Mean Field

Monday, October 31, 2022 10:24 AM

ELBO as a Lower Bound

(setting $p(z|x) = p(x,z)/p(x)$)

Theorem: for any q, $\log p(x) \geq \text{ELBO}(q)$
i.e. $\text{ELBO}(q)$ is a lower bound on $\log p(x)$

Proof #1:

Recall Jensen's Inequality: $f(E[x]) \geq E[f(x)]$, for concave f

$$\log p(x) = \log \int_z p(x,z)\, dz \quad \text{(marginal)}$$

$$= \log \int_z p(x,z) \frac{q(z)}{q(z)}\, dz \quad \text{(mult. by 1)}$$

$$= \log E_{q(z)}\left[p(x,z)/q(z) \right] \quad \text{(def. of expectation)}$$

$$\geq E_{q(z)}\left[\log\left(\frac{p(x,z)}{q(z)}\right)\right] \quad \text{(by Jensen's Ineq.)}$$

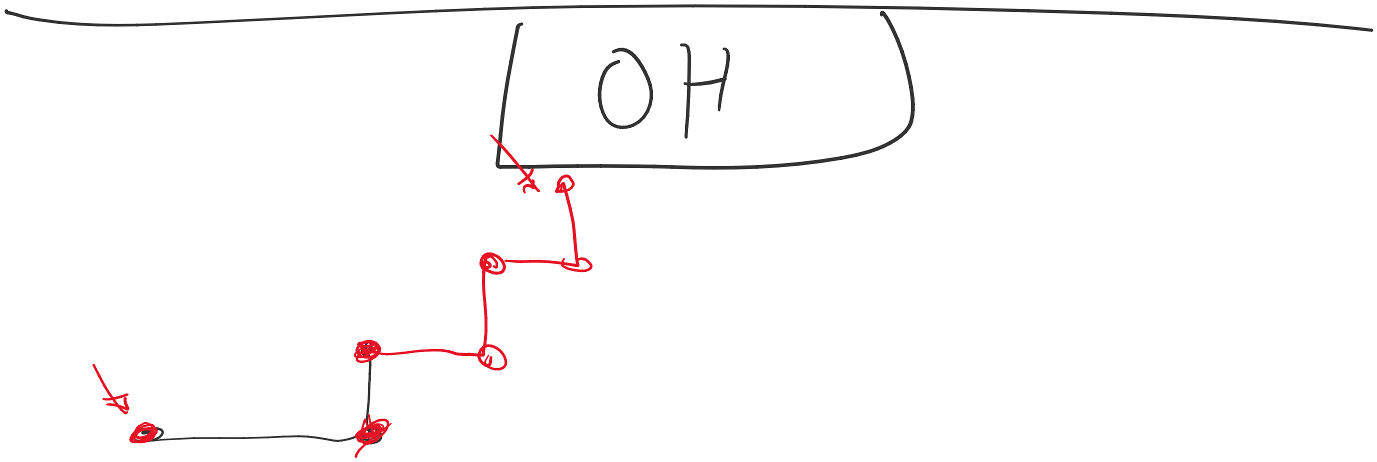
$$= E_{q(z)}[\log p(x,z)] - E_{q(z)}[\log q(z)] = \text{ELBO}(q)$$

$$\Rightarrow \log p(x) \geq \text{ELBO}(q)$$

Proof #2:

① $\log p(x) = KL(q\|p) + \text{ELBO}(q)$
② $KL(q\|p) \geq 0$ (without proof)
③ $\Rightarrow \log p(x) \geq \text{ELBO}(q)$

Takeaway: minimizing KL is the same as finding a tight $\text{ELBO}(q)$ lower bound for $\log p(x)$



$$A_{t,\cdot} \sim p(A_{t,\cdot} \mid \underbrace{x, y, A_{\neg t,\cdot}, B, \alpha, \beta}_{\text{data, hyperparameter}})$$