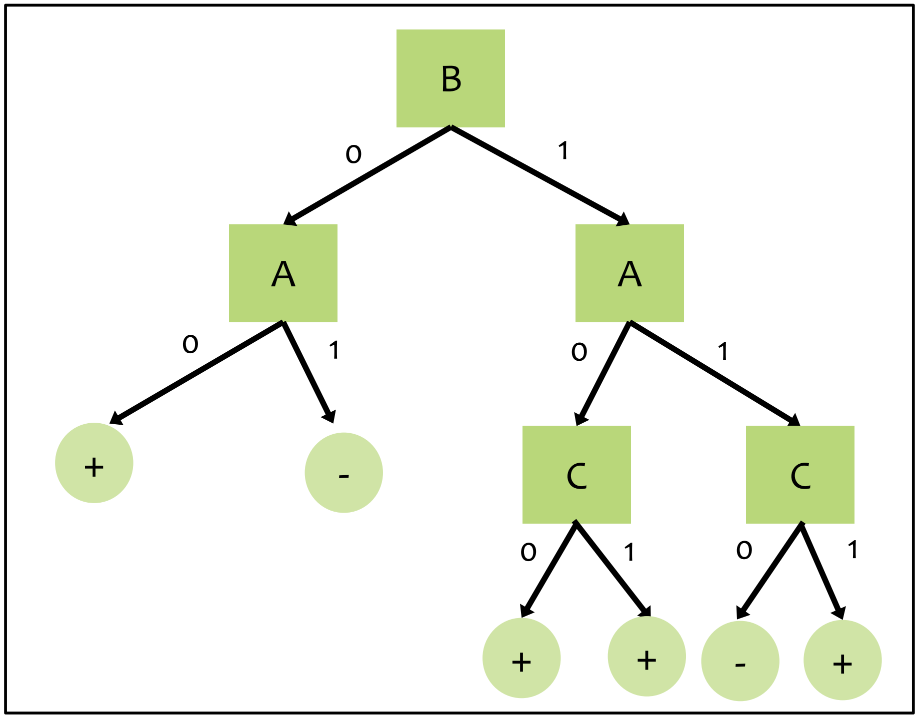
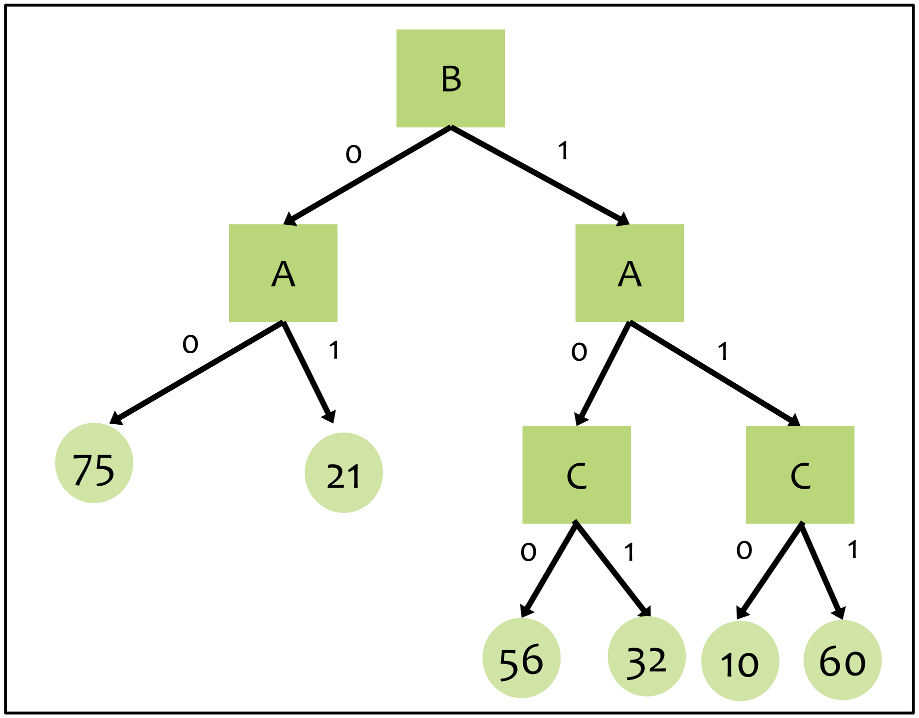
DECISION TREE REGRESSION

Decision Tree Regression

Decision Tree for Classification



Decision Tree for Regression

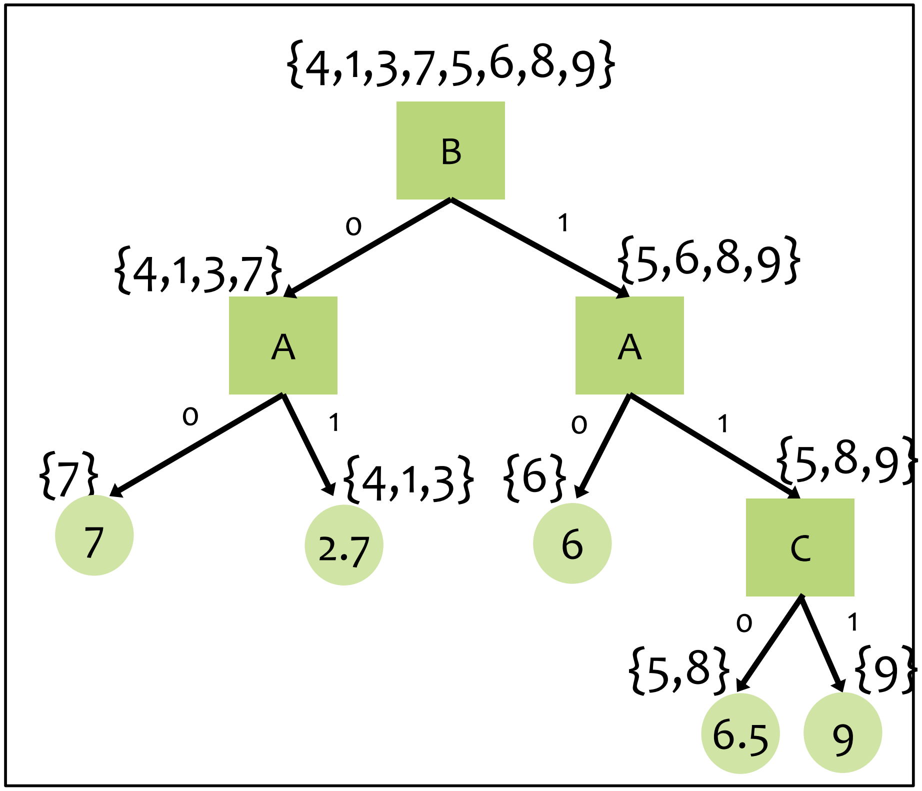


Decision Tree Regression

Dataset for Regression

Y	A	B	C
4	1	0	0
1	1	0	1
3	1	0	0
7	0	0	1
5	1	1	0
6	0	1	1
8	1	1	0
9	1	1	1

Decision Tree for Regression



During learning, choose the attribute that minimizes an appropriate splitting criterion (e.g. mean squared error, mean absolute error)

LINEAR FUNCTIONS, RESIDUALS, AND MEAN SQUARED ERROR

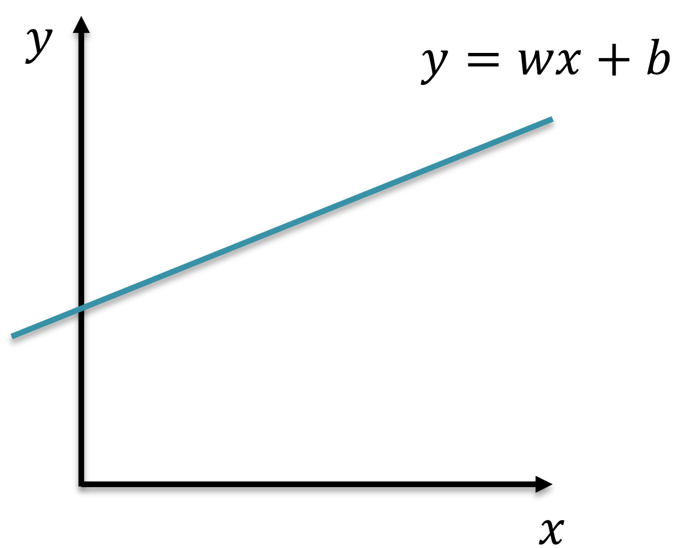
Linear Functions

Def: Regression is predicting real-valued outputs

$$\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^{n} \text{ with } \mathbf{x}^{(i)} \in \mathbb{R}^M, y^{(i)} \in \mathbb{R}$$

Common Misunderstanding:

Linear functions ≠ Linear decision boundaries



Linear Functions

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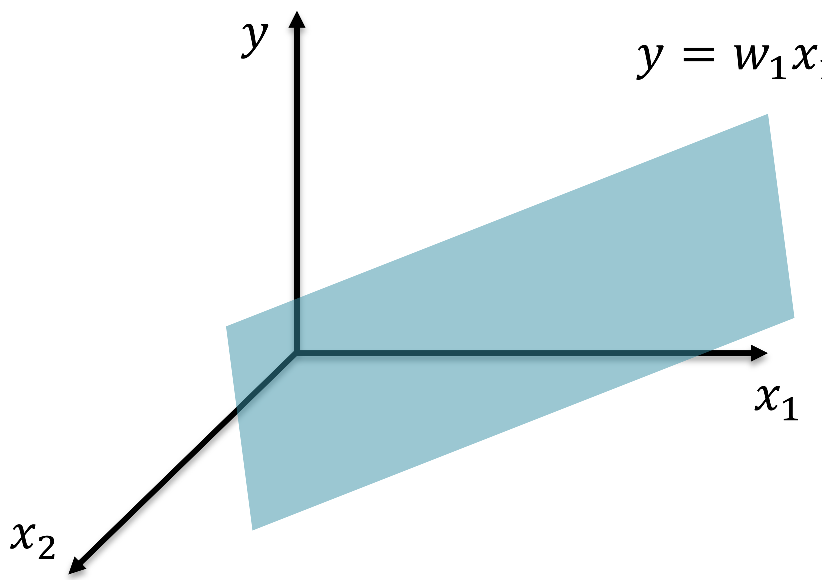
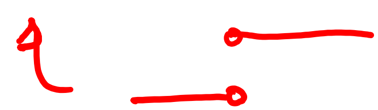


- A general linear function is
  $$y = \mathbf{w}^T \mathbf{x} + b$$
- A general linear decision boundary is
  $$y = \text{sign}(\mathbf{w}^T \mathbf{x} + b)$$



Regression Problems

Chalkboard

- Residuals
- Mean squared error

The Big Picture

OPTIMIZATION FOR ML

Unconstrained Optimization

- *Def*: In **unconstrained optimization**, we try minimize (or maximize) a function with *no constraints* on the inputs to the function

Given a function $J(\boldsymbol{\theta}), J : \mathbb{R}^M \rightarrow \mathbb{R}$

Our goal is to find $\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \mathbb{R}^M}{\operatorname{argmin}} J(\boldsymbol{\theta})$

For ML, these are the parameters

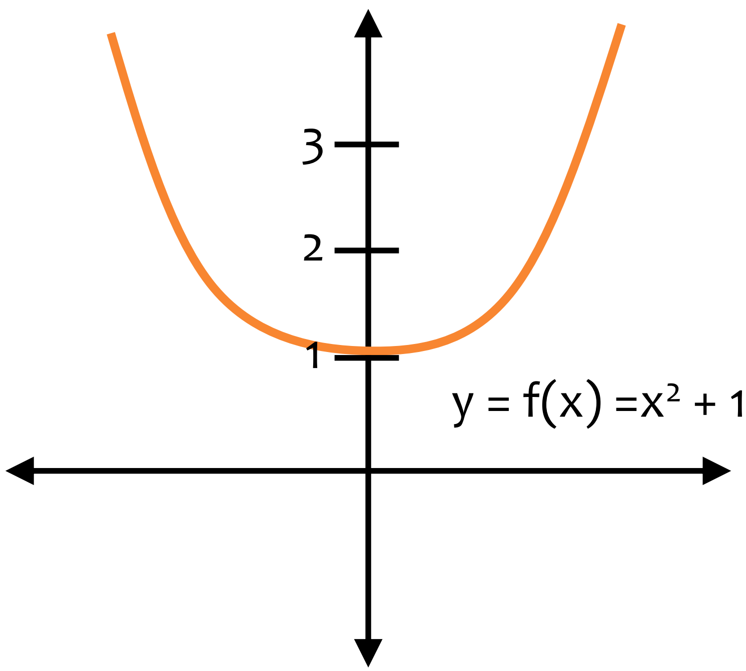
For ML, this is the objective function

Optimization for ML

Not quite the same setting as other fields…

- Function we are optimizing might not be the true goal
  (e.g. likelihood vs generalization error)
- Precision might not matter
  (e.g. data is noisy, so optimal up to 1e-16 might not help)
- Stopping early can help generalization error
  (i.e. "early stopping" is a technique for regularization – discussed more next time)

min vs. argmin



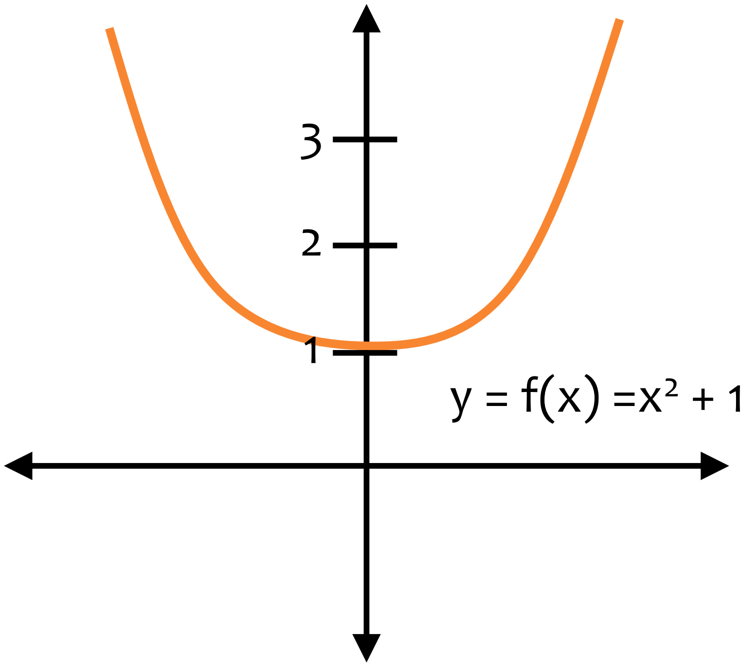
$v^* = \min_x f(x)$

$x^* = \text{argmin}_x f(x)$

Q1. 1. Question: What is v*?

Q2. 2. Question: What is x*?

min vs. argmin



$v^* = \min_x f(x)$

$x^* = \text{argmin}_x f(x)$

1. Question: What is v*?

   $v^* = 1$, the minimum value of the function

2. Question: What is x*?

   $x^* = 0$, the argument that yields the minimum value

OPTIMIZATION METHOD #0: RANDOM GUESSING

Notation Trick: Folding in the Intercept Term

$$\mathbf{x}' = [1, x_1, x_2, \ldots, x_M]^T$$

$$\boldsymbol{\theta} = [b, w_1, \ldots, w_M]^T$$

Notation Trick: fold the bias *b* and the weights ***w*** into a single vector **θ** by prepending a constant to ***x*** and increasing dimensionality by one!

$$h_{\mathbf{w},b}(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + b$$

$$h_{\boldsymbol{\theta}}(\mathbf{x}') = \boldsymbol{\theta}^T\mathbf{x}'$$

This convenience trick allows us to more compactly talk about linear functions as a simple dot product (without explicitly writing out the intercept term every time).

Linear Regression as Function Approximation

$\mathcal{D} = \{\mathbf{x}^{(i)}, y^{(i)}\}_{i=1}^N$
where $\mathbf{x} \in \mathbb{R}^M$ and $y \in \mathbb{R}$

1. Assume $\mathcal{D}$ generated as:

$$\mathbf{x}^{(i)} \sim p^*(\cdot)$$
$$y^{(i)} = h^*(\mathbf{x}^{(i)})$$

2. Choose hypothesis space, $\mathcal{H}$:
all linear functions in M-dimensional space

$$\mathcal{H} = \{h_{\boldsymbol{\theta}} : h_{\boldsymbol{\theta}}(\mathbf{x}) = \boldsymbol{\theta}^T \mathbf{x}, \boldsymbol{\theta} \in \mathbb{R}^M\}$$

3. Choose an objective function:
mean squared error (MSE)

$$J(\boldsymbol{\theta}) = \frac{1}{N} \sum_{i=1}^N e_i^2$$
$$= \frac{1}{N} \sum_{i=1}^N \left(y^{(i)} - h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)})\right)^2$$
$$= \frac{1}{N} \sum_{i=1}^N \left(y^{(i)} - \boldsymbol{\theta}^T \mathbf{x}^{(i)}\right)^2$$

4. Solve the unconstrained optimization problem via favorite method:
   - *gradient descent*
   - *closed form*
   - *stochastic gradient descent*
   - ...

$$\hat{\boldsymbol{\theta}} = \operatorname*{argmin}_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$$

5. Test time: given a new $\mathbf{x}$, make prediction $\hat{y}$

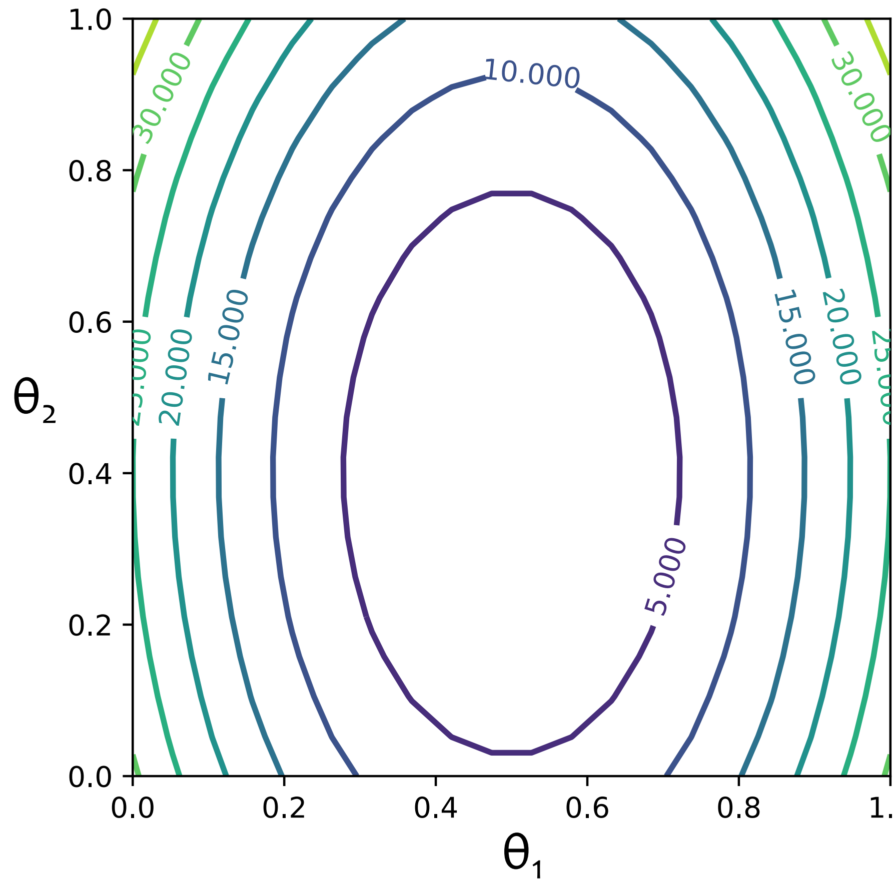
$$\hat{y} = h_{\hat{\boldsymbol{\theta}}}(\mathbf{x}) = \hat{\boldsymbol{\theta}}^T \mathbf{x}$$

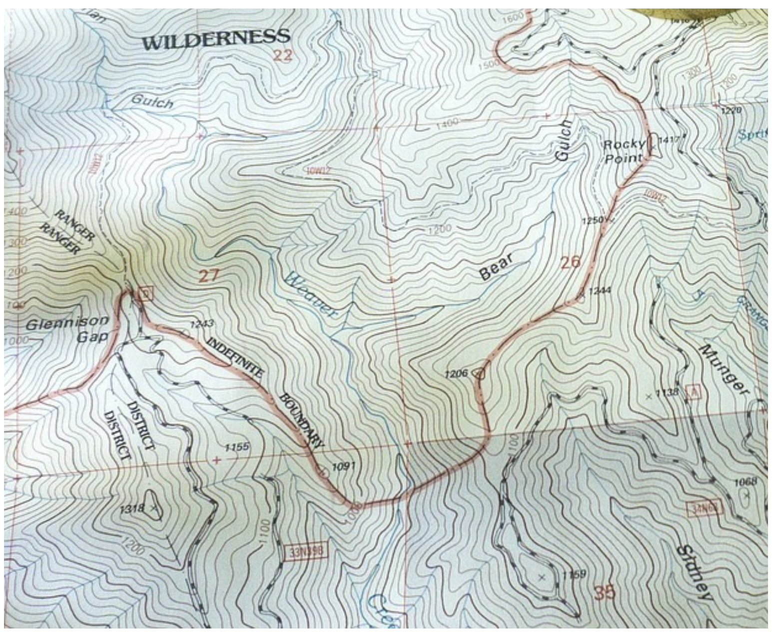
Contour Plots

Contour Plots

1. Each level curve labeled with value
2. Value label indicates the value of the function for all points lying on that level curve
3. Just like a topographical map, but for a function

$$J(\boldsymbol{\theta}) = J(\theta_1, \theta_2) = (10(\theta_1 - 0.5))^2 + (6(\theta_1 - 0.4))^2$$



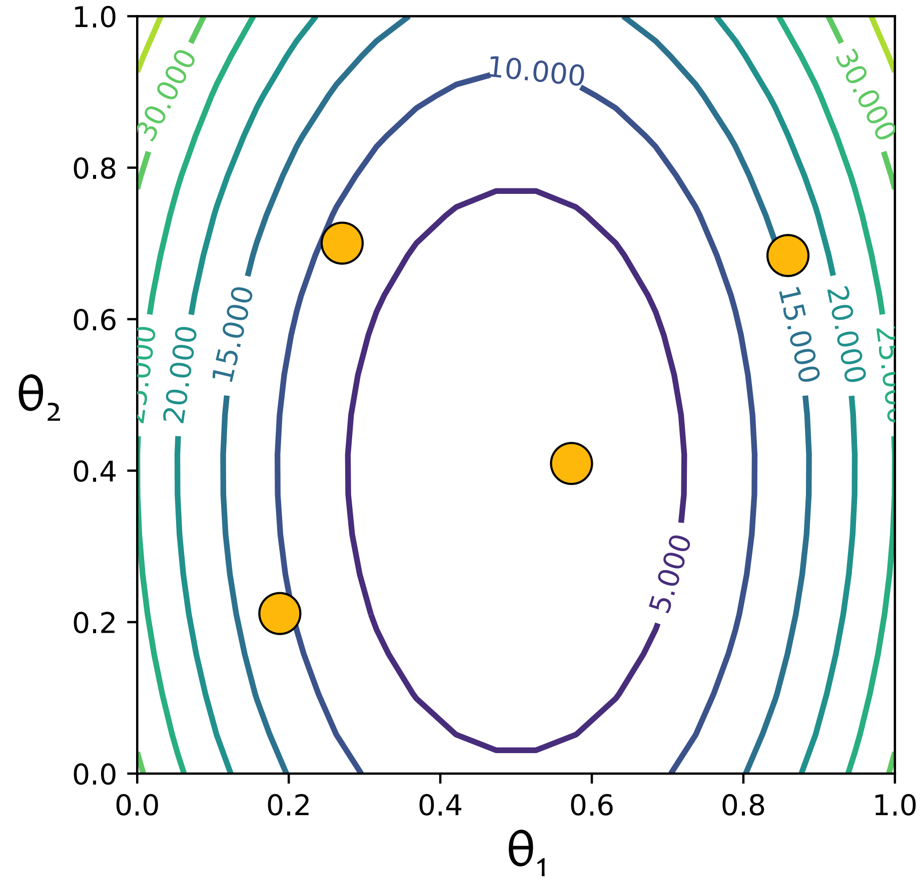


Optimization by Random Guessing

Optimization Method #0: Random Guessing

1. Pick a random $\boldsymbol{\theta}$
2. Evaluate $J(\boldsymbol{\theta})$
3. Repeat steps 1 and 2 many times
4. Return $\boldsymbol{\theta}$ that gives smallest $J(\boldsymbol{\theta})$

$J(\boldsymbol{\theta}) = J(\theta_1, \theta_2) = (10(\theta_1 - 0.5))^2 + (6(\theta_1 - 0.4))^2$



t	θ_1	θ_2	$J(\theta_1, \theta_2)$
1	0.2	0.2	10.4
2	0.3	0.7	7.2
3	0.6	0.4	1.0
4	0.9	0.7	16.2

Optimization by Random Guessing

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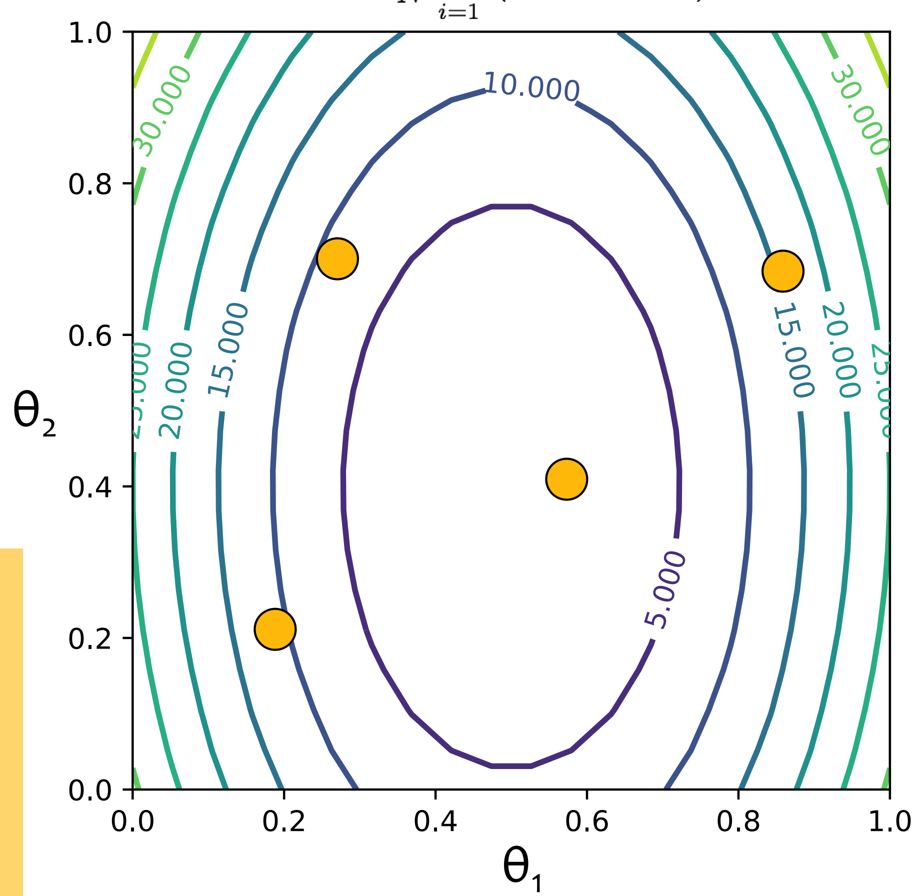
For Linear Regression:

- **objective function** is Mean Squared Error (MSE)
- MSE $= J(w, b)$
  $= J(\theta_1, \theta_2) = \frac{1}{N}\sum_{i=1}^{N}\left(y^{(i)} - \boldsymbol{\theta}^T\mathbf{x}^{(i)}\right)^2$
- contour plot: each line labeled with MSE – **lower means a better fit**
- **minimum** corresponds to parameters $(w,b) = (\theta_1, \theta_2)$ that **best fit** some training dataset

$$J(\boldsymbol{\theta}) = J(\theta_1, \theta_2) = \frac{1}{N}\sum_{i=1}^{N}\left(y^{(i)} - \boldsymbol{\theta}^T\mathbf{x}^{(i)}\right)^2$$



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1	0.2	0.2	10.4
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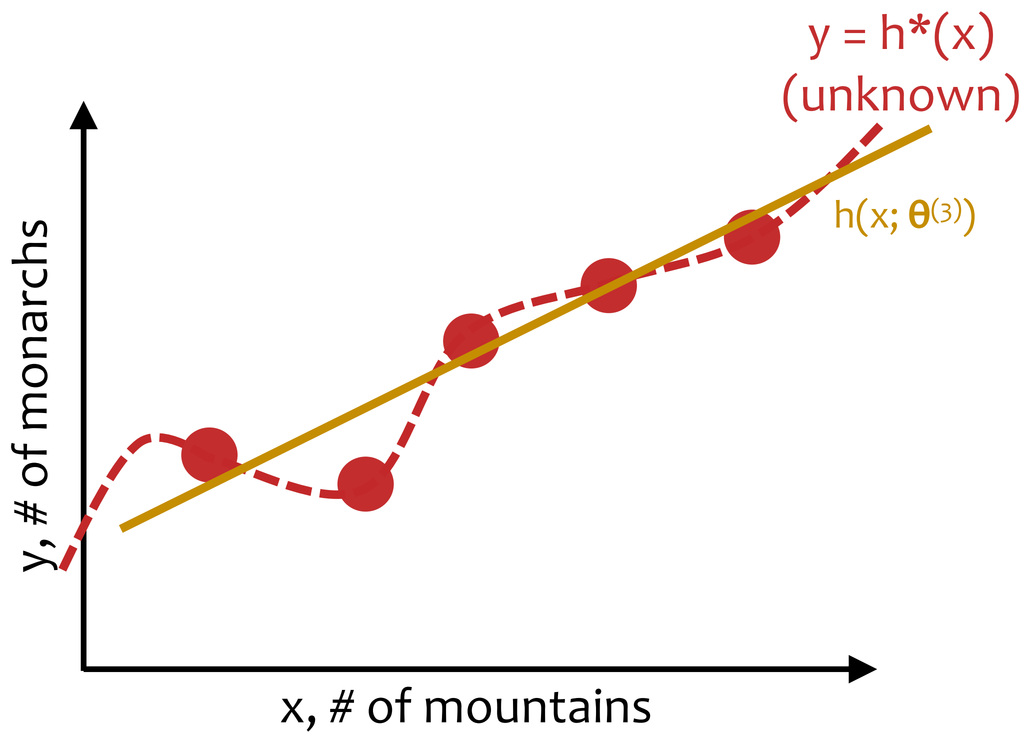
Counting Butterflies

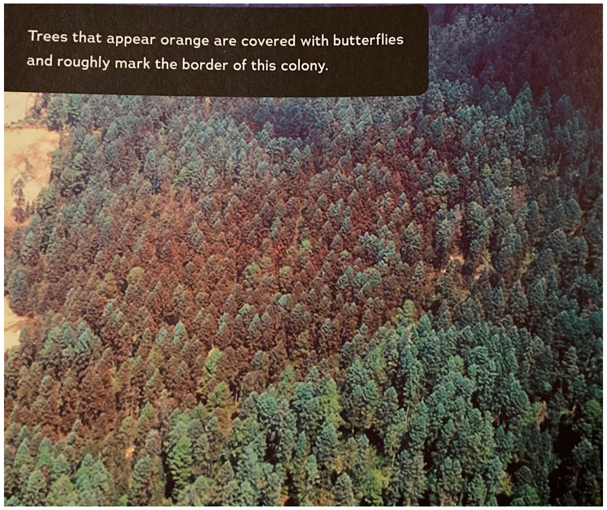


Linear Regression in High Dimensions

- In our discussions of linear regression, we will always assume there is just one output, y
- But our inputs will usually have many features:

  $$\mathbf{x} = [x_1, x_2, \ldots, x_M]^T$$

- For example:
  - suppose we had a drone take pictures of each section of forest
  - each feature could correspond to a pixel in this image such that $x_m = 1$ if the pixel is orange and $x_m = 0$ otherwise
  - the output y would be the number of butterflies in each picture

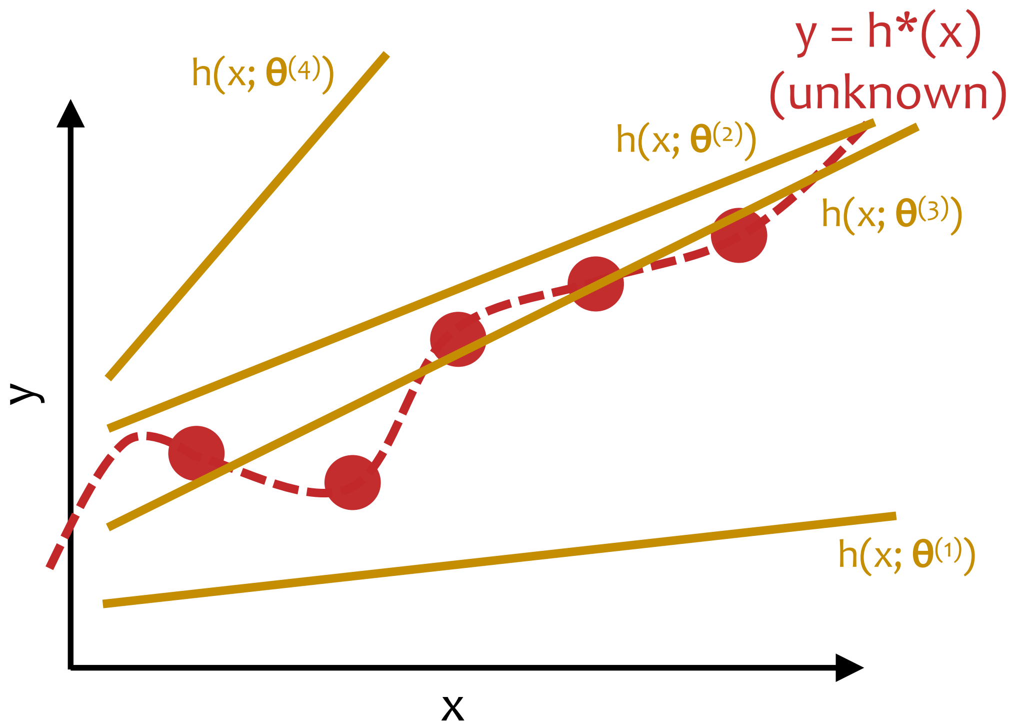


Q: How would you obtain ground truth data?

Linear Regression by Rand. Guessing

Optimization Method #0: Random Guessing

1. Pick a random $\boldsymbol{\theta}$
2. Evaluate $J(\boldsymbol{\theta})$
3. Repeat steps 1 and 2 many times
4. Return $\boldsymbol{\theta}$ that gives smallest $J(\boldsymbol{\theta})$



For Linear Regression:

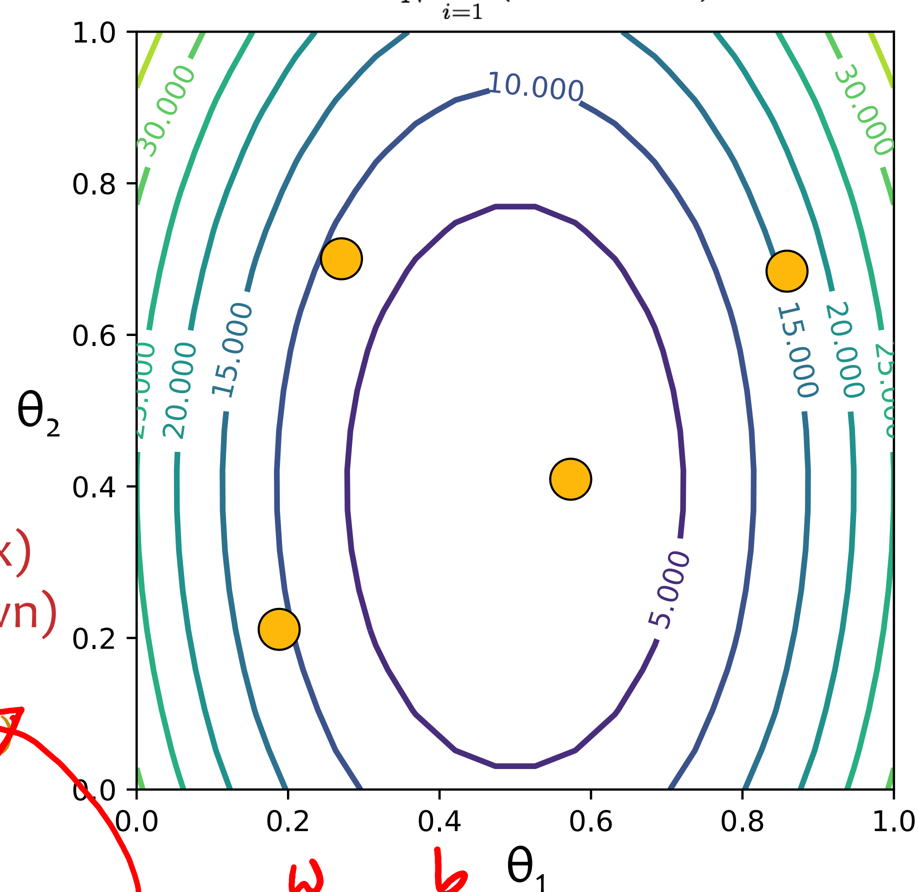
- target function h*(x) is **unknown**
- only have access to h*(x) through **training examples** $(x^{(i)},y^{(i)})$
- want $h(x; \boldsymbol{\theta}^{(t)})$ that **best approximates** h*(x)
- **enable generalization** w/inductive bias that restricts hypothesis class to **linear functions**

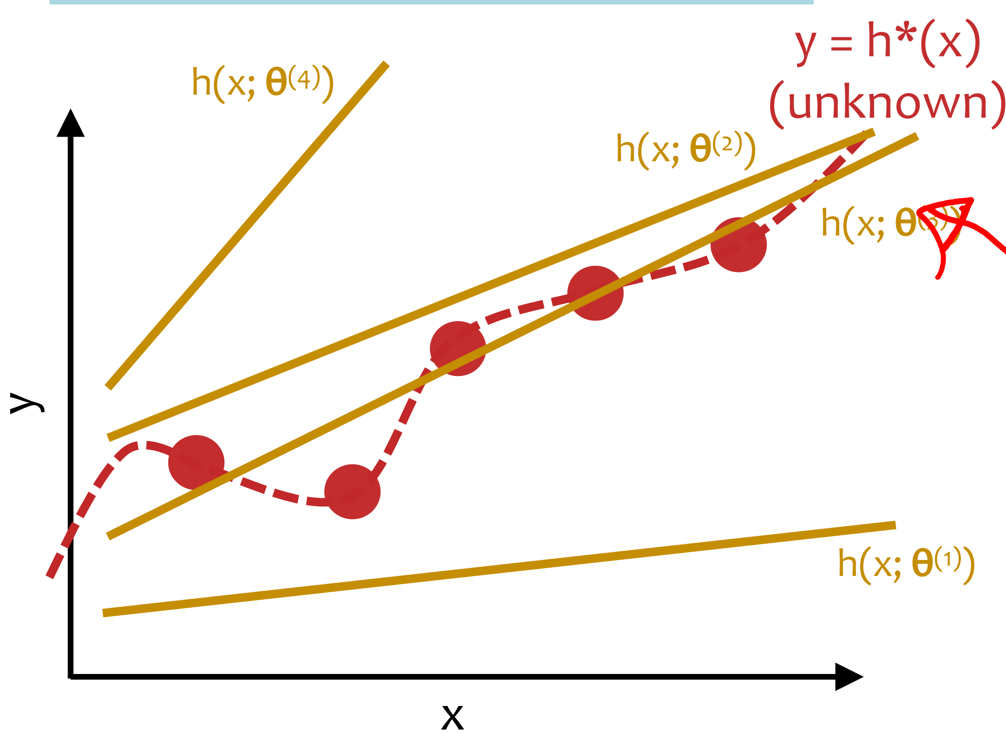
Linear Regression by Rand. Guessing

Optimization Method #0: Random Guessing

1. Pick a random **θ**
2. Evaluate J(**θ**)
3. Repeat steps 1 and 2 many times
4. Return **θ** that gives smallest J(**θ**)

$$J(\boldsymbol{\theta}) = J(\theta_1, \theta_2) = \frac{1}{N}\sum_{i=1}^{N}\left(y^{(i)} - \boldsymbol{\theta}^T \mathbf{x}^{(i)}\right)^2$$





t	θ_1 (ω)	θ_2 (b)	$J(\theta_1, \theta_2)$
1	0.2	0.2	10.4
2	0.3	0.7	7.2
3	0.6	0.4	1.0
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OPTIMIZATION METHOD #1: GRADIENT DESCENT

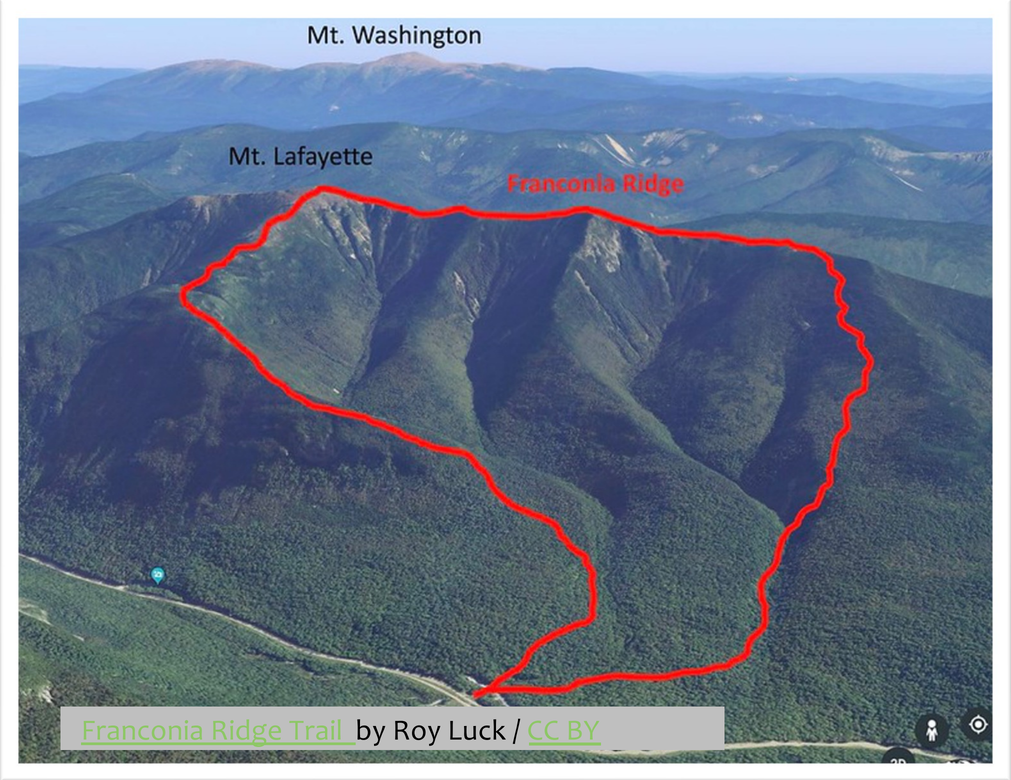
Optimization for ML

Chalkboard

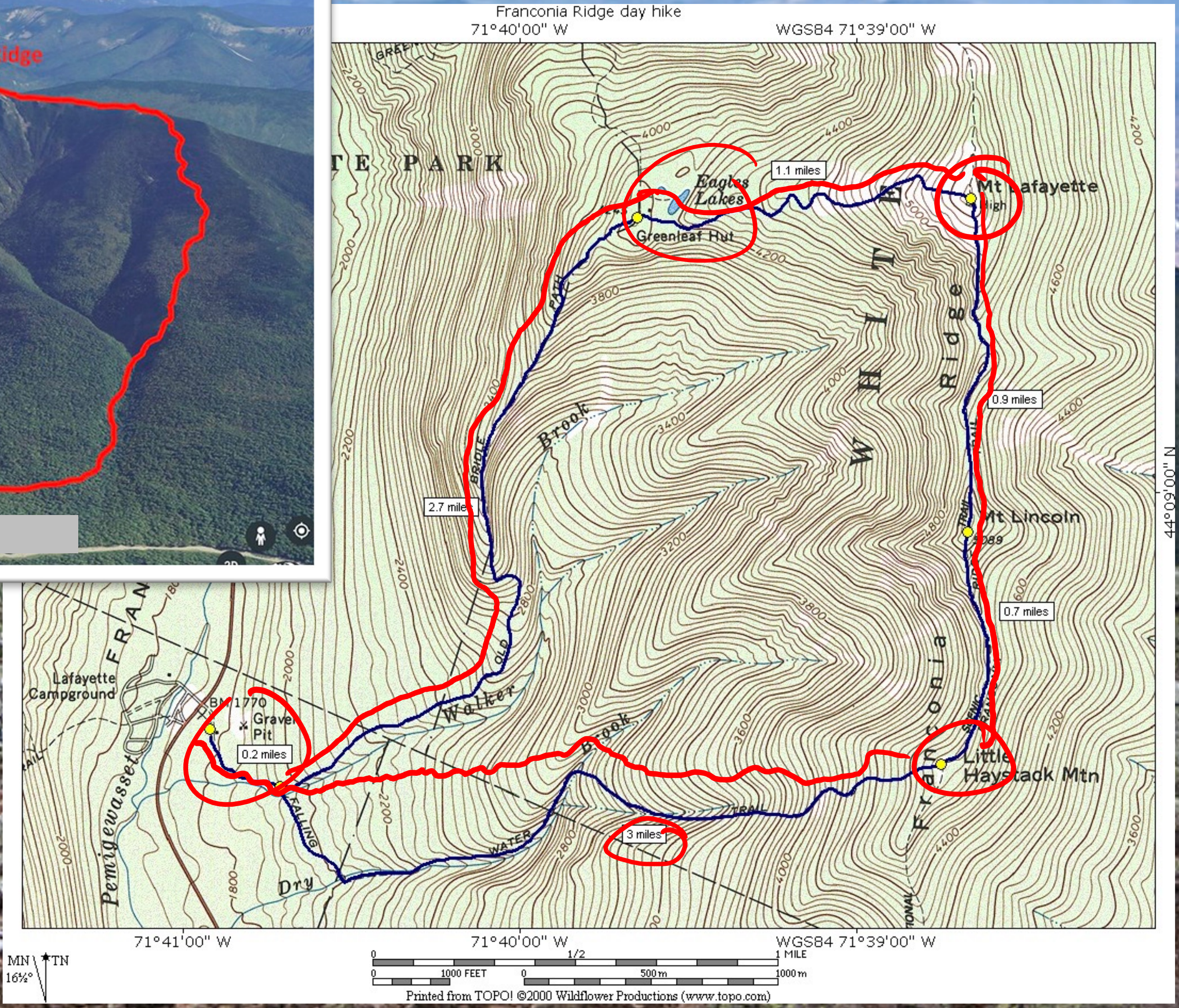
- Derivatives
- Gradient

Topographical Maps

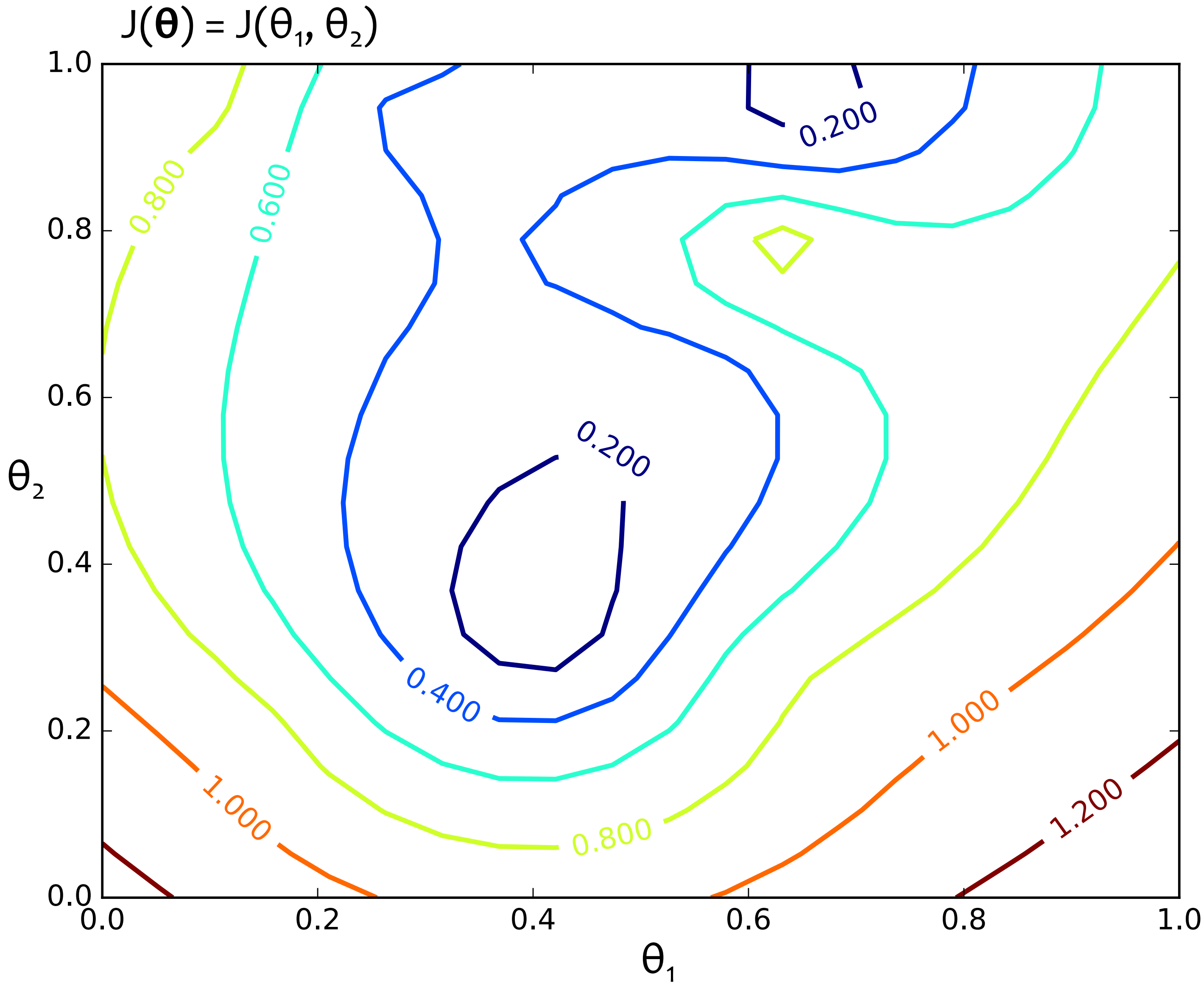
Topographical Maps



Franconia Ridge Trail by Roy Luck / CC BY

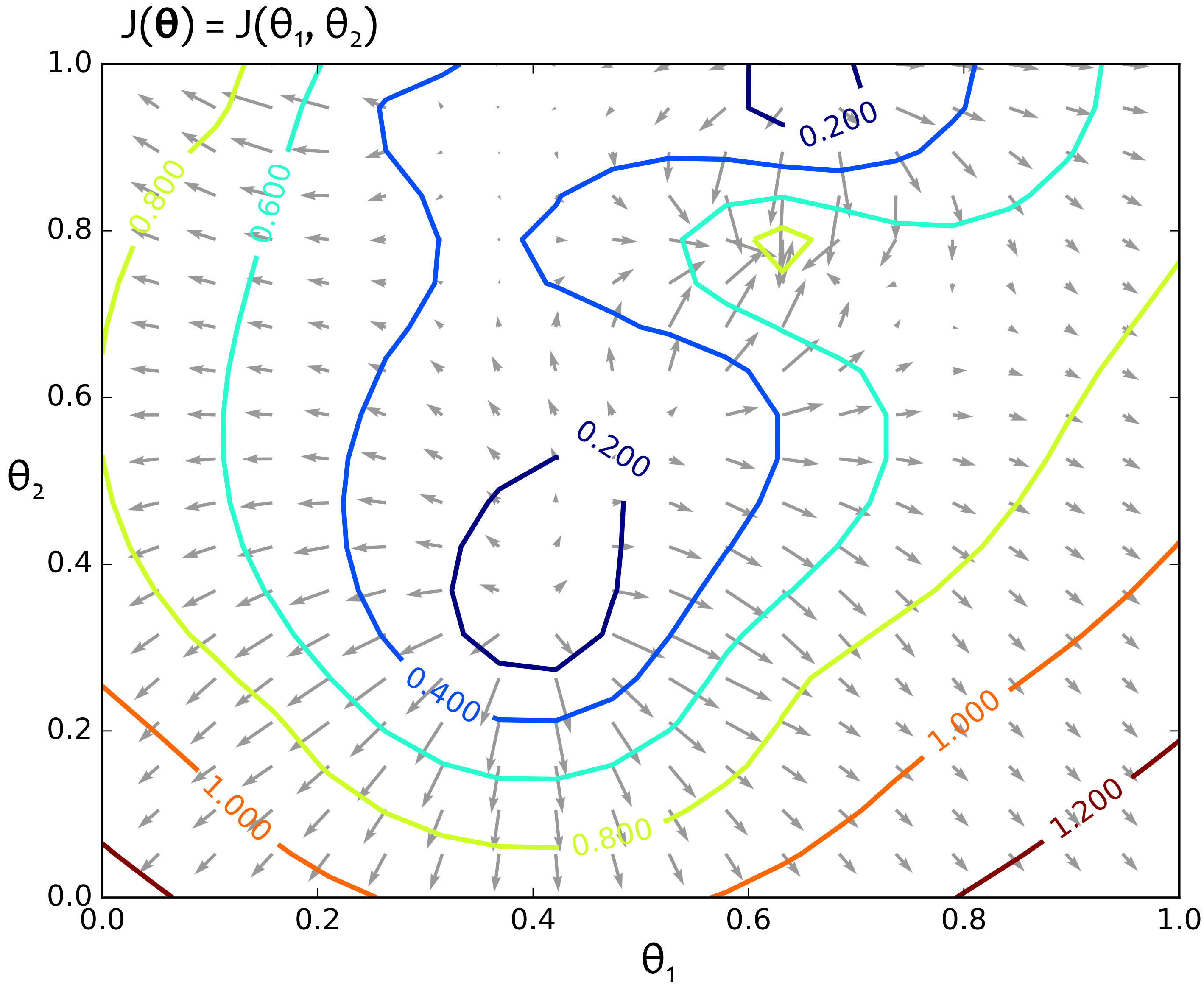


Gradients



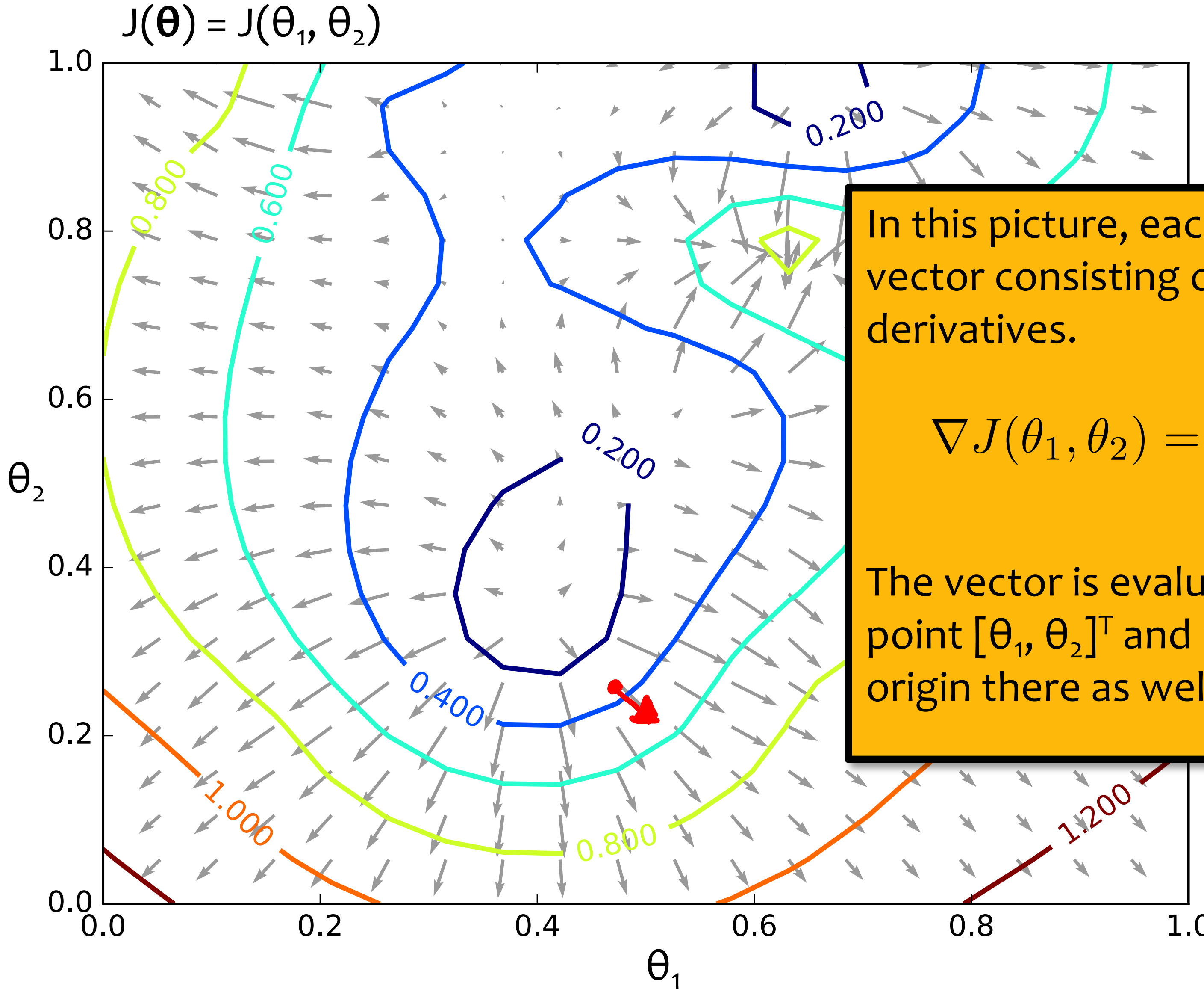
$J(\boldsymbol{\theta}) = J(\theta_1, \theta_2)$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
θ_1
θ_2
0.200
0.200
0.400
0.600
0.800
0.800
1.000
1.000
1.200

Gradients



These are the **gradients** that Gradient **Ascent** would follow.

Gradients



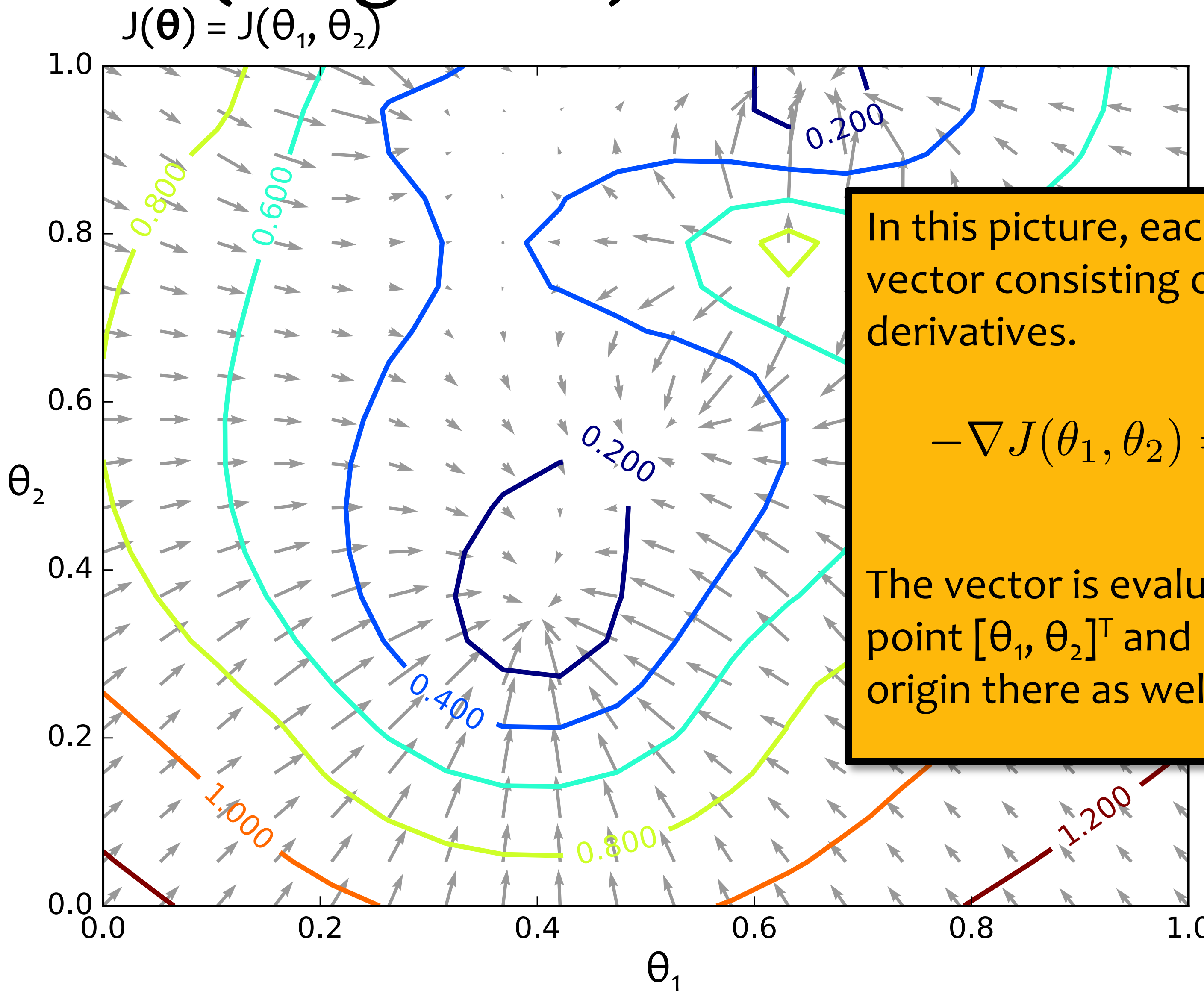
In this picture, each arrow is a 2D vector consisting of two partial derivatives.

$$\nabla J(\theta_1, \theta_2) = \begin{bmatrix} \frac{\partial J}{\partial \theta_1} \\ \frac{\partial J}{\partial \theta_2} \end{bmatrix}$$

The vector is evaluated at the point $[\theta_1, \theta_2]^T$ and plotted with its origin there as well.

These are the **gradients** that Gradient **Ascent** would follow.

(Negative) Gradients



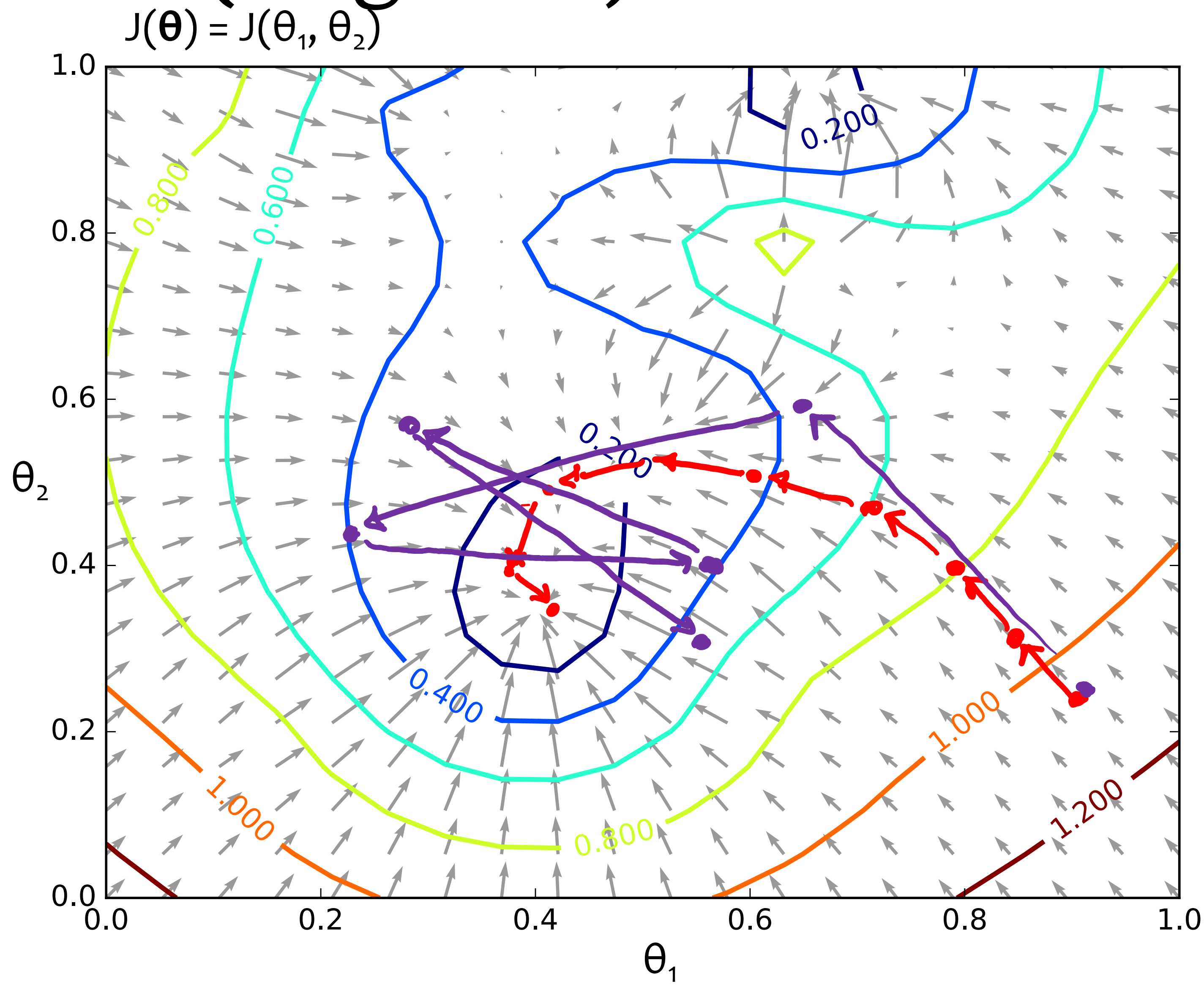
In this picture, each arrow is a 2D vector consisting of two partial derivatives.

$$-\nabla J(\theta_1, \theta_2) = \begin{bmatrix} -\frac{\partial J}{\partial \theta_1} \\ -\frac{\partial J}{\partial \theta_2} \end{bmatrix}$$

The vector is evaluated at the point $[\theta_1, \theta_2]^T$ and plotted with its origin there as well.

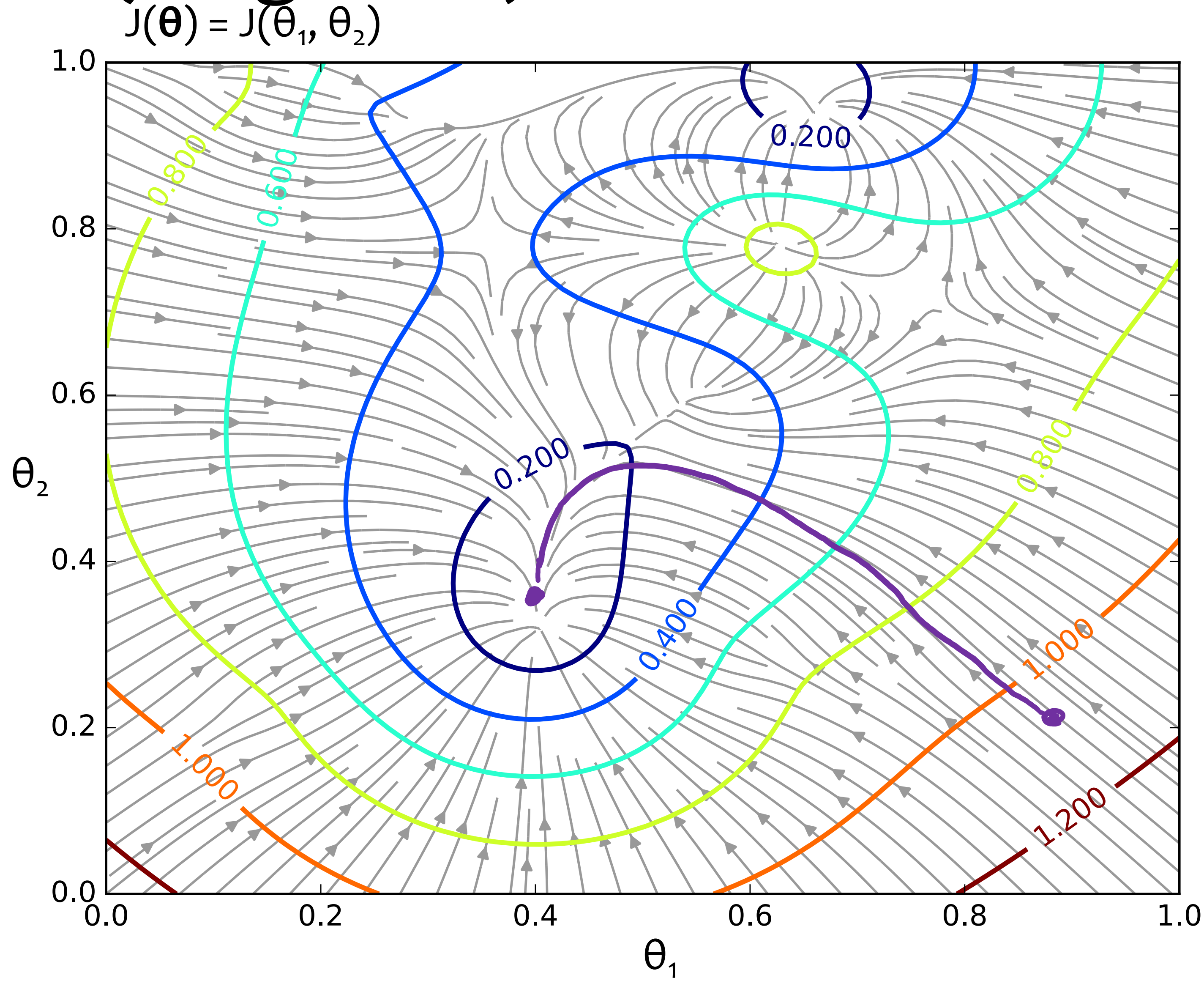
These are the **negative** gradients that Gradient **Descent** would follow.

(Negative) Gradients



These are the **negative** gradients that Gradient **Descent** would follow.

(Negative) Gradient *Paths*



Shown are the **paths** that Gradient Descent would follow if it were making **infinitesimally small steps.**

Gradient Descent

Chalkboard

- Gradient Descent Algorithm
- Details: starting point, stopping criterion, line search

Gradient Descent

Algorithm 1 Gradient Descent

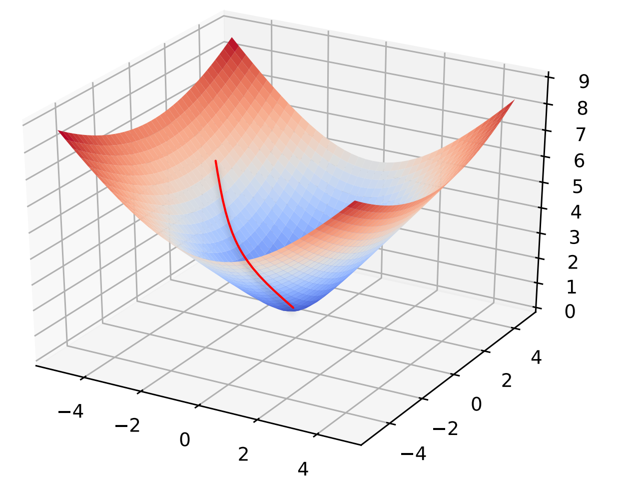
1: **procedure** GD($\mathcal{D}$, $\boldsymbol{\theta}^{(0)}$)

2: $\quad \boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$

3: $\quad$ **while** not converged **do**

4: $\quad\quad \boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \gamma \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$

5: $\quad$ **return** $\boldsymbol{\theta}$



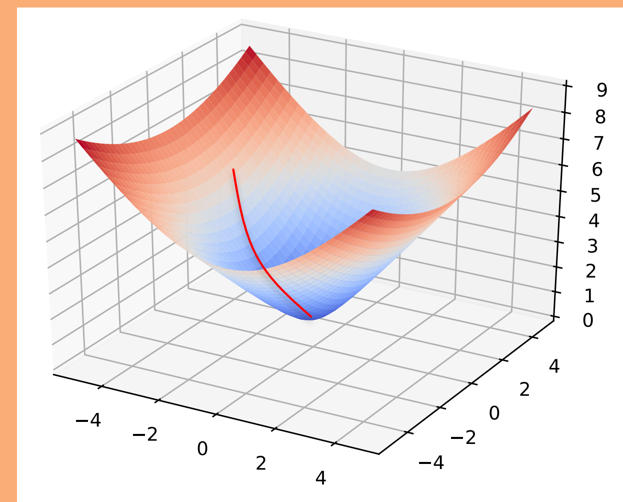
In order to apply GD to Linear Regression all we need is the **gradient** of the objective function (i.e. vector of partial derivatives).

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \begin{bmatrix} \frac{d}{d\theta_1} J(\boldsymbol{\theta}) \\ \frac{d}{d\theta_2} J(\boldsymbol{\theta}) \\ \vdots \\ \frac{d}{d\theta_M} J(\boldsymbol{\theta}) \end{bmatrix}$$

Gradient Descent

Algorithm 1 Gradient Descent

procedure GD($\mathcal{D}$, $\boldsymbol{\theta}^{(0)}$)
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$
    while not converged **do**
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \gamma \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$
    return $\boldsymbol{\theta}$



There are many possible ways to detect **convergence.** For example, we could check whether the L2 norm of the gradient is below some small tolerance.

$$||\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})||_2 \leq \epsilon$$

Alternatively we could check that the reduction in the objective function from one iteration to the next is small.

GRADIENT DESCENT FOR LINEAR REGRESSION

Linear Regression as Function Approximation

$\mathcal{D} = \{\mathbf{x}^{(i)}, y^{(i)}\}_{i=1}^{N}$
where $\mathbf{x} \in \mathbb{R}^M$ and $y \in \mathbb{R}$

1. Assume $\mathcal{D}$ generated as:

$$\mathbf{x}^{(i)} \sim p^*(\cdot)$$
$$y^{(i)} = h^*(\mathbf{x}^{(i)})$$

2. Choose hypothesis space, $\mathcal{H}$:
   all linear functions in M-dimensional space

$$\mathcal{H} = \{h_{\boldsymbol{\theta}} : h_{\boldsymbol{\theta}}(\mathbf{x}) = \boldsymbol{\theta}^T \mathbf{x}, \boldsymbol{\theta} \in \mathbb{R}^M\}$$

3. Choose an objective function:
   mean squared error (MSE)

$$J(\boldsymbol{\theta}) = \frac{1}{N} \sum_{i=1}^{N} e_i^2$$
$$= \frac{1}{N} \sum_{i=1}^{N} \left(y^{(i)} - h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)})\right)^2$$
$$= \frac{1}{N} \sum_{i=1}^{N} \left(y^{(i)} - \boldsymbol{\theta}^T \mathbf{x}^{(i)}\right)^2$$

4. Solve the unconstrained optimization problem via favorite method:
   - *gradient descent*
   - *closed form*
   - *stochastic gradient descent*
   - ...

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\text{argmin}}\, J(\boldsymbol{\theta})$$

5. Test time: given a new $\mathbf{x}$, make prediction $\hat{y}$

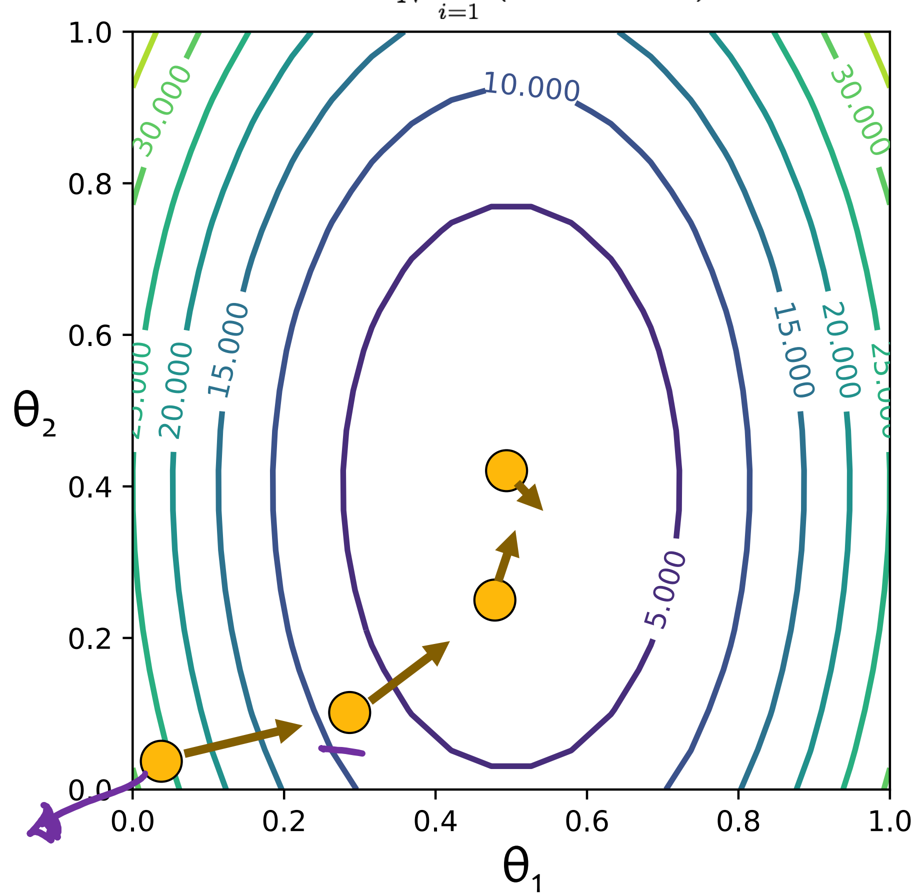
$$\hat{y} = h_{\hat{\boldsymbol{\theta}}}(\mathbf{x}) = \hat{\boldsymbol{\theta}}^T \mathbf{x}$$

Linear Regression by Gradient Desc.

Optimization Method #1: Gradient Descent

1. Pick a random $\boldsymbol{\theta}$
2. Repeat:
   a. Evaluate gradient $\nabla J(\boldsymbol{\theta})$
   b. Step opposite gradient
3. Return $\boldsymbol{\theta}$ that gives smallest $J(\boldsymbol{\theta})$

$$J(\boldsymbol{\theta}) = J(\theta_1, \theta_2) = \frac{1}{N}\sum_{i=1}^{N}\left(y^{(i)} - \boldsymbol{\theta}^T\mathbf{x}^{(i)}\right)^2$$



t	θ_1	θ_2	$J(\theta_1, \theta_2)$
1	0.01	0.02	25.2
2	0.30	0.12	8.7
3	0.51	0.30	1.5
4	0.59	0.43	0.2

Linear Regression by Gradient Desc.

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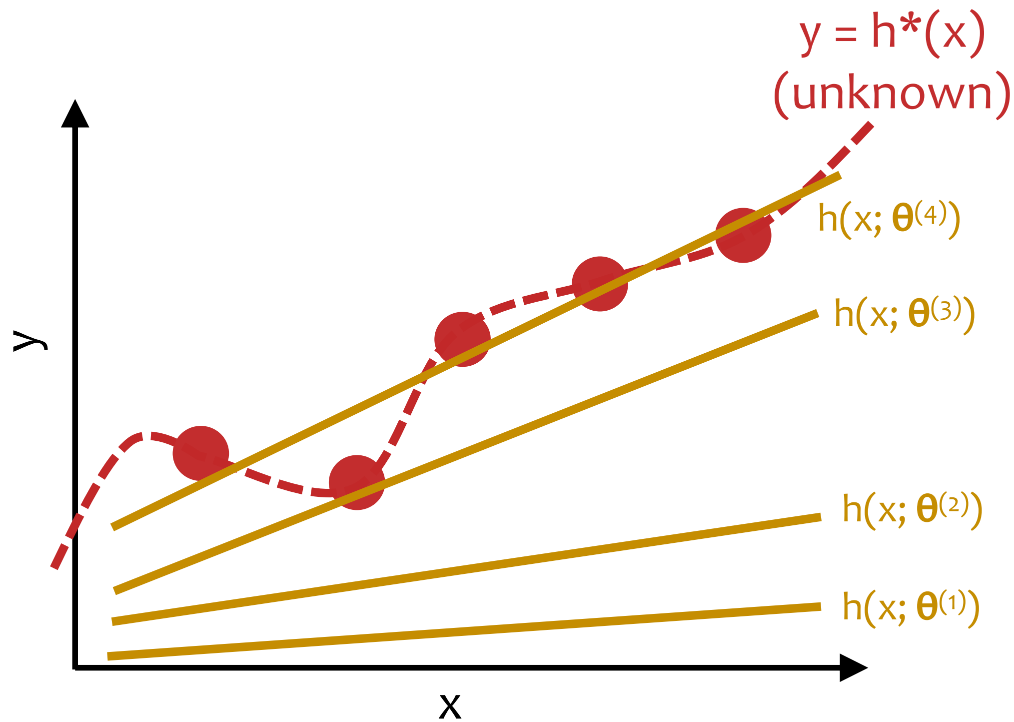

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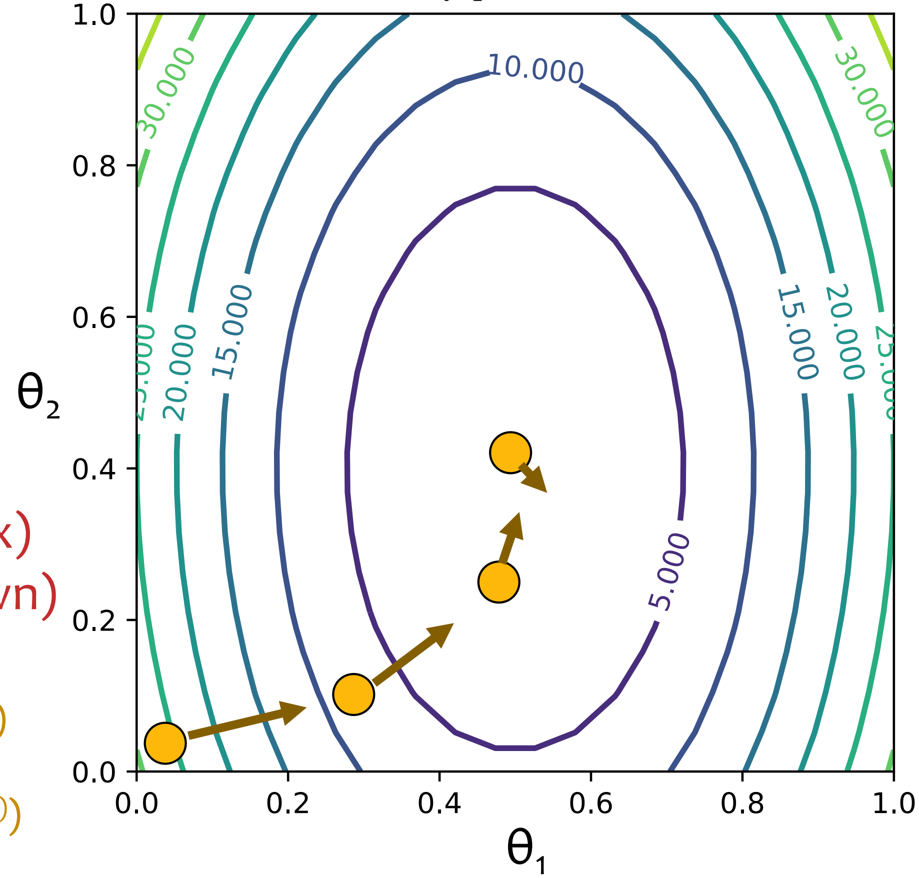

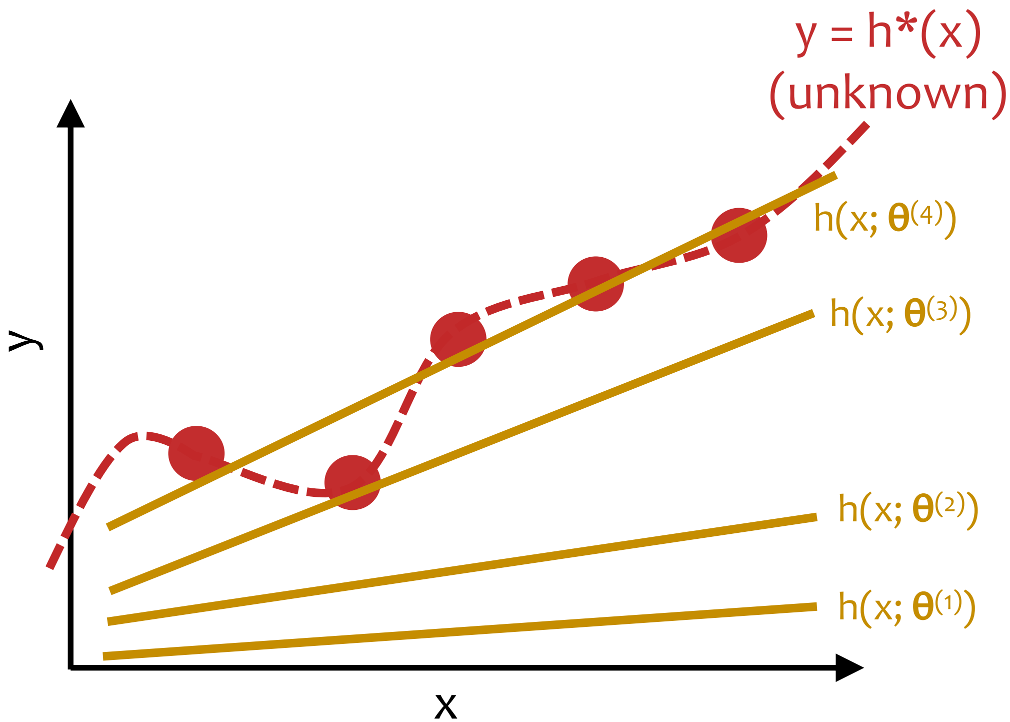

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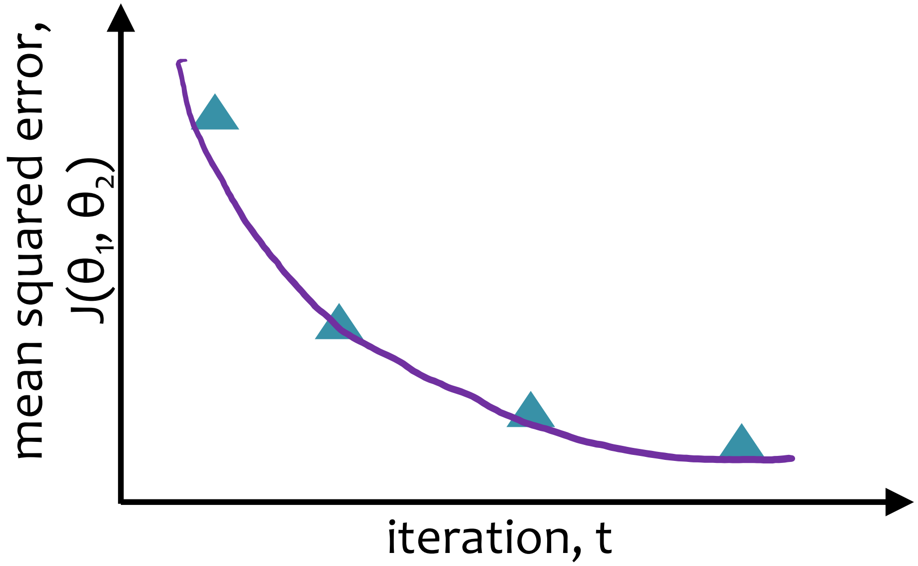

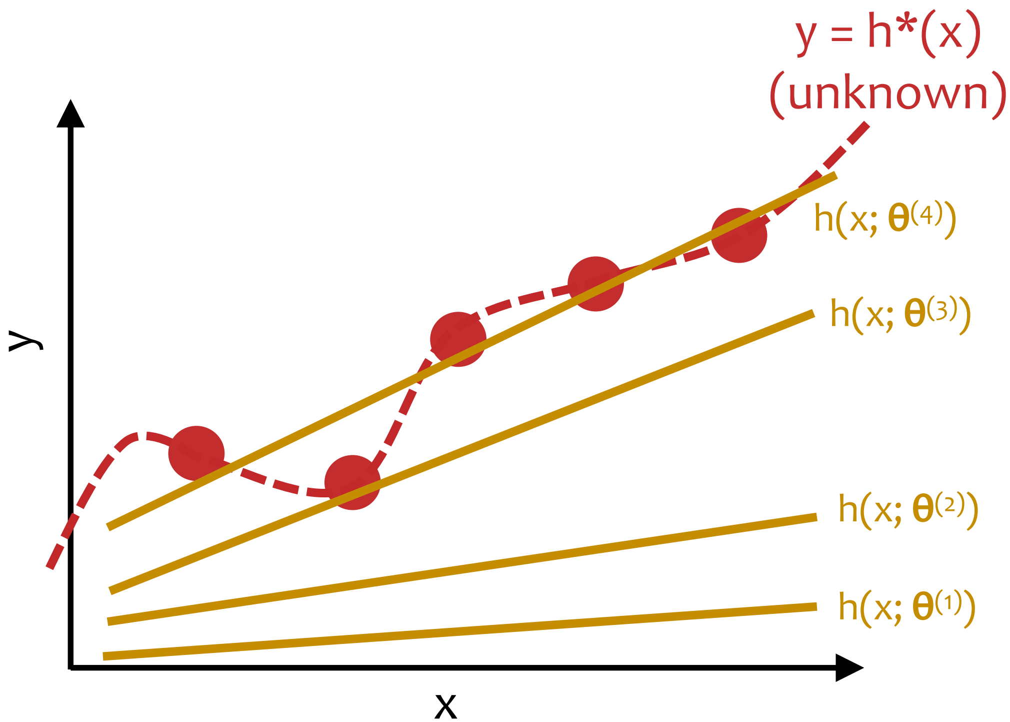

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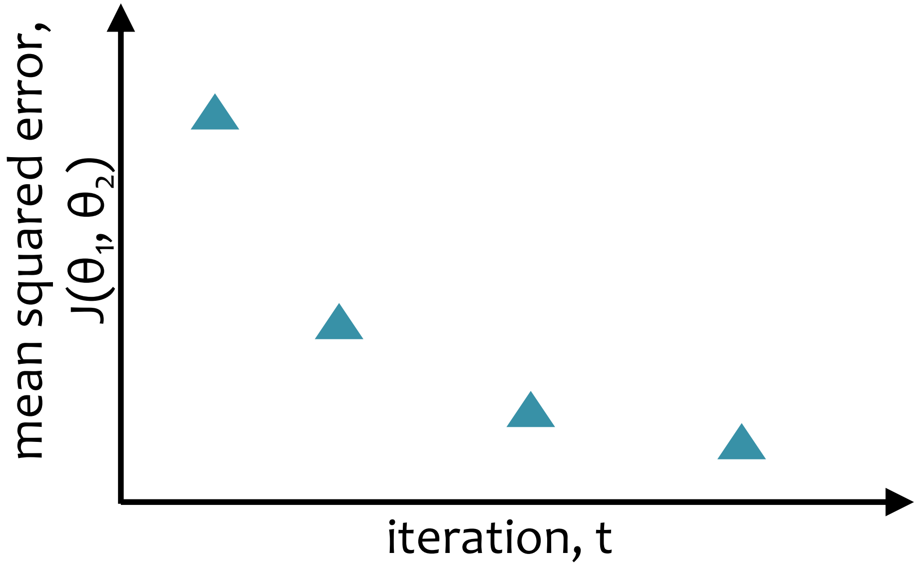

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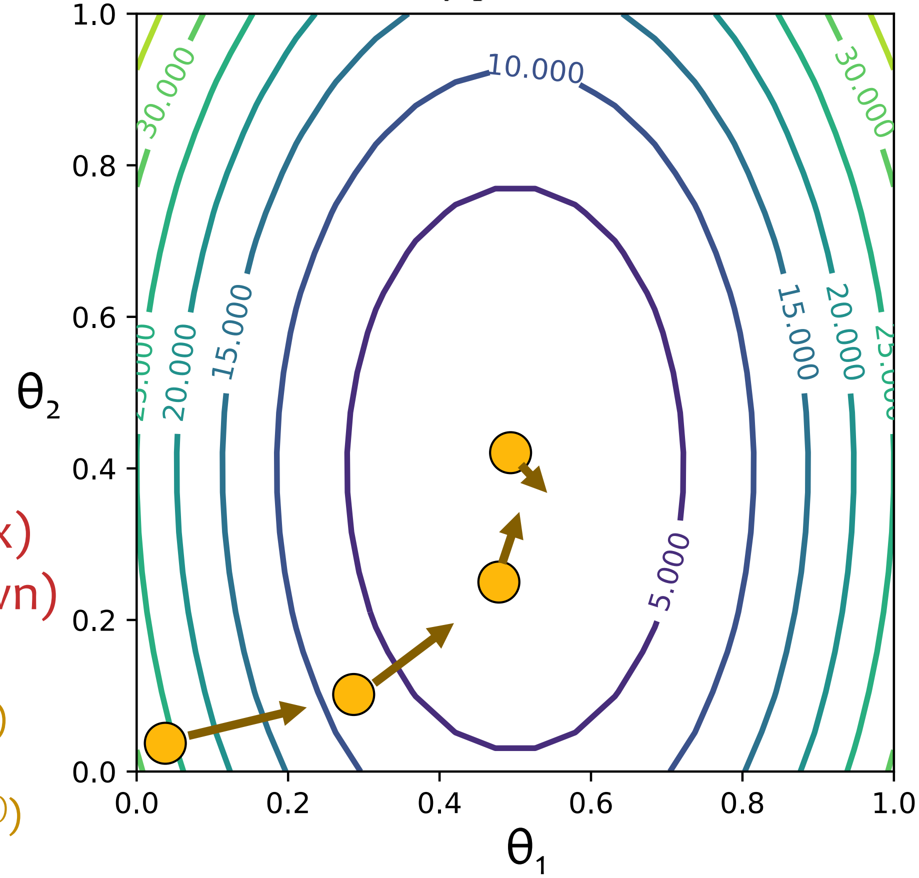

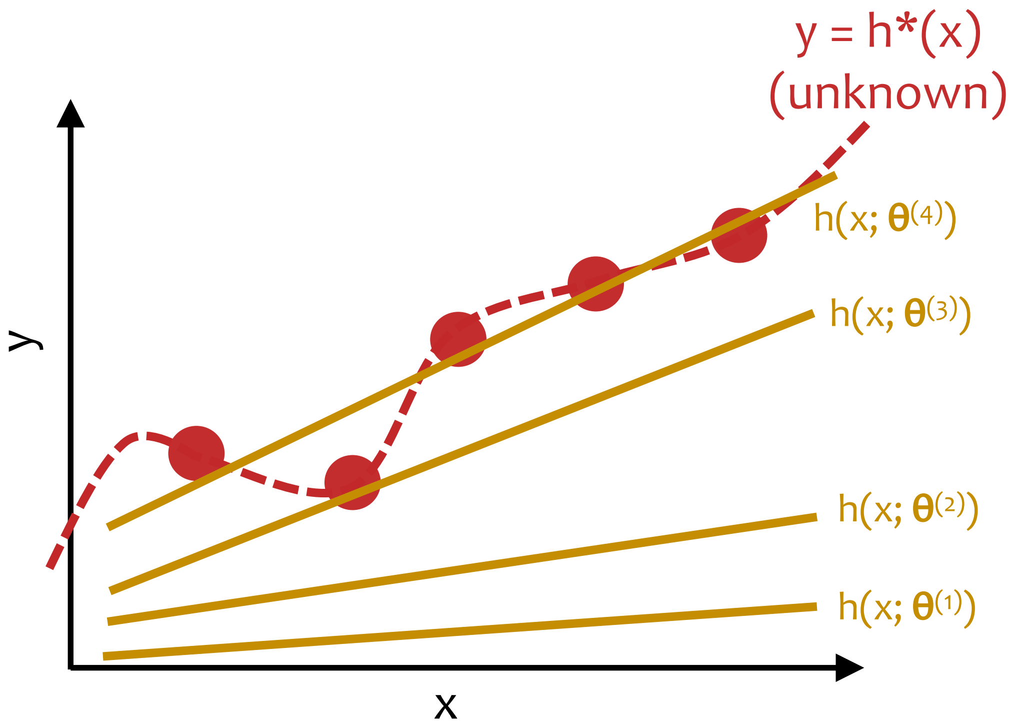

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Optimization for Linear Regression

Chalkboard

- – Computing the gradient for Linear Regression
- – Gradient Descent for Linear Regression

Gradient Calculation for Linear Regression

Derivative of $J^{(i)}(\boldsymbol{\theta})$:

$$\begin{aligned}
\frac{d}{d\theta_k} J^{(i)}(\boldsymbol{\theta}) &= \frac{d}{d\theta_k} \frac{1}{2} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)})^2 \\
&= \frac{1}{2} \frac{d}{d\theta_k} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)})^2 \\
&= (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \frac{d}{d\theta_k} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \\
&= (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \frac{d}{d\theta_k} \left(\sum_{j=1}^{K} \theta_j x_j^{(i)} - y^{(i)} \right) \\
&= (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_k^{(i)}
\end{aligned}$$

Derivative of $J(\boldsymbol{\theta})$:

$$\begin{aligned}
\frac{d}{d\theta_k} J(\boldsymbol{\theta}) &= \sum_{i=1}^{N} \frac{d}{d\theta_k} J^{(i)}(\boldsymbol{\theta}) \\
&= \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_k^{(i)}
\end{aligned}$$

Gradient of $J(\boldsymbol{\theta})$ [used by Gradient Descent]

$$\begin{aligned}
\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) &= \begin{bmatrix} \frac{d}{d\theta_1} J(\boldsymbol{\theta}) \\ \frac{d}{d\theta_2} J(\boldsymbol{\theta}) \\ \vdots \\ \frac{d}{d\theta_M} J(\boldsymbol{\theta}) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_1^{(i)} \\ \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_2^{(i)} \\ \vdots \\ \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_N^{(i)} \end{bmatrix} \\
&= \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \mathbf{x}^{(i)}
\end{aligned}$$

GD for Linear Regression

Gradient Descent for Linear Regression repeatedly takes steps opposite the gradient of the objective function

Algorithm 1 GD for Linear Regression

1: **procedure** GDLR($\mathcal{D}$, $\boldsymbol{\theta}^{(0)}$)
2: $\quad\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$ ▷ Initialize parameters
3: $\quad$**while** not converged **do**
4: $\quad\quad\mathbf{g} \leftarrow \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \mathbf{x}^{(i)}$ ▷ Compute gradient
5: $\quad\quad\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \gamma \mathbf{g}$ ▷ Update parameters
6: $\quad$**return** $\boldsymbol{\theta}$

Regression Loss Functions

In-Class Exercise:

Which of the following could be used as loss functions for training a linear regression model?

Select all that apply.

A. $\ell(\hat{y}, y) = (\hat{y})^2$

B. $\ell(\hat{y}, y) = |\hat{y} - y|$

C. $\ell(\hat{y}, y) = \frac{1}{2}(\hat{y} - y)^2$

D. $\ell(\hat{y}, y) = \frac{1}{4}(\hat{y} - y)^4$

E. $\ell(\hat{y}, y) = \begin{cases} \frac{1}{2}(\hat{y} - y)^2 & \text{if } |\hat{y} - y| \leq \delta \\ \delta|\hat{y} - y| - \frac{1}{2}\delta^2 & \text{otherwise} \end{cases}$

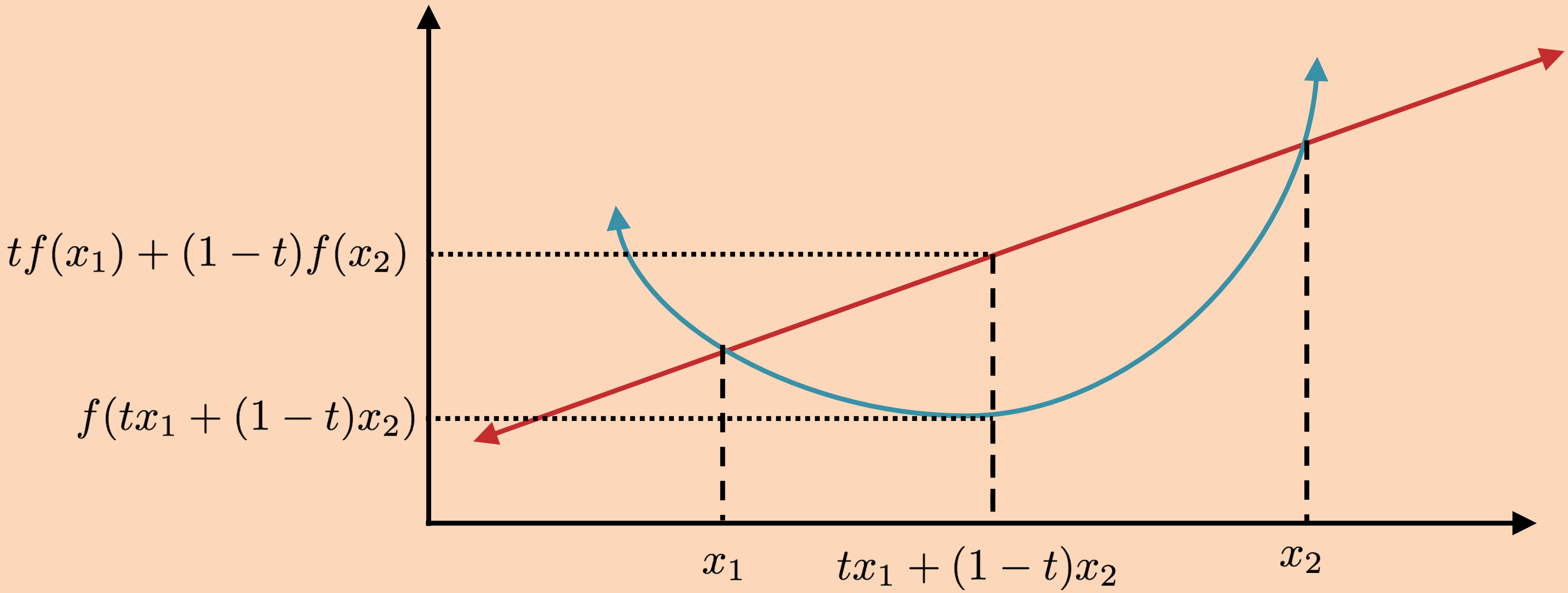
F. $\ell(\hat{y}, y) = \log(\cosh(\hat{y} - y))$

CONVEXITY

Convexity

Function $f : \mathbb{R}^M \rightarrow \mathbb{R}$ is **convex**
if $\forall\ \mathbf{x}_1 \in \mathbb{R}^M, \mathbf{x}_2 \in \mathbb{R}^M, 0 \leq t \leq 1$:

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \leq tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2)$$

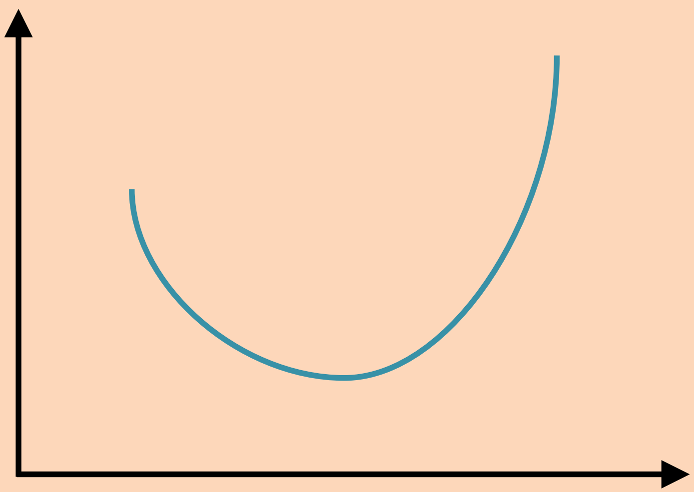


Convexity

Suppose we have a function $f(x) : \mathcal{X} \rightarrow \mathcal{Y}$.

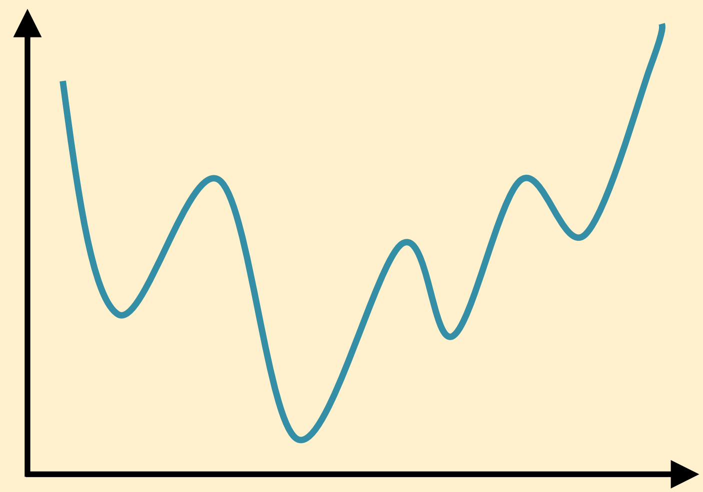
- The value x^* is a **global minimum** of f iff $f(x^*) \leq f(x), \forall x \in \mathcal{X}$.
- The value x^* is a **local minimum** of f iff $\exists \epsilon$ s.t. $f(x^*) \leq f(x), \forall x \in [x^* - \epsilon, x^* + \epsilon]$.

Convex Function



- Each **local minimum** is a **global minimum**

Nonconvex Function



- A *nonconvex* function is **not convex**
- Each **local minimum** is **not** necessarily a **global minimum**

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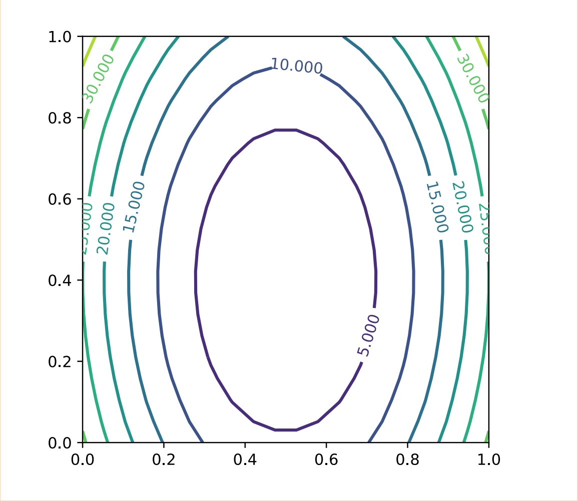
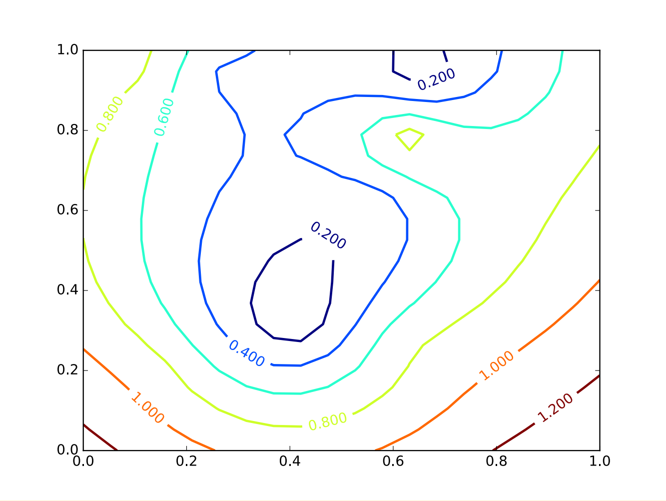
Convex Function



- Each **local minimum** is a **global minimum**

Nonconvex Function



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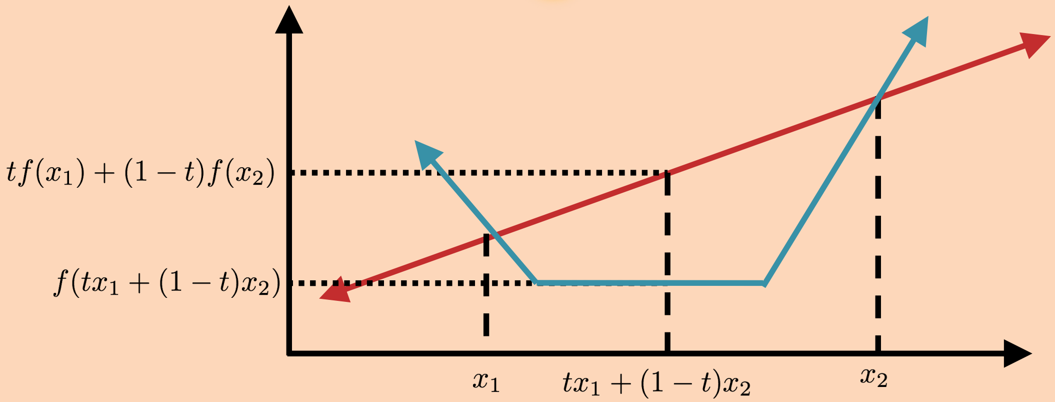
- Each **local minimum** is **not** necessarily a **global minimum**

Convexity

Function $f: \mathbb{R}^M \rightarrow \mathbb{R}$ is **convex**
if $\forall\ \mathbf{x}_1 \in \mathbb{R}^M, \mathbf{x}_2 \in \mathbb{R}^M, 0 \leq t \leq 1$:

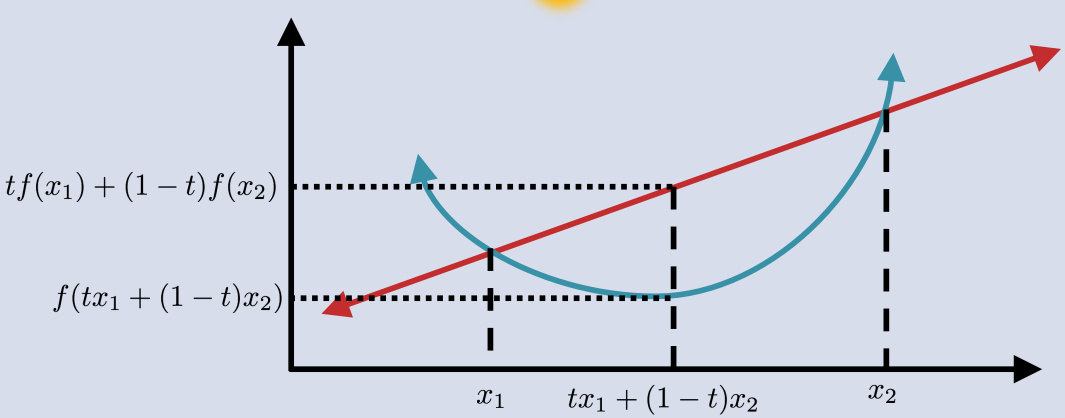
$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \leq t f(\mathbf{x}_1) + (1-t) f(\mathbf{x}_2)$$



Each **local minimum** of a **convex** function is also a **global minimum.**

Function $f: \mathbb{R}^M \rightarrow \mathbb{R}$ is **strictly convex**
if $\forall\ \mathbf{x}_1 \in \mathbb{R}^M, \mathbf{x}_2 \in \mathbb{R}^M, 0 \leq t \leq 1$:

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) < t f(\mathbf{x}_1) + (1-t) f(\mathbf{x}_2)$$



A **strictly convex** function has a **unique global minimum.**

CONVEXITY AND LINEAR REGRESSION

Convexity and Linear Regression

The **Mean Squared Error** function, which we minimize for learning the parameters of Linear Regression, **is convex**!

…but in the general case it is **not strictly convex.**

Gradient Descent & Convexity

- Gradient descent is a **local optimization algorithm**
- If the function is **nonconvex**, it will find a local minimum, not necessarily a global minimum
- If the function is **convex**, it will find a global minimum