# Markov Chains + Bayesian Inference

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## Markov Chains

Q: How to define a M.C. s.t. it will yield samples from a desired dist. $p^*$?

for a DGM or UGM this is the learned distribution

A:

We need two properties:

**Property #1:** $p^*$ is the invariant distribution of the M.C.

We say that $p^*$ is invariant w.r.t. the transition matrix R (aka. Markov chain) if:

$$p^*(x) = \sum_{x'} R(x \leftarrow x') p^*(x')$$

OR

$$p^*(x) = \int_{x'} R(x \leftarrow x') p^*(x') \, dx'$$

Consider the marginal of $\vec{x}^{(t+1)}$

$$p(\vec{x}^{(t+1)}) = \sum_{\vec{x}^{(1)}} \sum_{\vec{x}^{(2)}} \cdots \sum_{\vec{x}^{(t)}} p(\vec{x}^{(t+1)}, \vec{x}^{(t)}, \dots, \vec{x}^{(1)})$$

$$= \dots$$

by induction

$$= \sum_{x^{(t)}} p(x^{(t+1)} | x^{(t)}) \, p(x^{(t)})$$

**Property #2:** $p^*$ is the equilibrium dist. of the M.C.

Given an arbitrary initial state $\vec{x}^{(1)}$

Suppose $p^*(x)$ is invariant wrt R

Then $p^*(x)$ is the equilibrium dist if:

$$\lim_{t \to \infty} p(\vec{x}^{(t)}) = p^*(\vec{x}^{(t)})$$

$$\lim_{t \to \infty} p(\vec{x}^{(t)}) = p^*(\vec{x}^{(t)})$$

AKA. the M.C. is <u>ergodic</u>

Q: How do prove that our ~~MCMC~~ MCMC algorithms have these properties?

A: Prove these instead...

## Sufficient Conditions for Prop. #1 and Prop #2

① R satisfies <u>detailed balance</u> if

$$R(x' \leftarrow x) p^*(x) = R(x \leftarrow x') p^*(x')$$

D.B. $\Rightarrow$ $p^*$ is invariant wrt R.

② If $p^*(x)$ is invariant wrt R and

$$v = \min_{x, x' \text{ s.t. } p^*(x) > 0} \frac{R(x' \leftarrow x)}{p^*(x)}$$

and

$v > 0$

then $p^*$ is the equilibrium dist for R.

Q: What does this have to do with MCMC algos?

A:

## Transition Matrix for Metropolis-Hastings as M.C.

$$R(x^{(t+1)} \leftarrow x^{(t)}) = q(x^{(t+1)} \mid x^{(t)}) \underline{\min\left(1, \frac{p(\vec{x}^{(t+1)})}{p(\vec{x}^{(t)})} \frac{q(\vec{x}^{(t)} \mid \vec{x}^{(t+1)})}{q(\vec{x}^{(t+1)} \mid \vec{x}^{(t)})}\right)}$$

Q1: fill in the blank

Exercise for HW4:
prove that M-H satisfies detailed balance.