1 Probability and Statistics

You should be familiar with event notations for probabilities, i.e. \( P(A \cup B) \) and \( P(A \cap B) \), where \( A \) and \( B \) are binary events.

In this class, however, we will mainly be dealing with random variable notations, where \( A \) and \( B \) are random variables that can take on different states, i.e. \( a_1, a_2, \) and \( b_1, b_2 \), respectively. Below are some notation equivalents, as well as basic probability rules to keep in mind.

- \( P(A = a_1 \cap B = b_1) = P(A = a_1, B = b_1) = p(a_1, b_1) \)
- \( P(A = a_1 \cup B = b_1) = \sum_{b \in B} p(a_1, b) + \sum_{a \in A} p(a, b_1) - p(a_1, b_1) \)
- \( p(a_1 \mid b_1) = \frac{p(a_1, b_1)}{p(b_1)} \)
- \( p(a_1) = \sum_{b \in B} p(a_1, b) \)

1. Two random variables, \( A \) and \( B \), each can take on two values, \( a_1, a_2, \) and \( b_1, b_2, \) respectively. \( a_1 \) and \( b_2 \) are considered disjoint (mutually exclusive). \( P(A = a_1) = 0.5, P(B = b_2) = 0.5. \)
   - What is \( p(a_1, b_2) \) ?
   - What is \( p(a_1, b_1) \) ?
   - What is \( p(a_1 \mid b_2) \) ?

2. Now, instead, \( a_1 \) and \( b_2 \) are not disjoint, but the two random variables \( A \) and \( B \) are independent.
   - What is \( p(a_1, b_2) \) ?
   - What is \( p(a_1, b_1) \) ?
   - What is \( p(a_1 \mid b_2) \) ?

3. A student is looking at her activity tracker (Fitbit/Apple Watch) data and she notices that she seems to sleep better on days that she exercises. They observe the following:
Exercise Good Sleep Probability
yes yes 0.3
yes no 0.2
no no 0.4
no yes 0.1

• What is the $P(\text{GoodSleep} = \text{yes} \mid \text{Exercise} = \text{yes})$?

• Why doesn’t $P(\text{GoodSleep} = \text{yes}, \text{Exercise} = \text{yes}) = P(\text{GoodSleep} = \text{yes}) \cdot P(\text{Exercise} = \text{yes})$?

• The student merges her activity tracker data with her food logs and finds that the $P(\text{Eatwell} = \text{yes} \mid \text{Exercise} = \text{yes}, \text{GoodSleep} = \text{yes})$ is 0.25. What is the probability of all three happening on the same day?

4. What is the expectation of $X$ where $X$ is a single roll of a fair 6-sided dice ($S = \{1, 2, 3, 4, 5, 6\}$)? What is the variance of $X$?

5. Imagine that we had a new dice where the sides were $S = \{3, 4, 5, 6, 7, 8\}$. How do the expectation and the variance compare to our original dice?

2 Calculus

1. If $f(x) = x^3e^x$, find $f'(x)$.

2. If $f(x) = e^x$, $g(x) = 4x^2 + 2$, find $h'(x)$, where $h(x) = f(g(x))$.

3. If $f(x, y) = y \log(1 - x) + (1 - y) \log(x)$, $x \in (0, 1)$, evaluate $\frac{\partial f(x,y)}{\partial x}$ at the point $(\frac{1}{2}, \frac{1}{2})$.

4. Find $\frac{\partial}{\partial w_j} x^T w$, where $x$ and $w$ are $M$-dimensional real-valued vectors and $1 \leq j \leq M$. 
3. **Vectors, Matrices, and Geometry**

1. **Inner Product**: \( \mathbf{u} = [6 \ 1 \ 2]^T, \mathbf{v} = [3 \ -10 \ -2]^T \), what is the inner product of \( \mathbf{u} \) and \( \mathbf{v} \)? What is the geometric interpretation?

2. **Cauchy-Schwarz inequality** (Optional): Given \( \mathbf{u} = [3 \ 1 \ 2]^T, \mathbf{v} = [3 \ -1 \ 4]^T \), what is \( ||\mathbf{u}||^2 \) and \( ||\mathbf{v}||^2 \)? What is \( \mathbf{u} \cdot \mathbf{v} \)? How do \( \mathbf{u} \cdot \mathbf{v} \) and \( ||\mathbf{u}||^2 ||\mathbf{v}||^2 \) compare? Is this always true?

3. **Matrix algebra.** Generally, if \( \mathbf{A} \in \mathbb{R}^{M \times N} \) and \( \mathbf{B} \in \mathbb{R}^{N \times P} \), then \( \mathbf{AB} \in \mathbb{R}^{M \times P} \) and \( (\mathbf{AB})_{ij} = \sum_k A_{ik} B_{kj} \).

   Given \( \mathbf{A} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 1 & -1 \\ 3 & -2 & 2 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \).

   - What is \( \mathbf{AB} \)? Does \( \mathbf{BA} = \mathbf{AB} \)? What is \( \mathbf{Bu} \)?
   - What is rank of \( \mathbf{A} \)?
   - What is \( \mathbf{A}^T \)?
   - Calculate \( \mathbf{uv}^T \).
   - What are the eigenvalues of \( \mathbf{A} \)?

4. **Geometry**: Given a line \( 2x + y = 2 \) in the two-dimensional plane,

   - If a given point \((\alpha, \beta)\) satisfies \(2\alpha + \beta > 2\), where does it lie relative to the line?
   - What is the relationship of vector \( \mathbf{v} = [2, 1]^T \) to this line?
   - What is the distance from origin to this line?
4 CS Fundamentals

1. For each \((f, g)\) functions below, is \(f(n) \in \mathcal{O}(g(n))\) or \(g(n) \in \mathcal{O}(f(n))\) or both?
   - \(f(n) = \log_2(n), g(n) = \log_3(n)\)
   - \(f(n) = 2^n, g(n) = 3^n\)
   - \(f(n) = \frac{n}{50}, g(n) = \log_{10}(n)\)

2. Find the DFS traversal and BFS traversal of the following binary tree. What are the time complexities of the traversals?