10-418/10-618 Machine Learning for Structured Data

Machine Learning Department
School of Computer Science
Carnegie Mellon University

Belief Propagation
+
Learning fully observable MRFs and CRFs

Matt Gormley
Lecture 9
Sep. 28, 2022

Reminders

- Homework 2: Learning to Search for RNNs
  - Out: Sun, Sep 18
  - Written (except for Empirical Questions)
    - Due: Thu, Sep 29 at 11:59pm
  - Programming + Empirical Questions
    - Due: NEVER?
- Homework 3: General Graph CRF Module
  - Out: Thu, Sep 29
  - Due: Mon, Oct 10 at 11:59pm

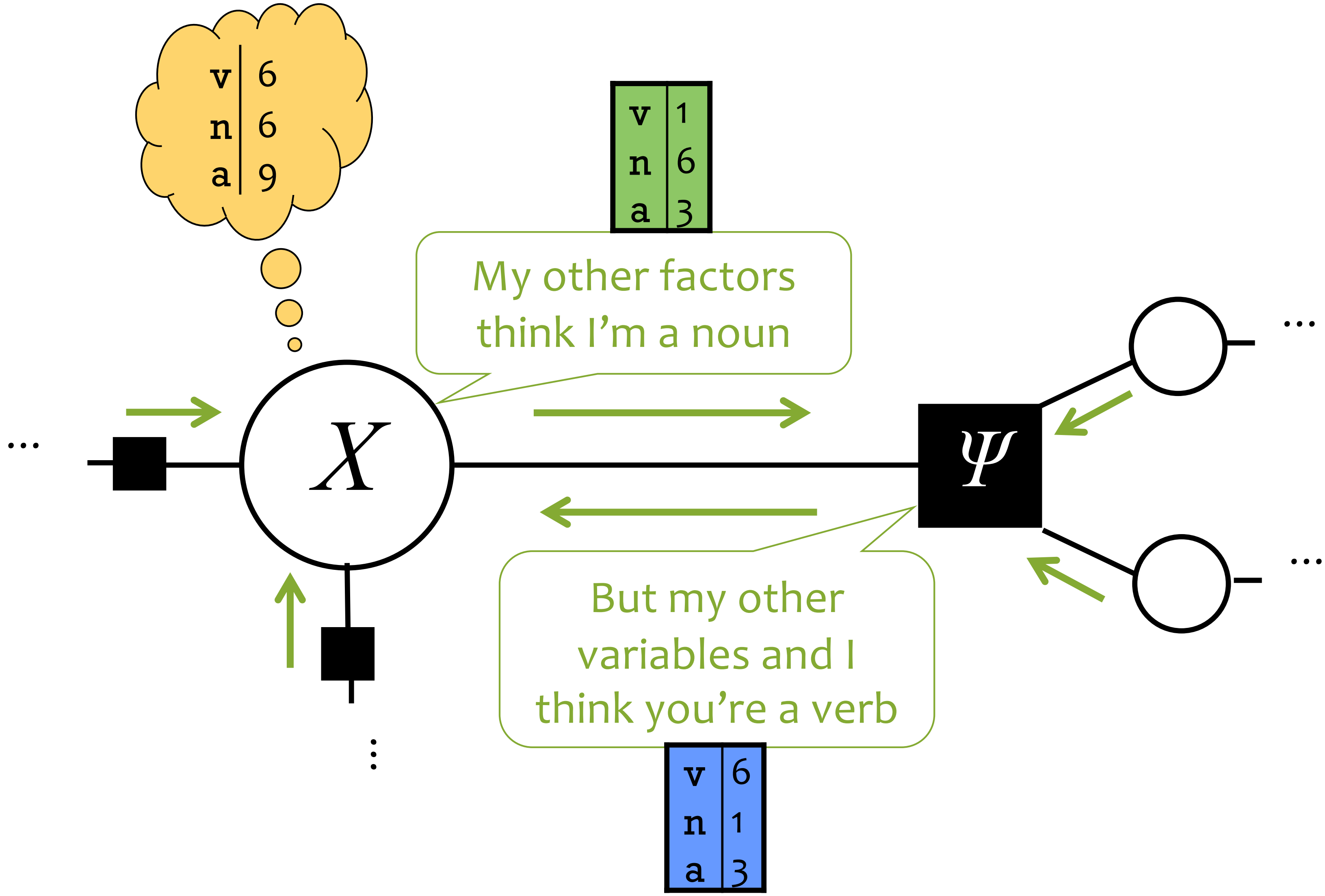
Reminders

- **Homework 2: Learning to Search for RNNs**
  - Out: Sun, Sep 18
  - Written (except for Empirical Questions)
    - Due: Thu, Sep 29 at 11:59pm
  - Programming + Empirical Questions
    - Due: Mon, Oct 24 at 9:00am
- **Homework 3: General Graph CRF Module**
  - Out: Thu, Sep 29
  - Due: Mon, Oct 10 at 11:59pm

Exact marginal inference for factor trees

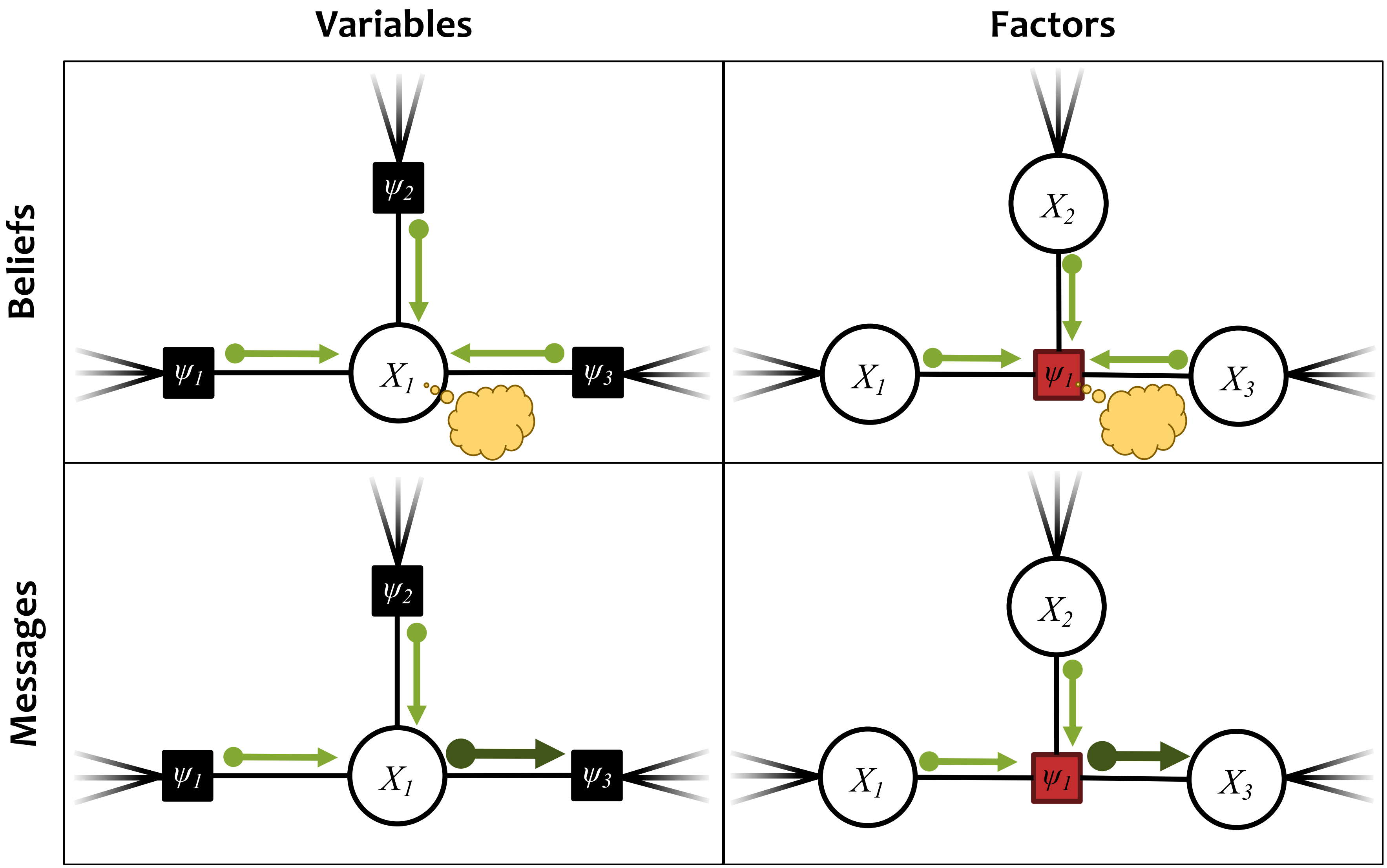
SUM-PRODUCT BELIEF PROPAGATION

Message Passing in Belief Propagation



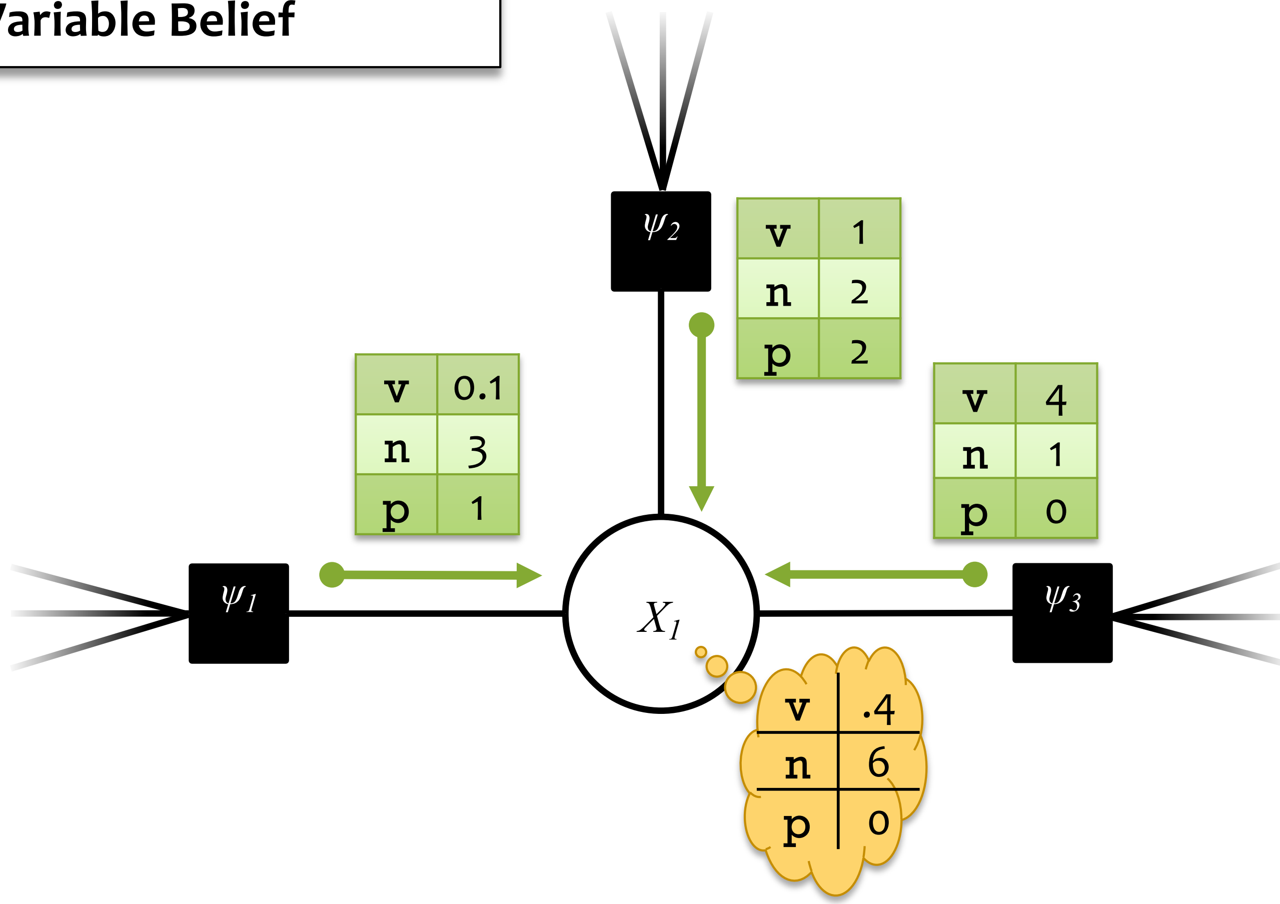
Both of these messages judge the possible values of variable X.
Their product = belief at X = product of all 3 messages to X.

Sum-Product Belief Propagation



Sum-Product Belief Propagation

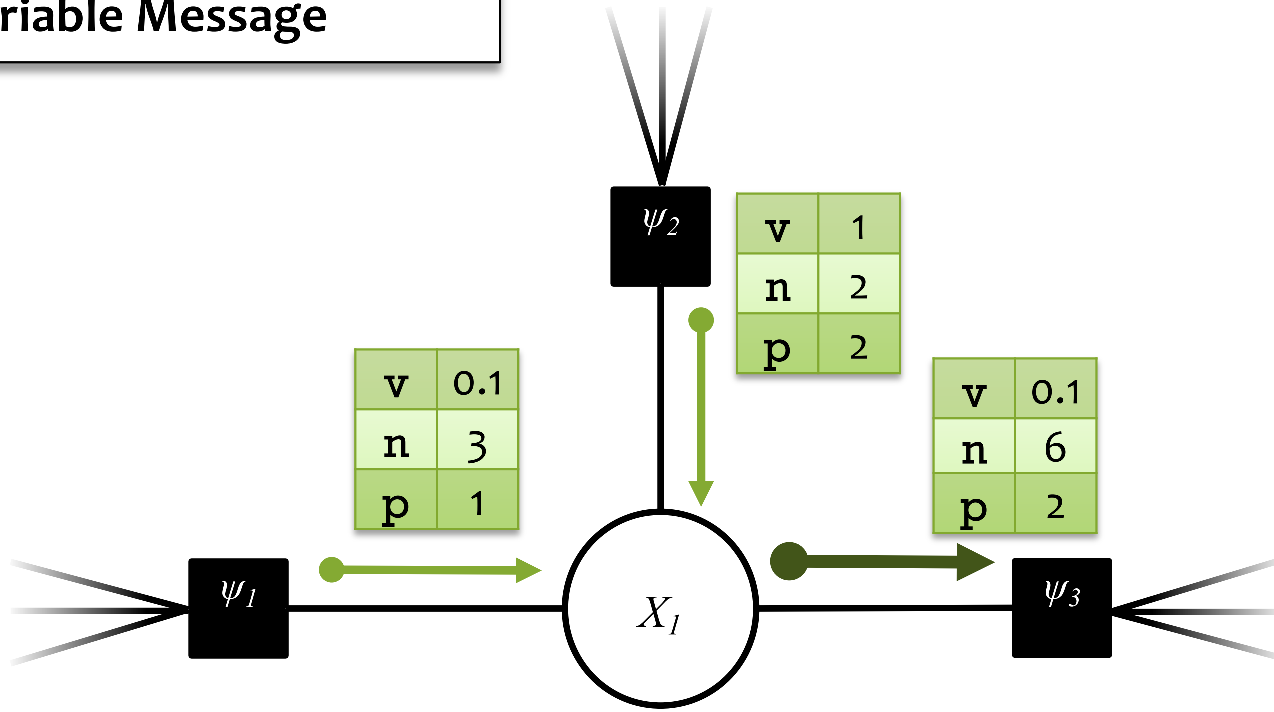
Variable Belief



$$b_i(x_i) = \prod_{\alpha \in \mathcal{N}(i)} \mu_{\alpha \to i}(x_i)$$

Sum-Product Belief Propagation

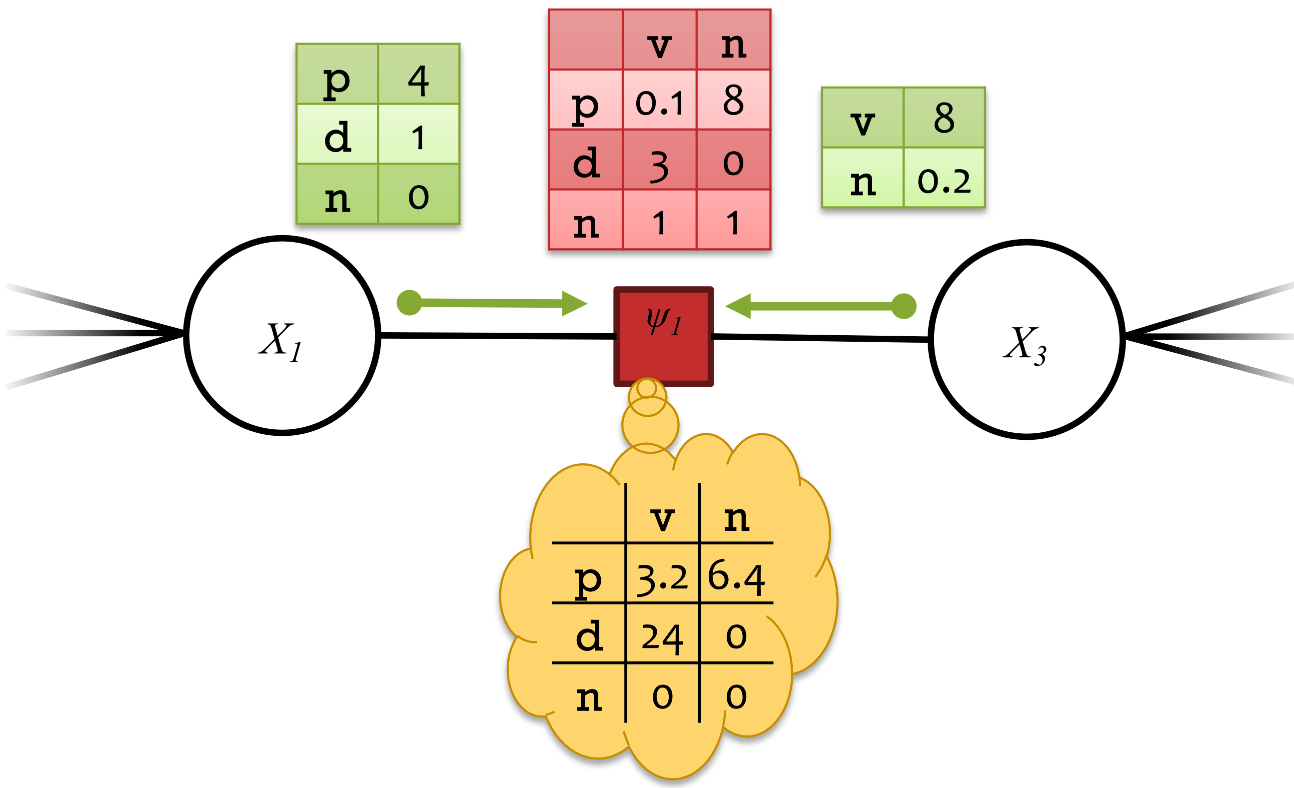
Variable Message



$$\mu_{i \to \alpha}(x_i) = \prod_{\alpha \in \mathcal{N}(i) \backslash \alpha} \mu_{\alpha \to i}(x_i)$$

Sum-Product Belief Propagation

Factor Belief



p	4
d	1
n	0

	v	n
p	0.1	8
d	3	0
n	1	1

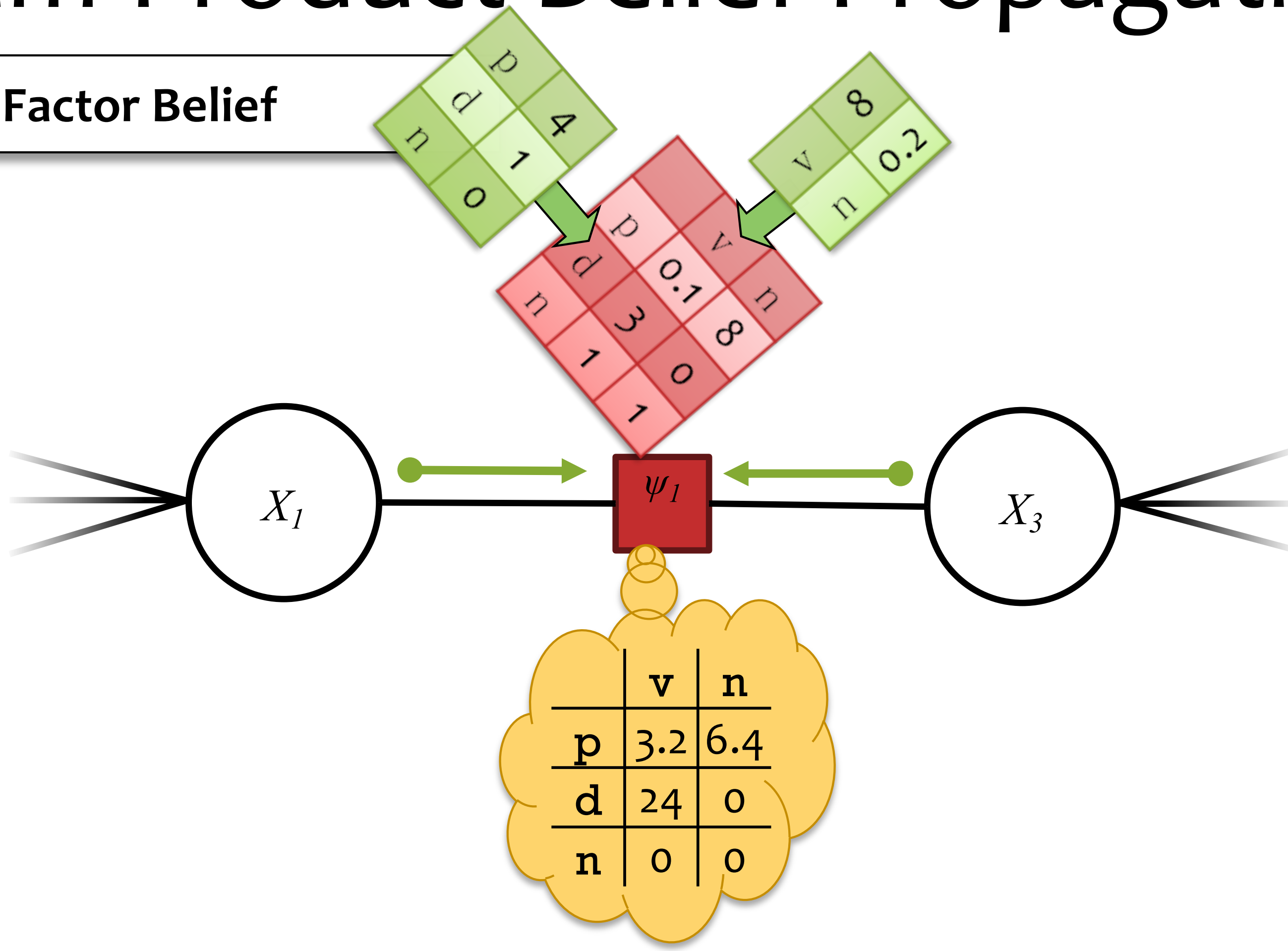
v	8
n	0.2

	v	n
p	3.2	6.4
d	24	0
n	0	0

$$b_\alpha(\boldsymbol{x_\alpha}) = \psi_\alpha(\boldsymbol{x_\alpha}) \prod_{i \in \mathcal{N}(\alpha)} \mu_{i \to \alpha}(\boldsymbol{x_\alpha}[i])$$

Sum-Product Belief Propagation

Factor Belief



$$b_\alpha(\boldsymbol{x}_\alpha) = \psi_\alpha(\boldsymbol{x}_\alpha) \prod_{i \in \mathcal{N}(\alpha)} \mu_{i \to \alpha}(\boldsymbol{x}_\alpha[i])$$

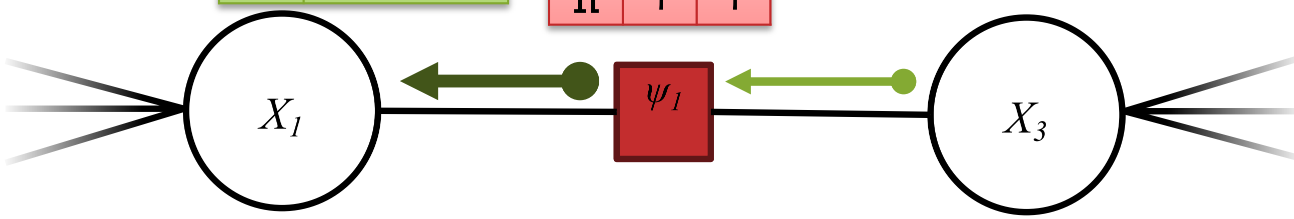
Sum-Product Belief Propagation

Factor Message

p	0.8 + 0.16
d	24 + 0
n	8 + 0.2

	v	n
p	0.1	8
d	3	0
n	1	1

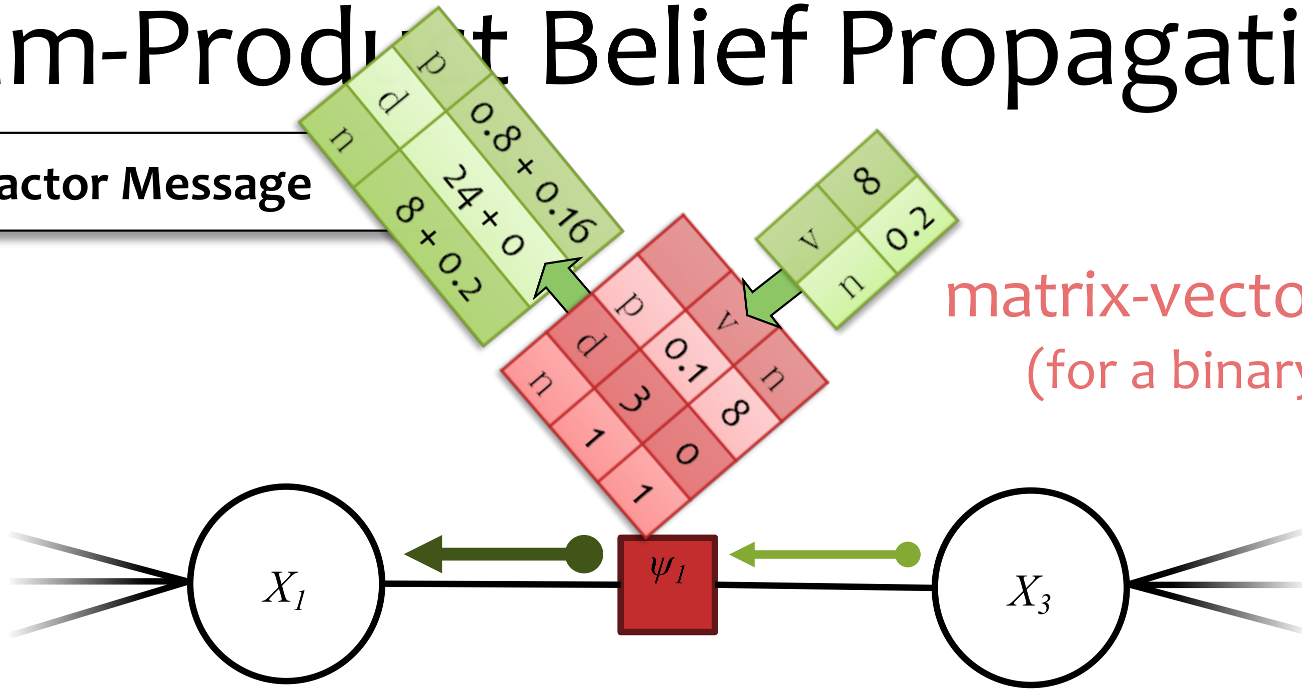
v	8
n	0.2



$$\mu_{\alpha \to i}(x_i) = \sum_{\boldsymbol{x}_{\boldsymbol{\alpha}} : \boldsymbol{x}_{\boldsymbol{\alpha}}[i] = x_i} \psi_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) \prod_{j \in \mathcal{N}(\alpha) \setminus i} \mu_{j \to \alpha}(\boldsymbol{x}_{\boldsymbol{\alpha}}[i])$$

Sum-Product Belief Propagation

Factor Message



matrix-vector product
(for a binary factor)

$$\mu_{\alpha \to i}(x_i) = \sum_{\boldsymbol{x}_{\boldsymbol{\alpha}} : \boldsymbol{x}_{\boldsymbol{\alpha}}[i] = x_i} \psi_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) \prod_{j \in \mathcal{N}(\alpha) \setminus i} \mu_{j \to \alpha}(\boldsymbol{x}_{\boldsymbol{\alpha}}[i])$$

Sum-Product Belief Propagation

Input: a factor graph with no cycles

Output: exact marginals for each variable and factor

Algorithm:

1. Initialize the messages to the uniform distribution.

$$\mu_{i \to \alpha}(x_i) = 1 \qquad \mu_{\alpha \to i}(x_i) = 1$$

1. Choose a root node.
2. Send messages from the **leaves** to the **root**.
   Send messages from the **root** to the **leaves**.

$$\mu_{i \to \alpha}(x_i) = \prod_{\alpha \in \mathcal{N}(i) \setminus \alpha} \mu_{\alpha \to i}(x_i) \qquad \mu_{\alpha \to i}(x_i) = \sum_{\boldsymbol{x}_{\boldsymbol{\alpha}} : \boldsymbol{x}_{\boldsymbol{\alpha}}[i] = x_i} \psi_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) \prod_{j \in \mathcal{N}(\alpha) \setminus i} \mu_{j \to \alpha}(\boldsymbol{x}_{\boldsymbol{\alpha}}[i])$$

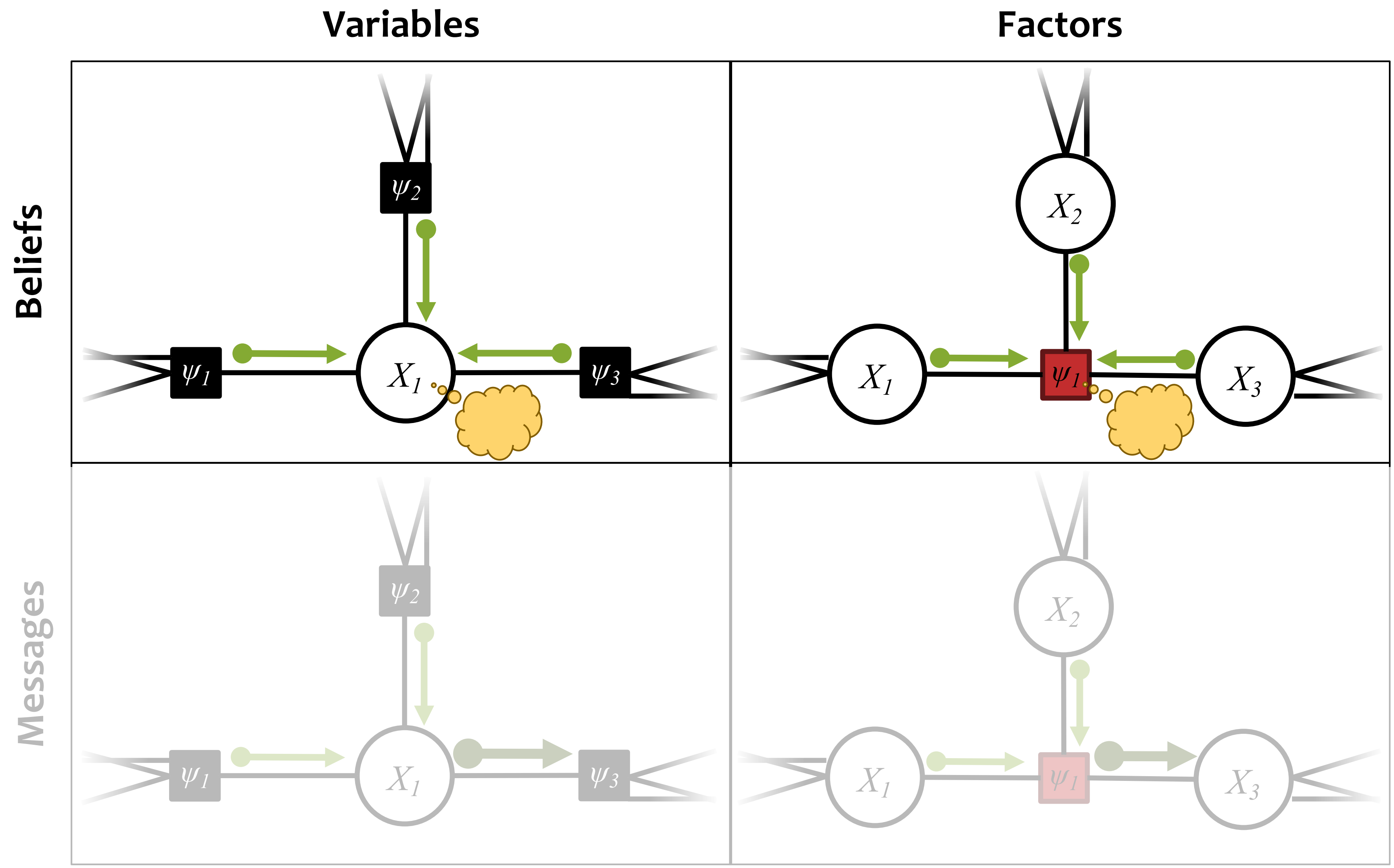
1. Compute the beliefs (unnormalized marginals).

$$b_i(x_i) = \prod_{\alpha \in \mathcal{N}(i)} \mu_{\alpha \to i}(x_i) \qquad b_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) = \psi_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) \prod_{i \in \mathcal{N}(\alpha)} \mu_{i \to \alpha}(\boldsymbol{x}_{\boldsymbol{\alpha}}[i])$$

2. Normalize beliefs and return the **exact** marginals.

$$p_i(x_i) \propto b_i(x_i) \qquad p_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) \propto b_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}})$$

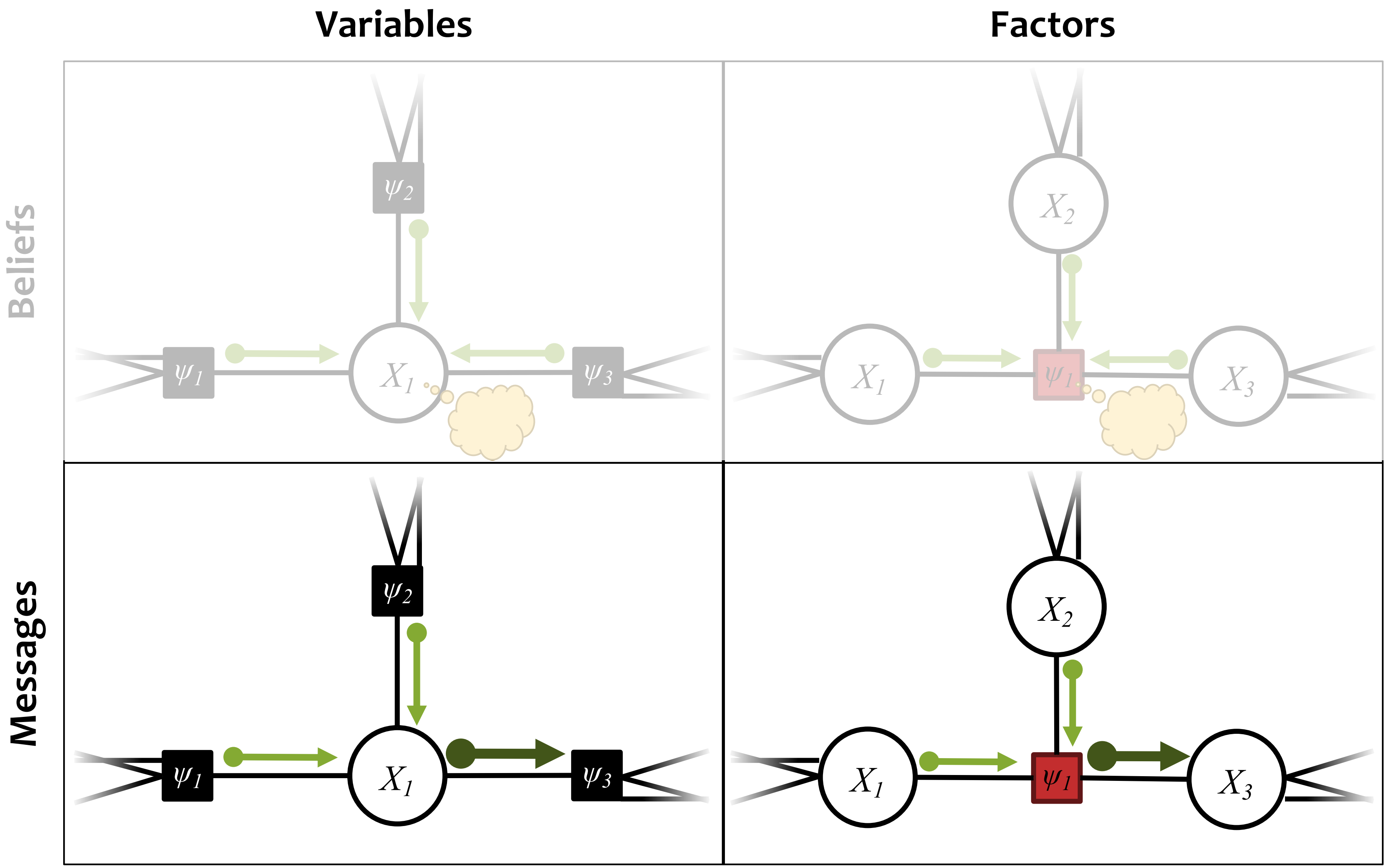
Sum-Product Belief Propagation



$$b_i(x_i) = \prod_{\alpha \in \mathcal{N}(i)} \mu_{\alpha \to i}(x_i)$$

$$b_\alpha(\boldsymbol{x_\alpha}) = \psi_\alpha(\boldsymbol{x_\alpha}) \prod_{i \in \mathcal{N}(\alpha)} \mu_{i \to \alpha}(\boldsymbol{x_\alpha}[i])$$

Sum-Product Belief Propagation



$$\mu_{i \to \alpha}(x_i) = \prod_{\alpha \in \mathcal{N}(i) \setminus \alpha} \mu_{\alpha \to i}(x_i)$$

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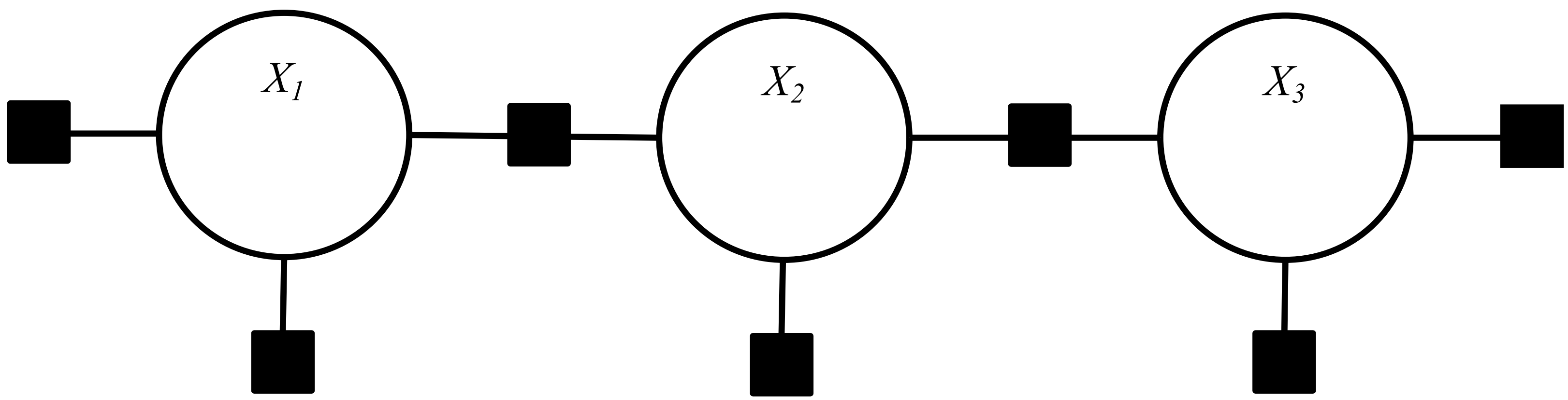
FORWARD BACKWARD AS SUM-PRODUCT BP

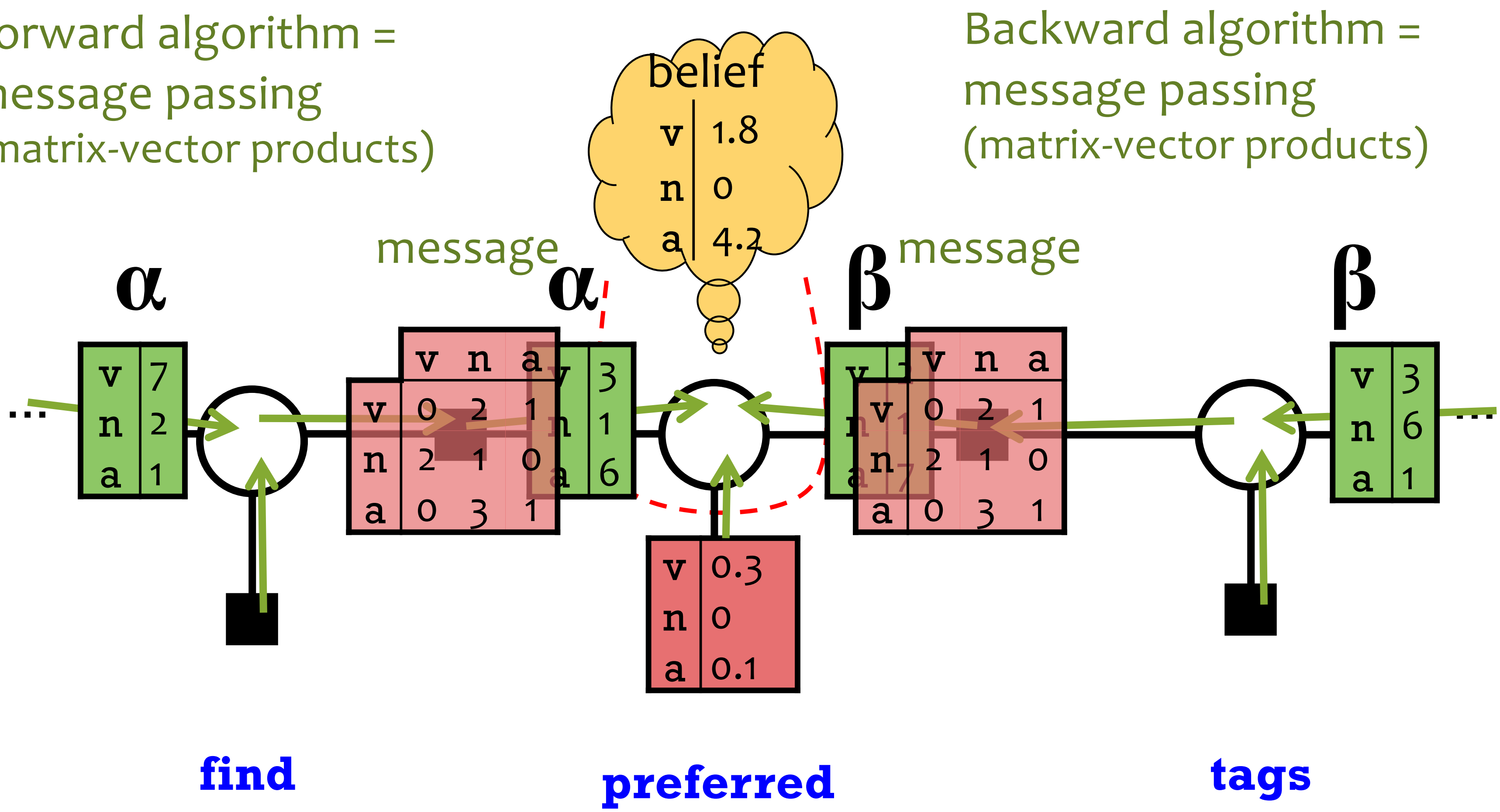
CRF Tagging Model



find | preferred | tags

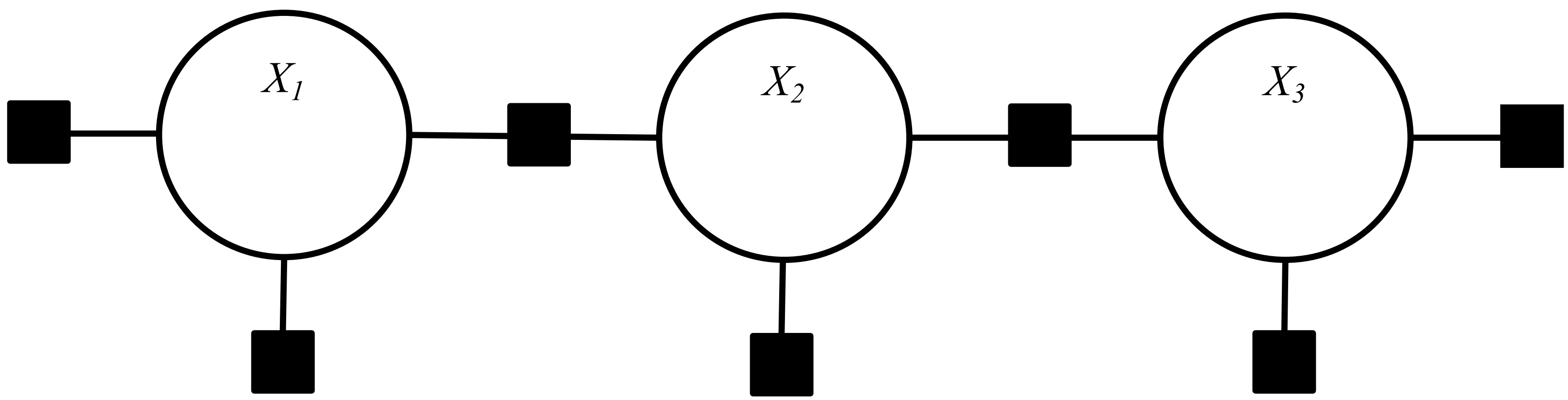
Could be verb or noun | *Could be adjective or verb* | *Could be noun or verb*

CRF Tagging by Belief Propagation



- Forward-backward is a message passing algorithm.
- It's the simplest case of belief propagation.

So Let’s Review Forward-Backward …



find **preferred** **tags**

Could be verb or noun *Could be adjective or verb* *Could be noun or verb*

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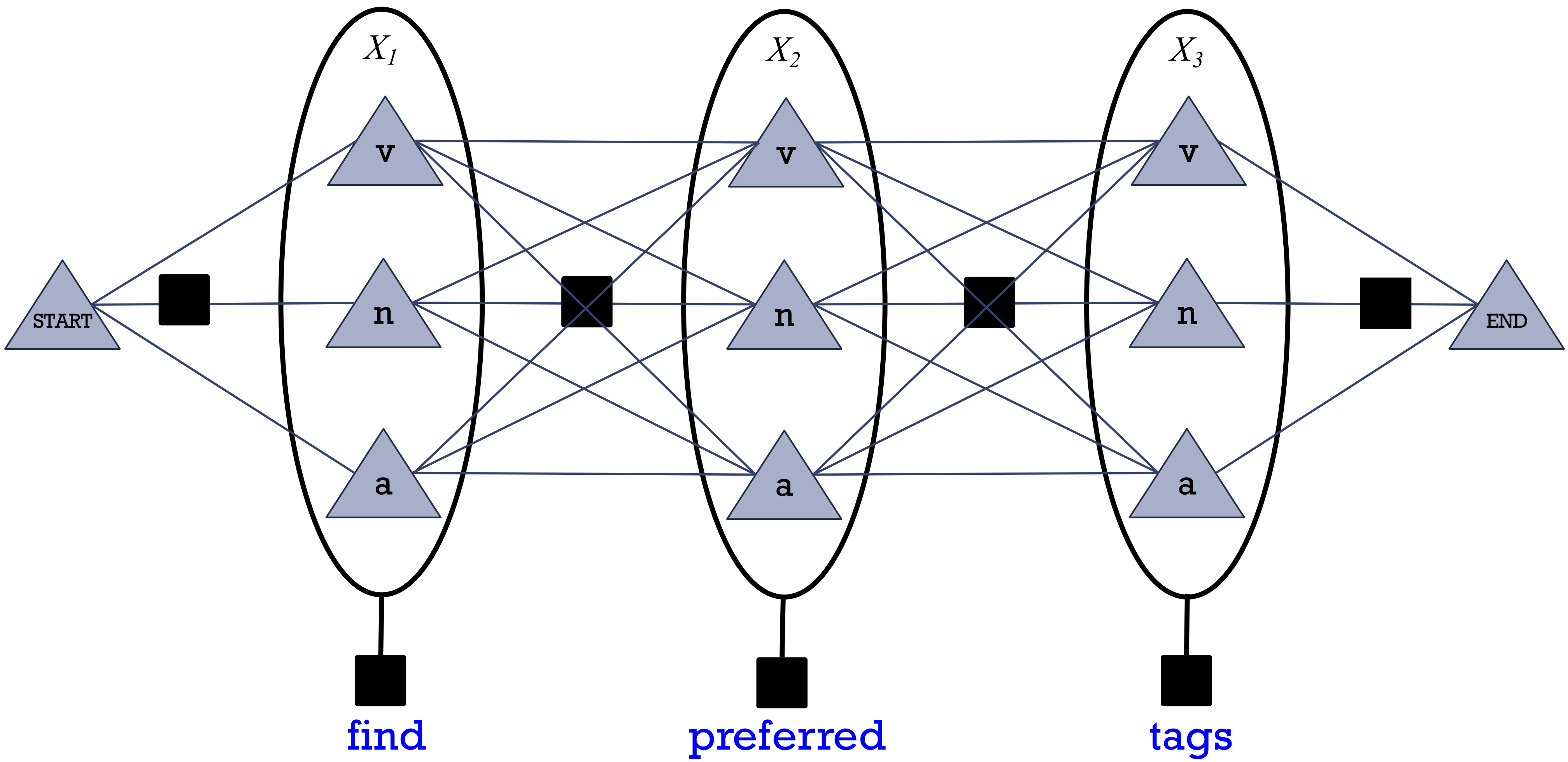
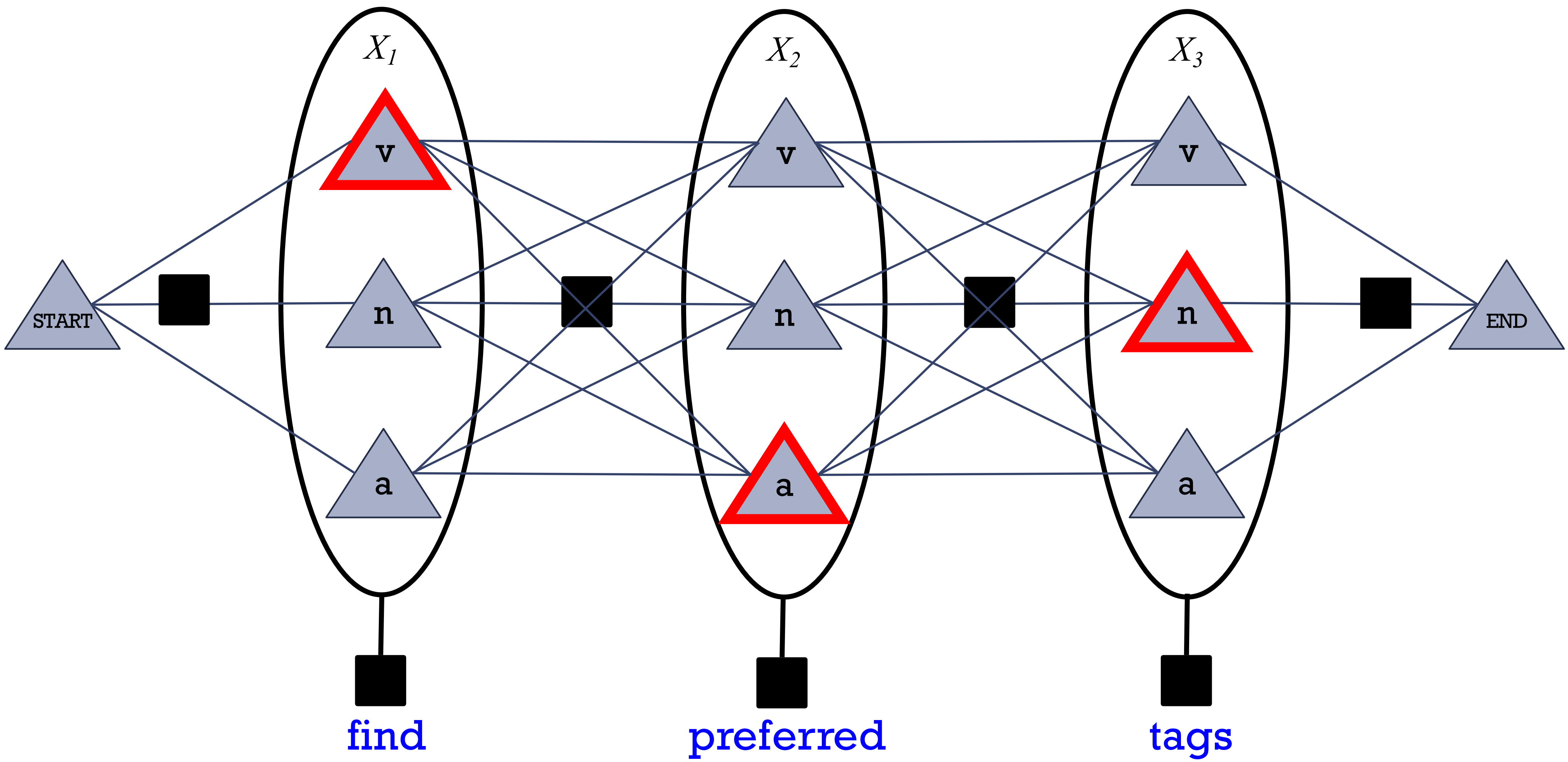


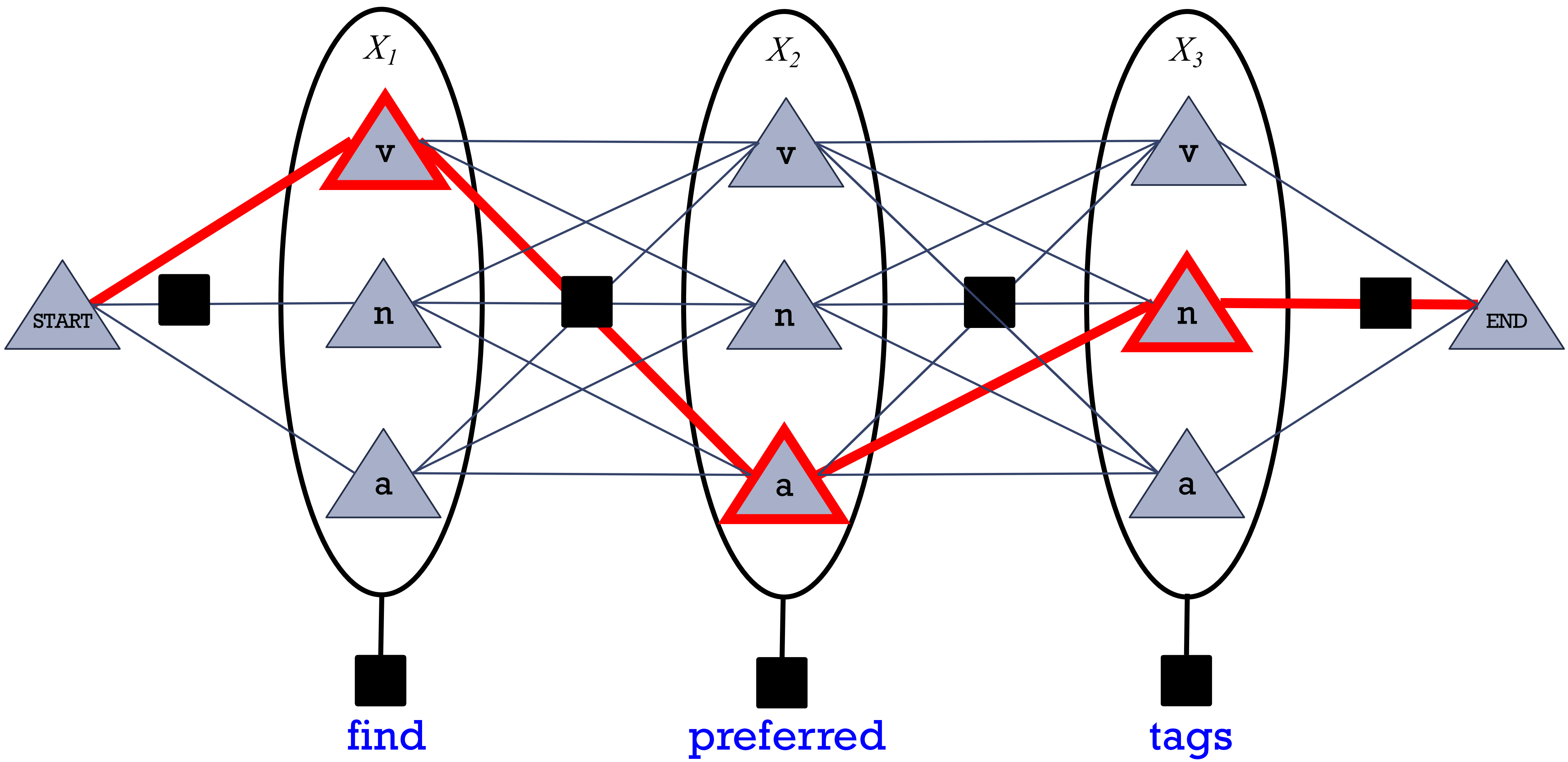
- Show the possible *values* for each variable

So Let's Review Forward-Backward …



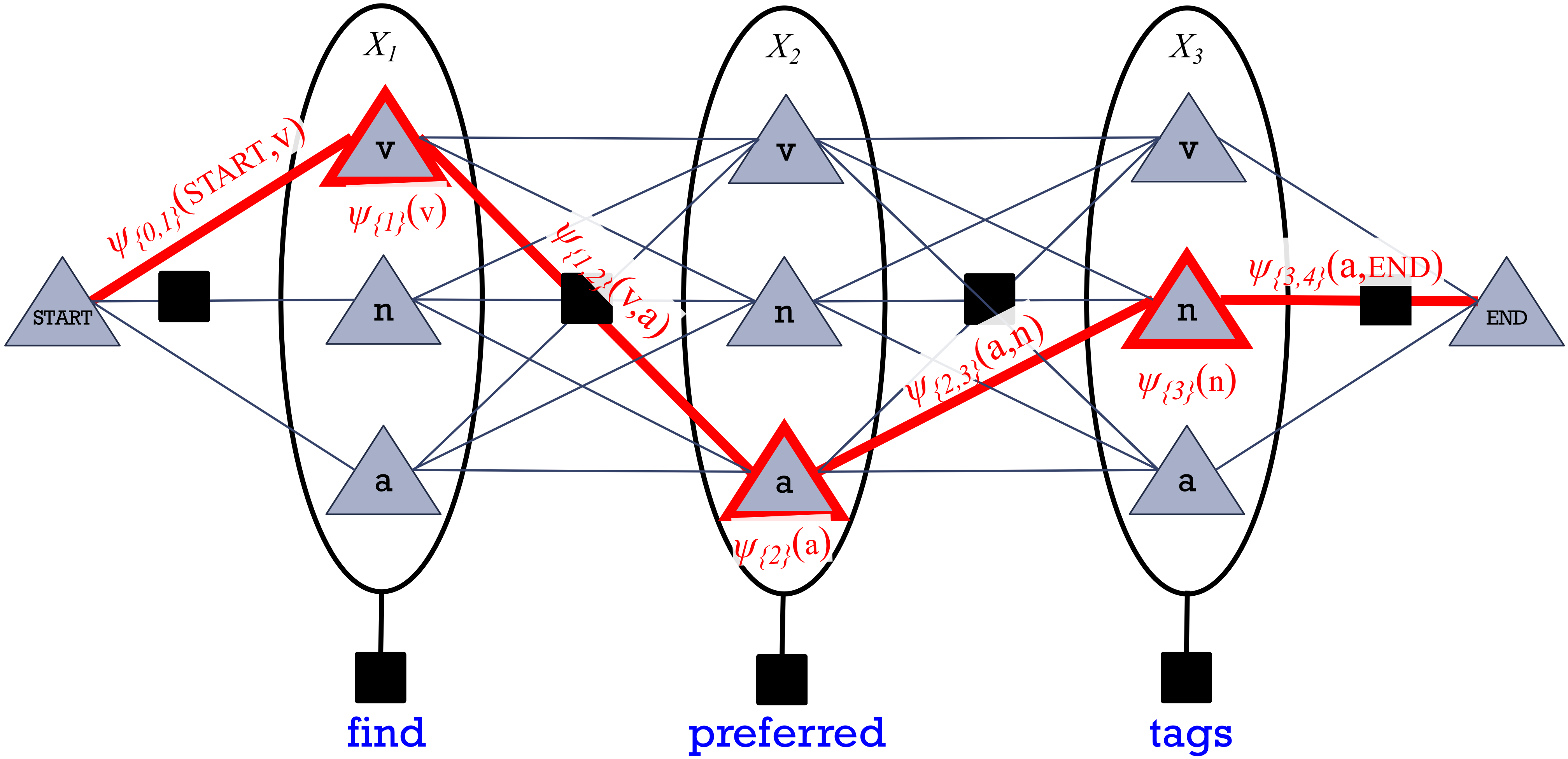
- Let's show the possible *values* for each variable
- One possible assignment

So Let's Review Forward-Backward …



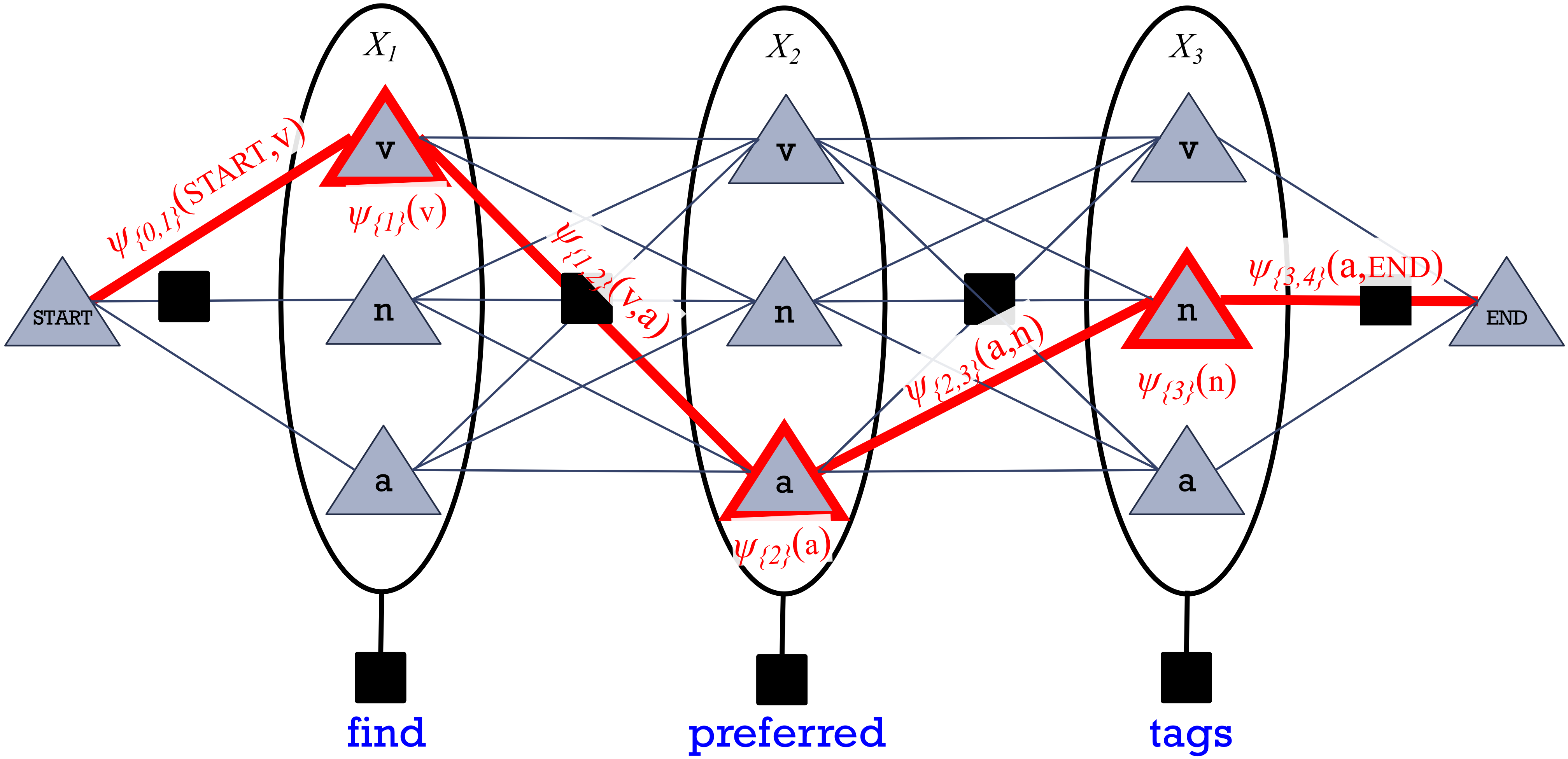
- Let's show the possible *values* for each variable
- One possible assignment
- And what the 7 factors think of it …

Viterbi Algorithm: Most Probable Assignment



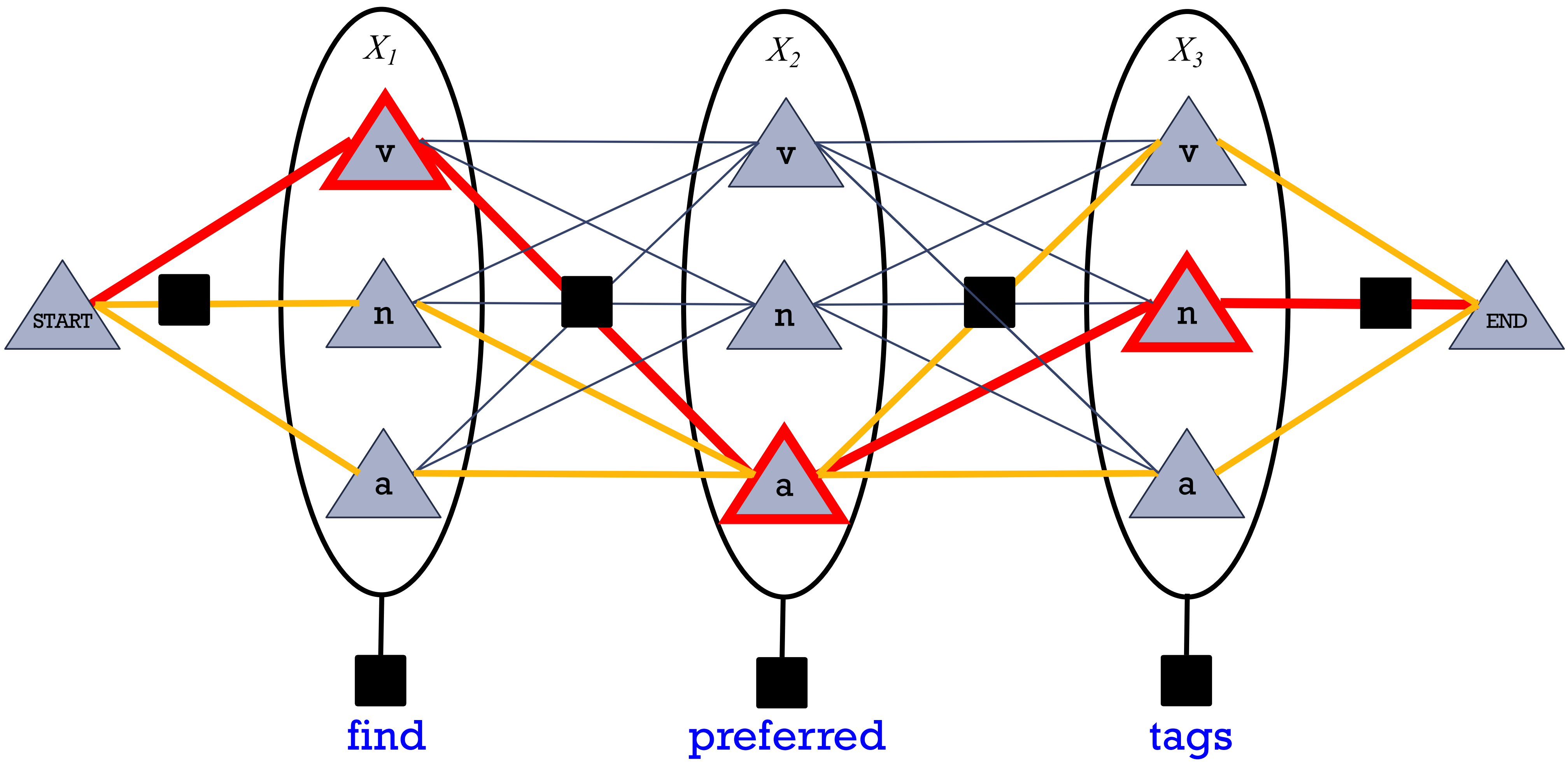
- So p(**v a n**) = (1/Z) * product of 7 numbers
- Numbers associated with edges and nodes of path
- Most probable assignment = **path with highest product**

Viterbi Algorithm: Most Probable Assignment



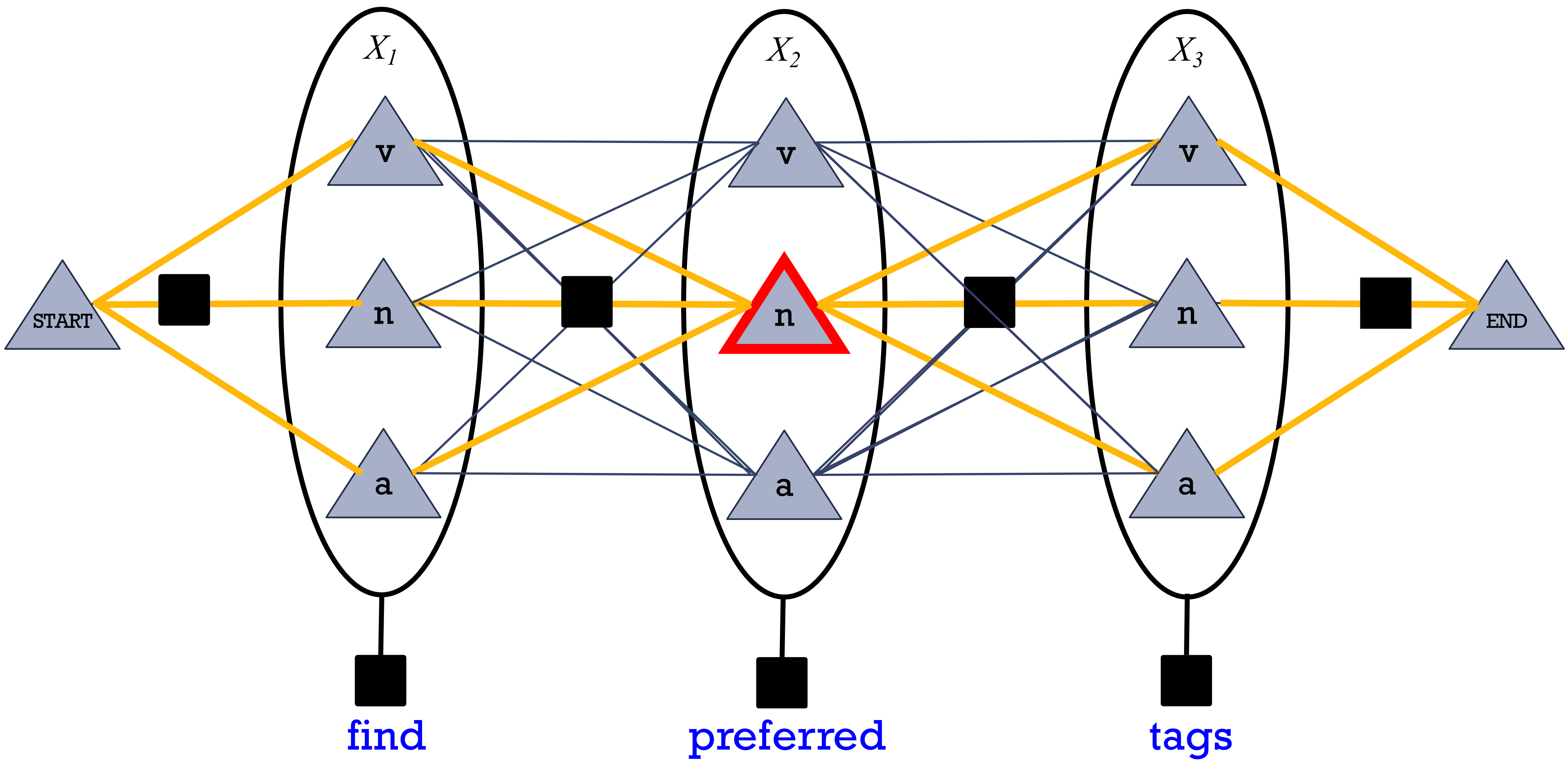
- So p(**v a n**) = (1/Z) * product weight of one path

Forward-Backward Algorithm: Finds Marginals



- So p(**v a n**) = (1/Z) * product weight of one path
- Marginal probability $p(X_2 = a)$
  = (1/Z) * total weight of *all* paths through **a**

Forward-Backward Algorithm: Finds Marginals



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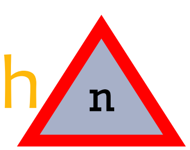
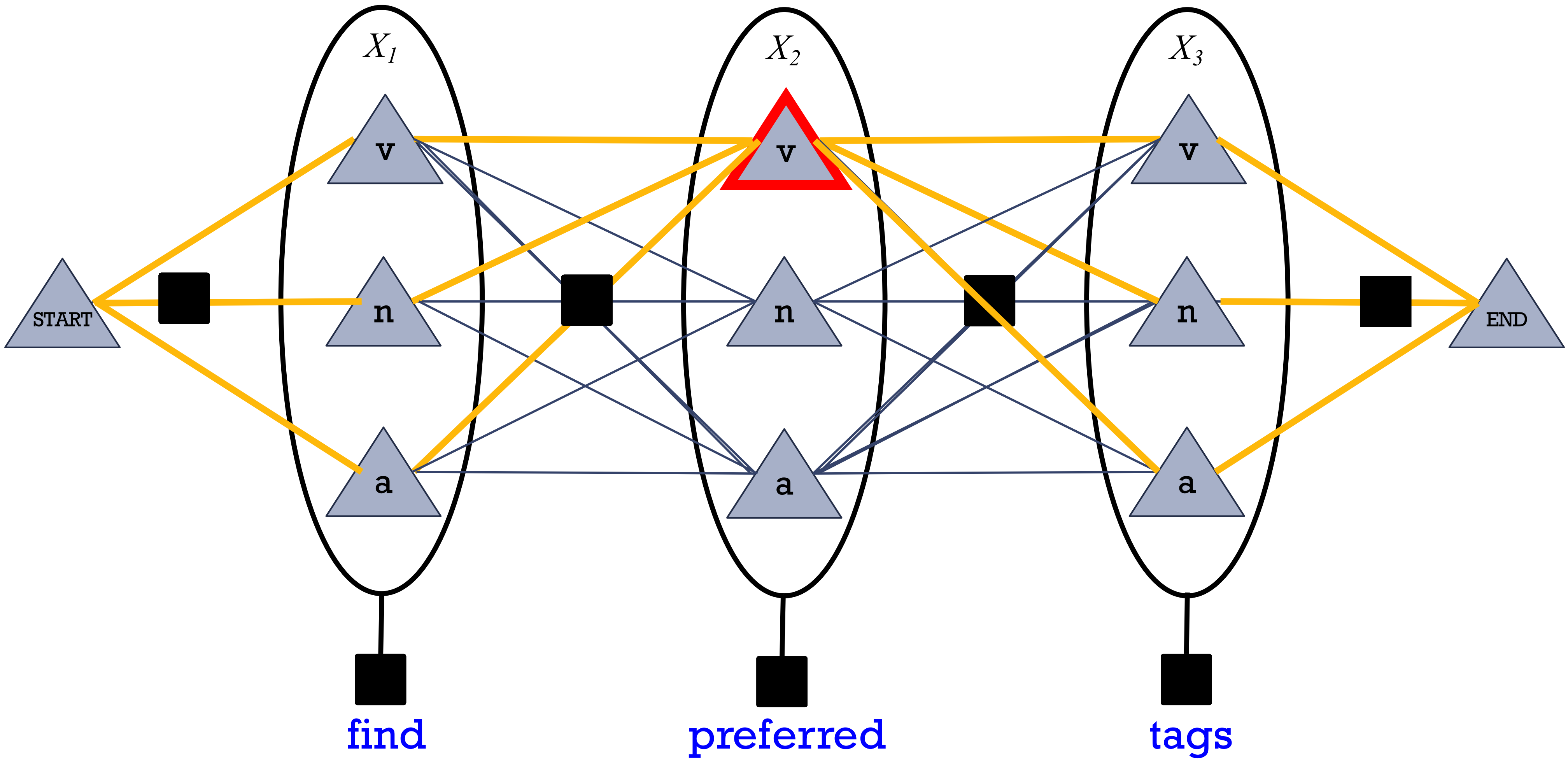


Forward-Backward Algorithm: Finds Marginals



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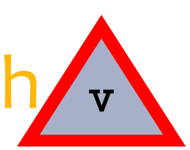
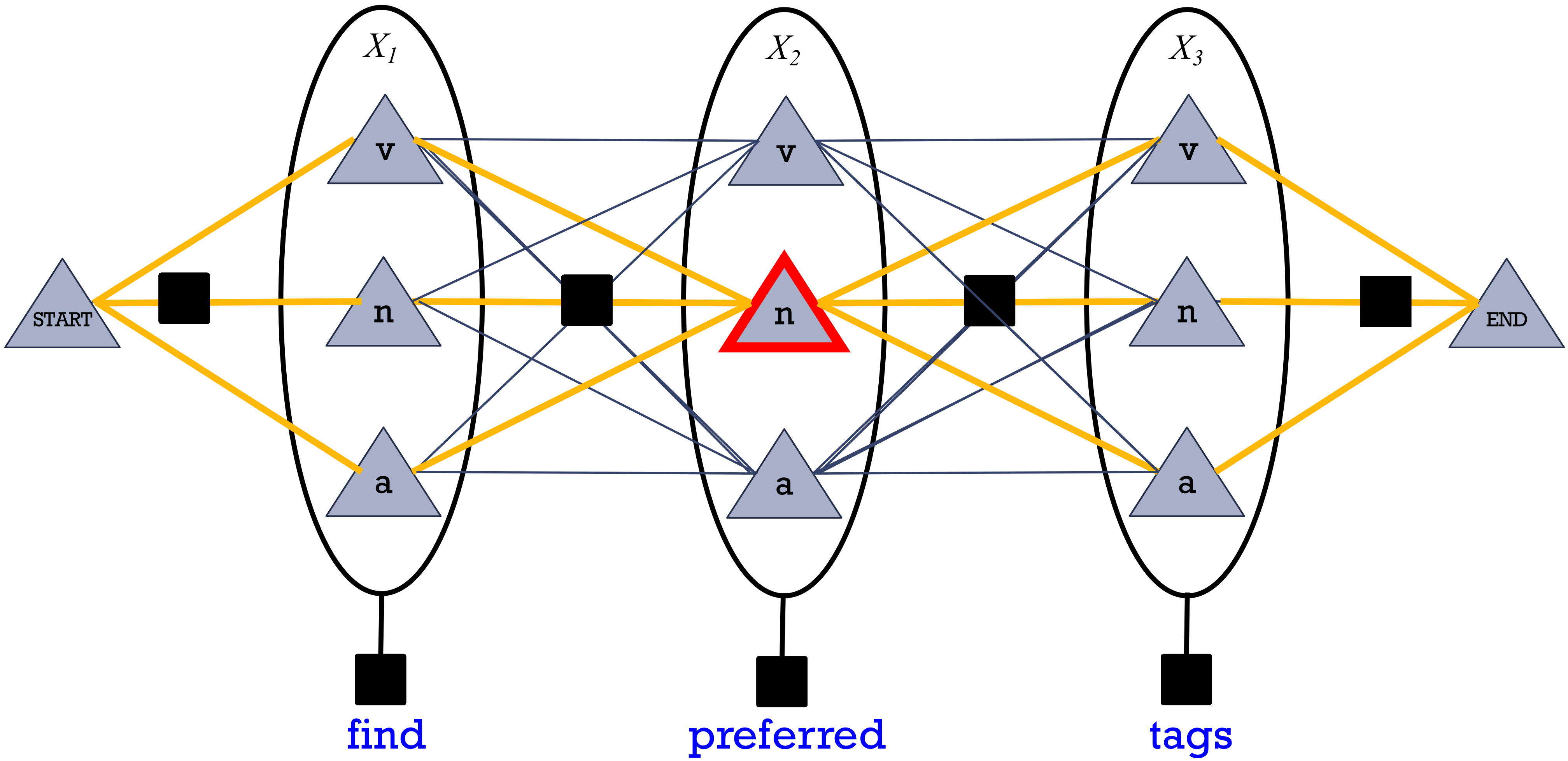
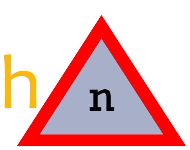
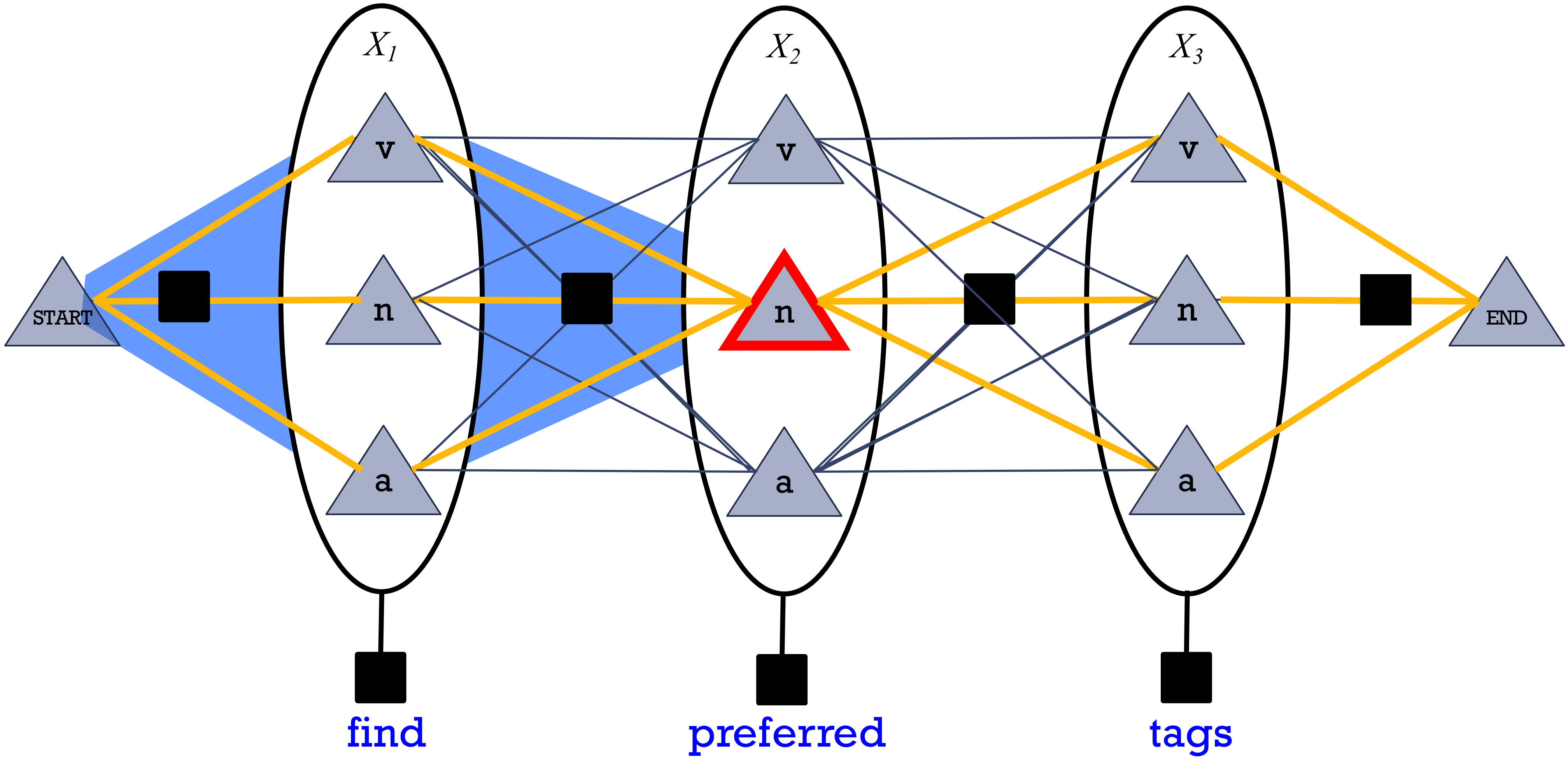
  = (1/Z) * total weight of *all* paths through 

Forward-Backward Algorithm: Finds Marginals



- So p(**v a n**) = (1/Z) * product weight of one path
- Marginal probability $p(X_2 = \mathrm{a})$
  = (1/Z) * total weight of *all* paths through 

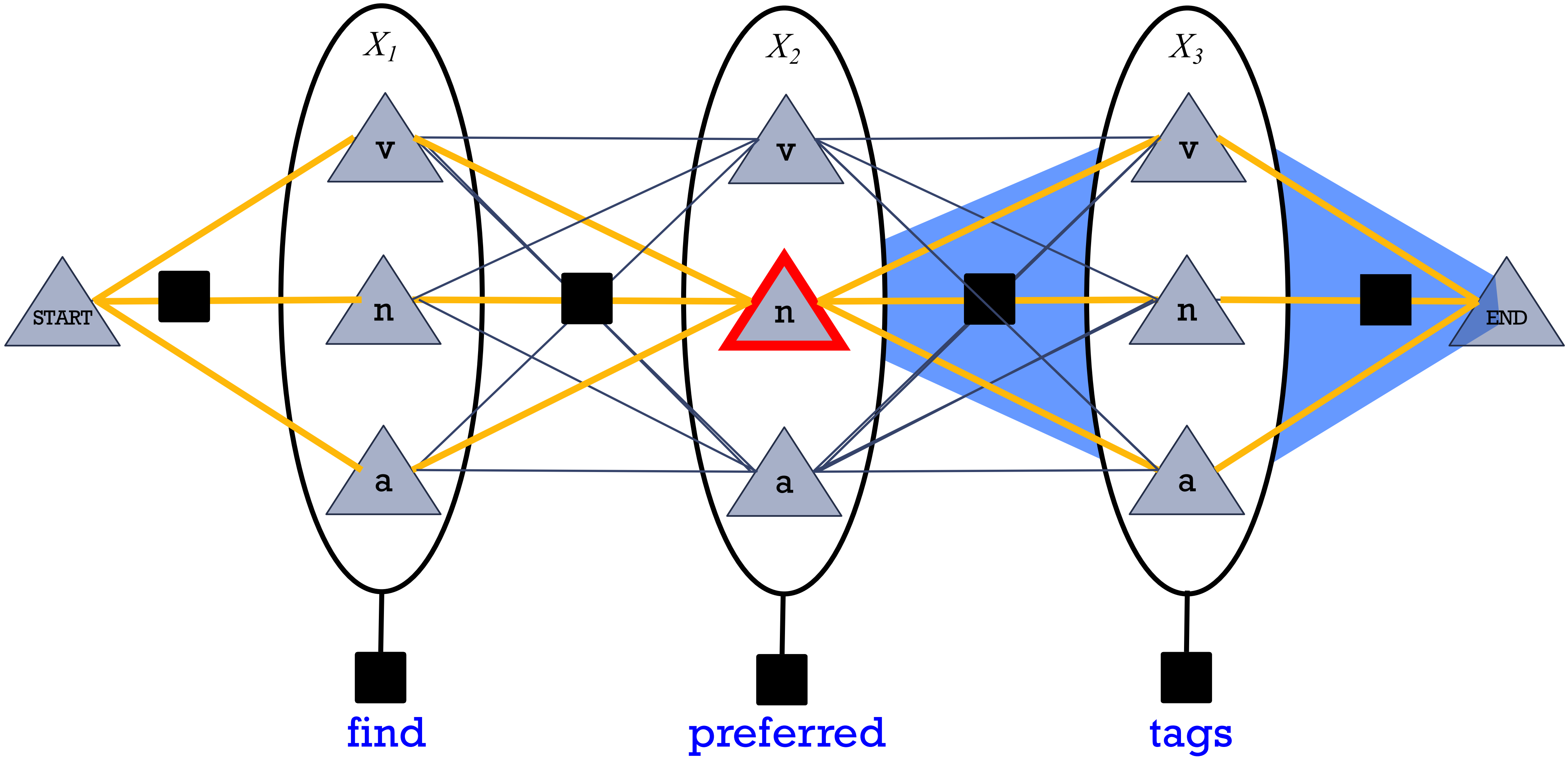
Forward-Backward Algorithm: Finds Marginals



$\alpha_2(\mathbf{n})$ = total weight of these path *prefixes*

(found by dynamic programming: matrix-vector products)

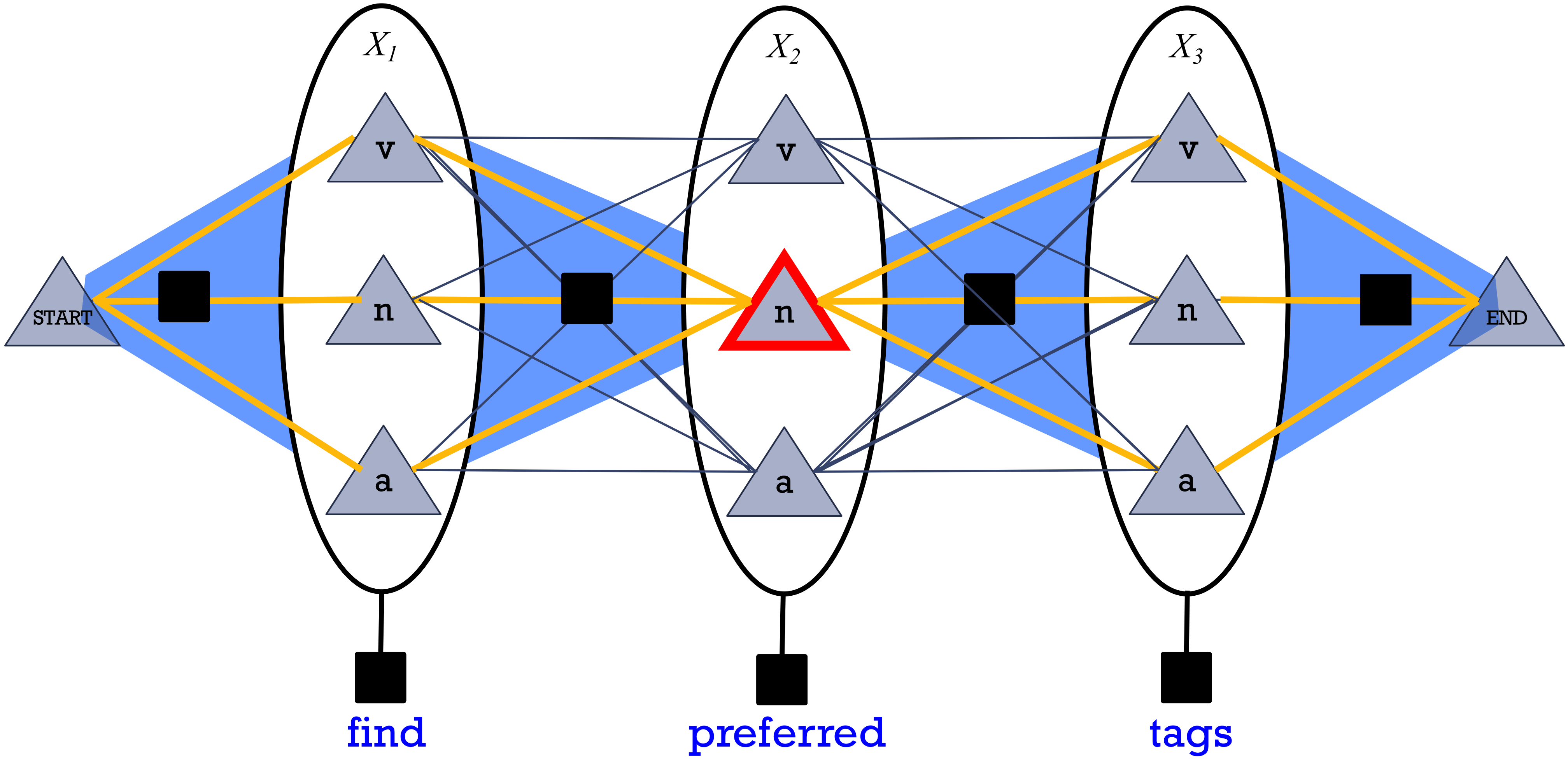
Forward-Backward Algorithm: Finds Marginals



$\beta_2(\mathbf{n})$ = total weight of these path *suffixes*

(found by dynamic programming: matrix-vector products)

Forward-Backward Algorithm: Finds Marginals



$\alpha_2(\mathbf{n})$ = total weight of these path *prefixes* (a + b + c)

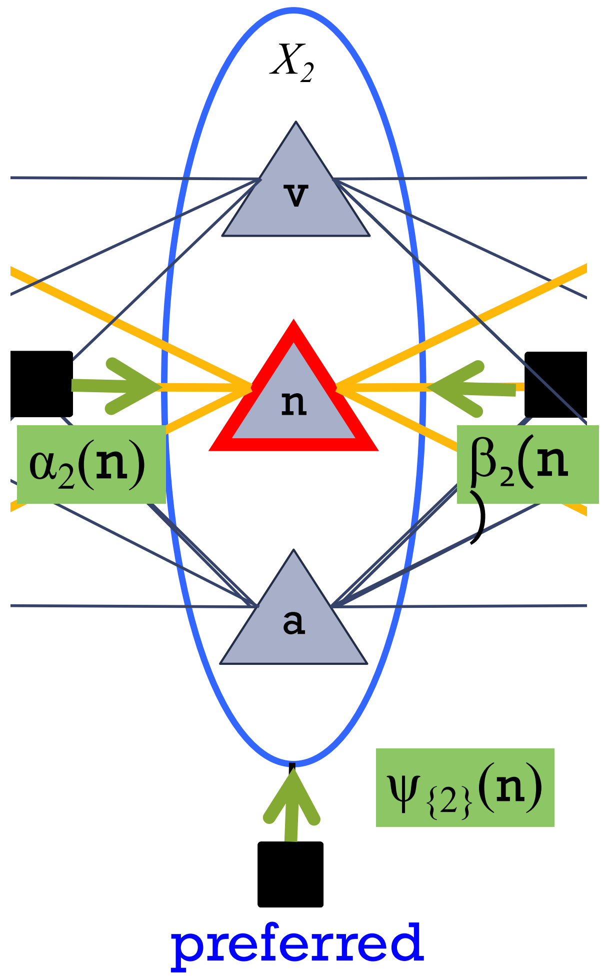
$\beta_2(\mathbf{n})$ = total weight of these path *suffixes* (x + y + z)

Product gives ax+ay+az+bx+by+bz+cx+cy+cz = total weight of paths

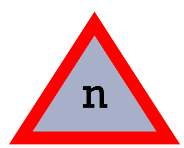
Forward-Backward Algorithm: Finds Marginals

Oops! The weight of a path through a state also includes a weight at that state.
So $\alpha(\mathbf{n})\cdot\beta(\mathbf{n})$ isn't enough.

The extra weight is the opinion of the unigram factor at this variable.

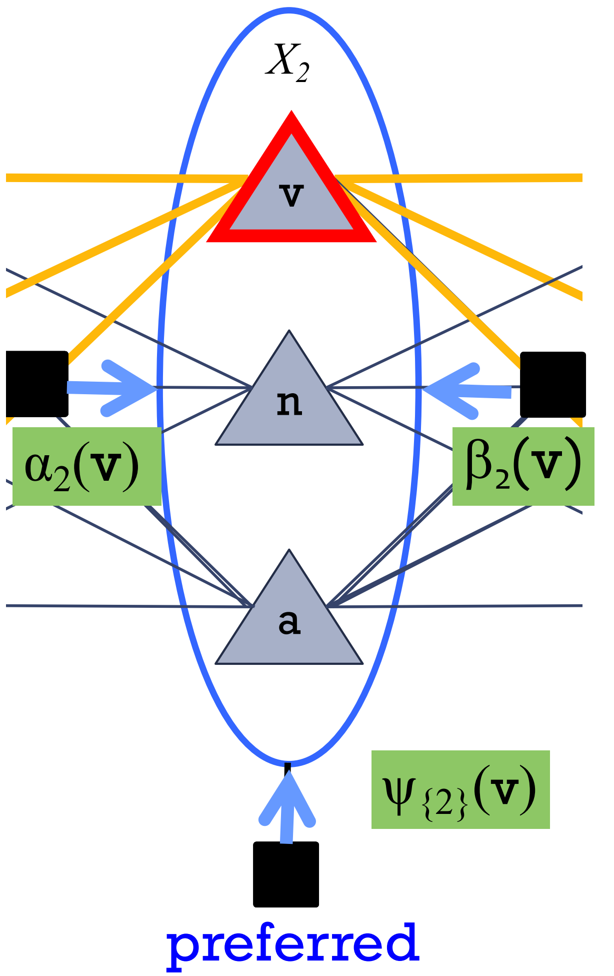


"belief that $X_2 = \mathbf{n}$"

total weight of *all* paths through 

$= \alpha_2(\mathbf{n}) \; \psi_{\{2\}}(\mathbf{n}) \; \beta_2(\mathbf{n})$

Forward-Backward Algorithm: Finds Marginals



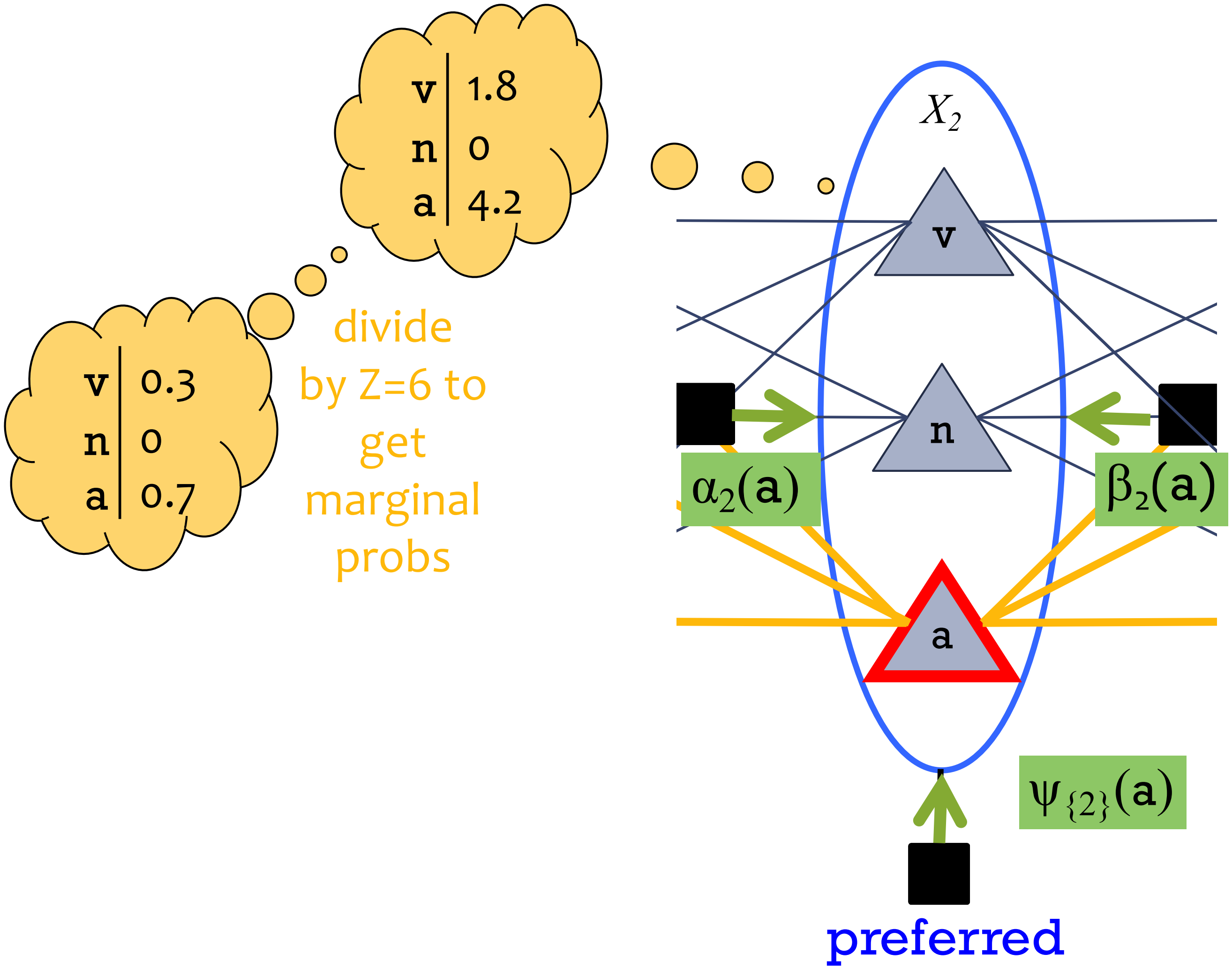
"belief that X_2 = **v**"

"belief that X_2 = **n**"

total weight of *all* paths through **v**

= $\alpha_2(\mathbf{v})\ \psi_{\{2\}}(\mathbf{v})\ \beta_2(\mathbf{v})$

Forward-Backward Algorithm: Finds Marginals



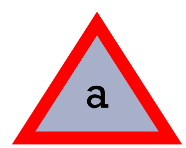
“belief that X_2 = **v**”

“belief that X_2 = **n**”

“belief that X_2 = **a**”

---

sum = Z
(total probability of *all* paths)

total weight of *all* paths through 

= $\alpha_2(\mathtt{a})$ $\psi_{\{2\}}(\mathtt{a})$ $\beta_2(\mathtt{a})$

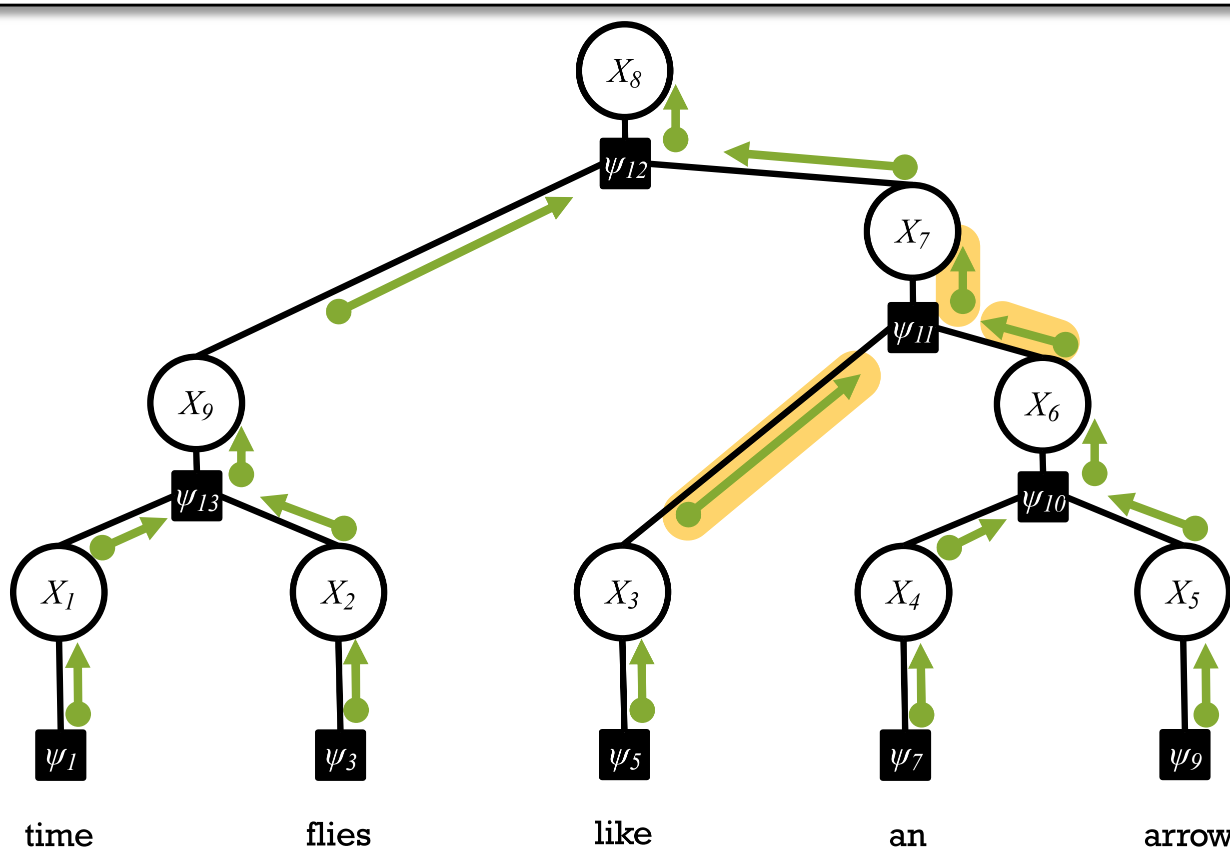
BP AS DYNAMIC PROGRAMMING

(Acyclic) Belief Propagation

In a factor graph with no cycles:

1. Pick any node to serve as the root.
2. Send messages from the **leaves** to the **root**.
3. Send messages from the **root** to the **leaves**.

A node computes an outgoing message along an edge
only after it has received incoming messages along all its other edges.



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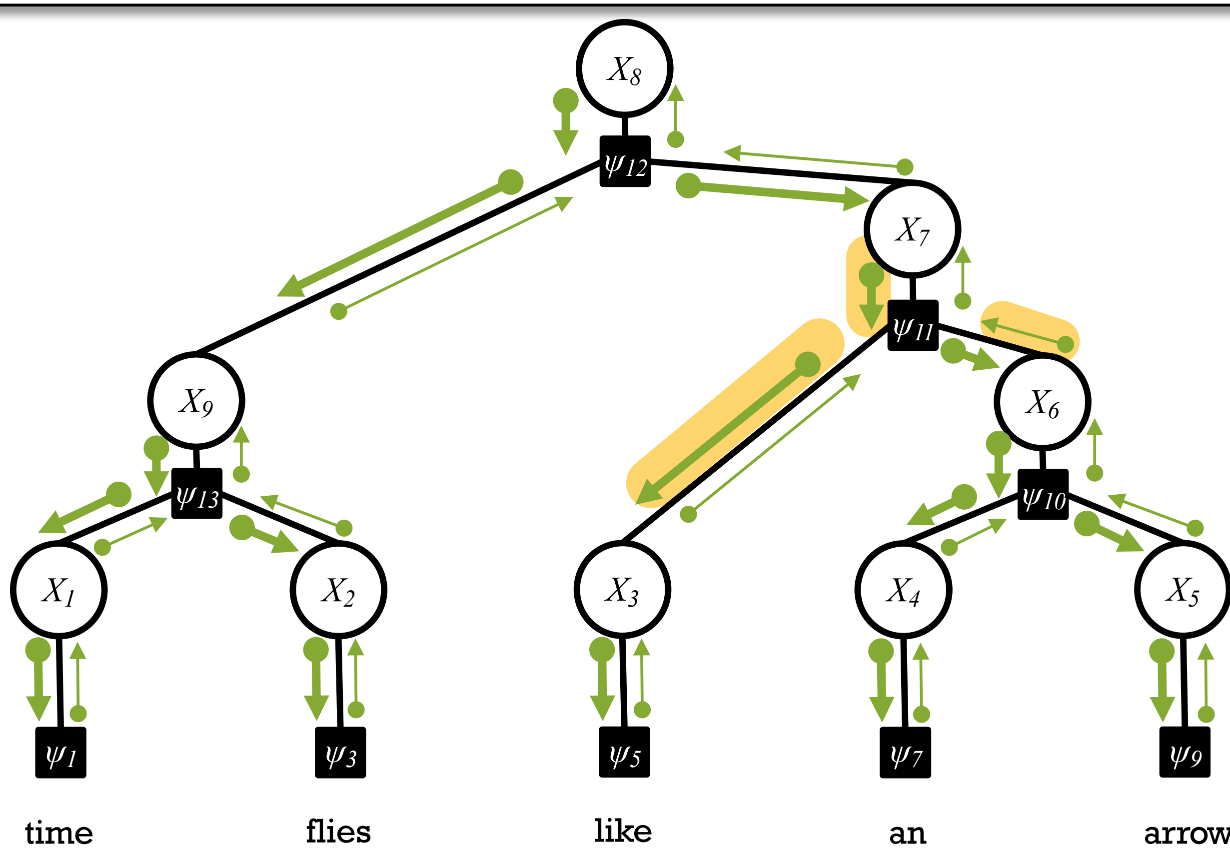
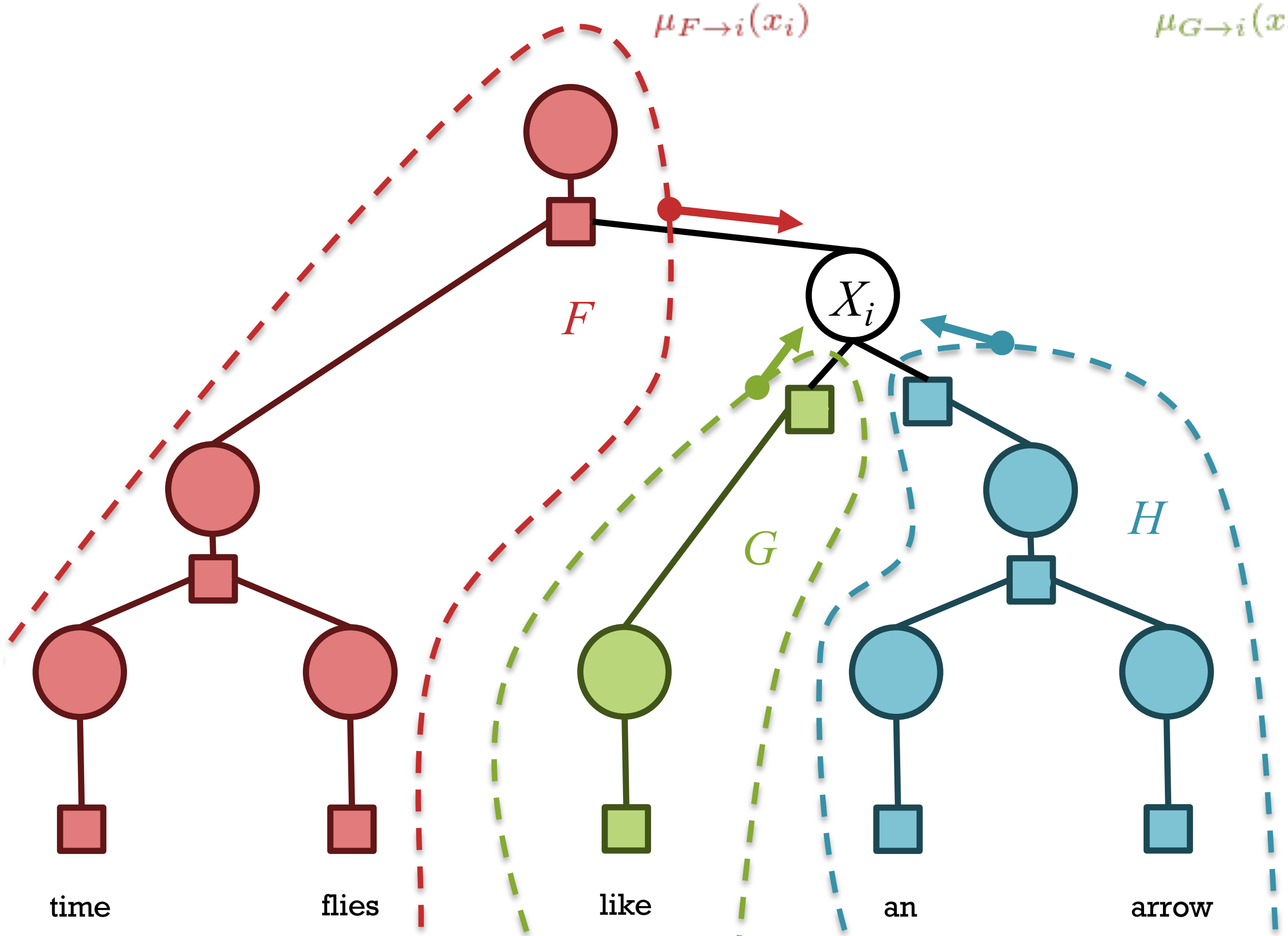


Acyclic BP as Dynamic Programming

$$p(X_i = x_i) \propto b_i(x_i) = \sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha} \psi_\alpha(\boldsymbol{x}_\alpha)$$

$$= \underbrace{\left(\sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha \subseteq F} \psi_\alpha(\boldsymbol{x}_\alpha) \right)}_{\mu_{F\rightarrow i}(x_i)} \underbrace{\left(\sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha \subseteq G} \psi_\alpha(\boldsymbol{x}_\alpha) \right)}_{\mu_{G\rightarrow i}(x_i)} \underbrace{\left(\sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha \subseteq H} \psi_\alpha(\boldsymbol{x}_\alpha) \right)}_{\mu_{H\rightarrow i}(x_i)}$$



Subproblem:
Inference using just the factors in subgraph H

Figure adapted from Burkett & Klein (2012)

Acyclic BP as Dynamic Programming

$$p(X_i = x_i) \propto b_i(x_i) = \sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha} \psi_\alpha(\boldsymbol{x}_\alpha)$$

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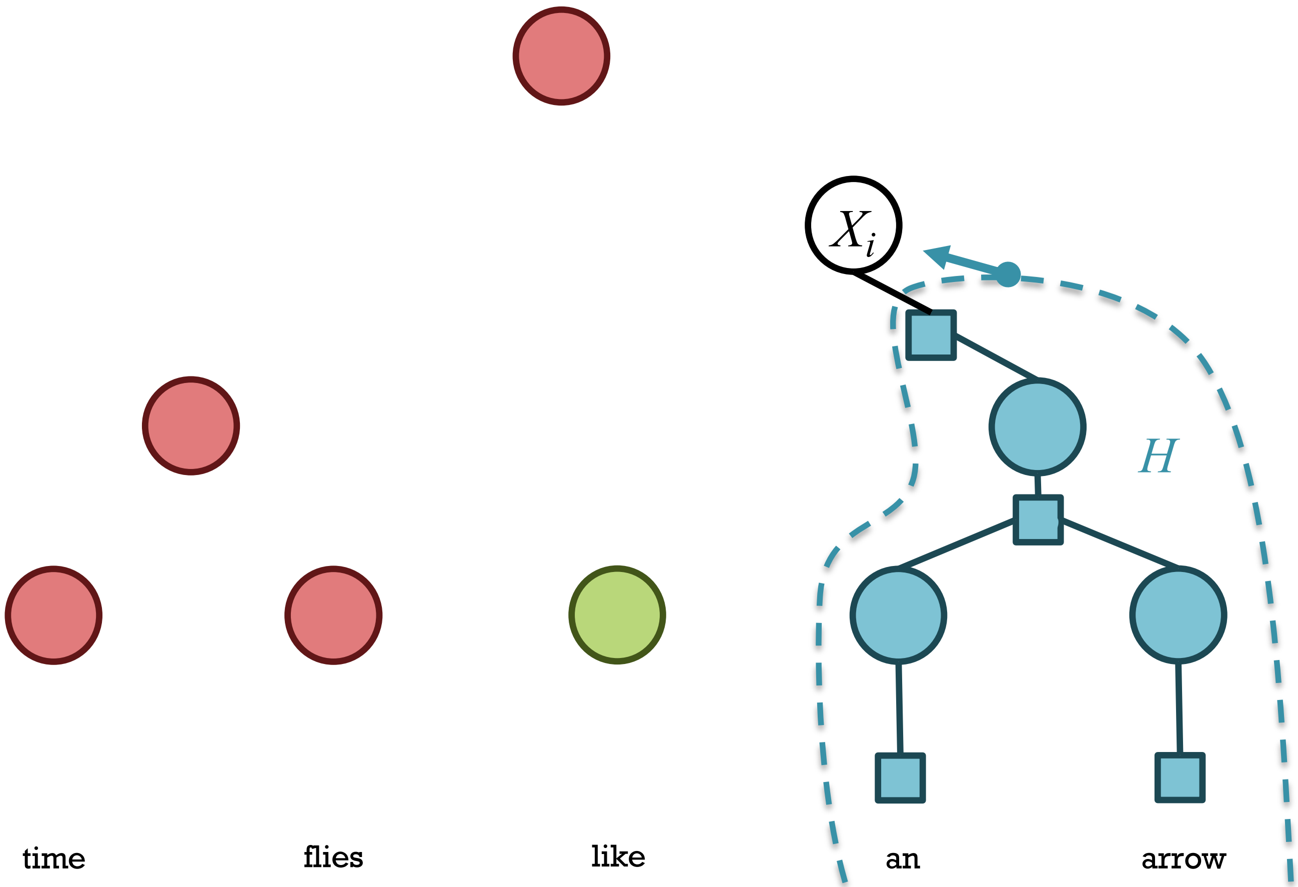

Subproblem:
Inference using just the factors in subgraph H

The marginal of X_i in that smaller model is the message sent to X_i from subgraph H

*Message **to** a variable*

Acyclic BP as Dynamic Programming

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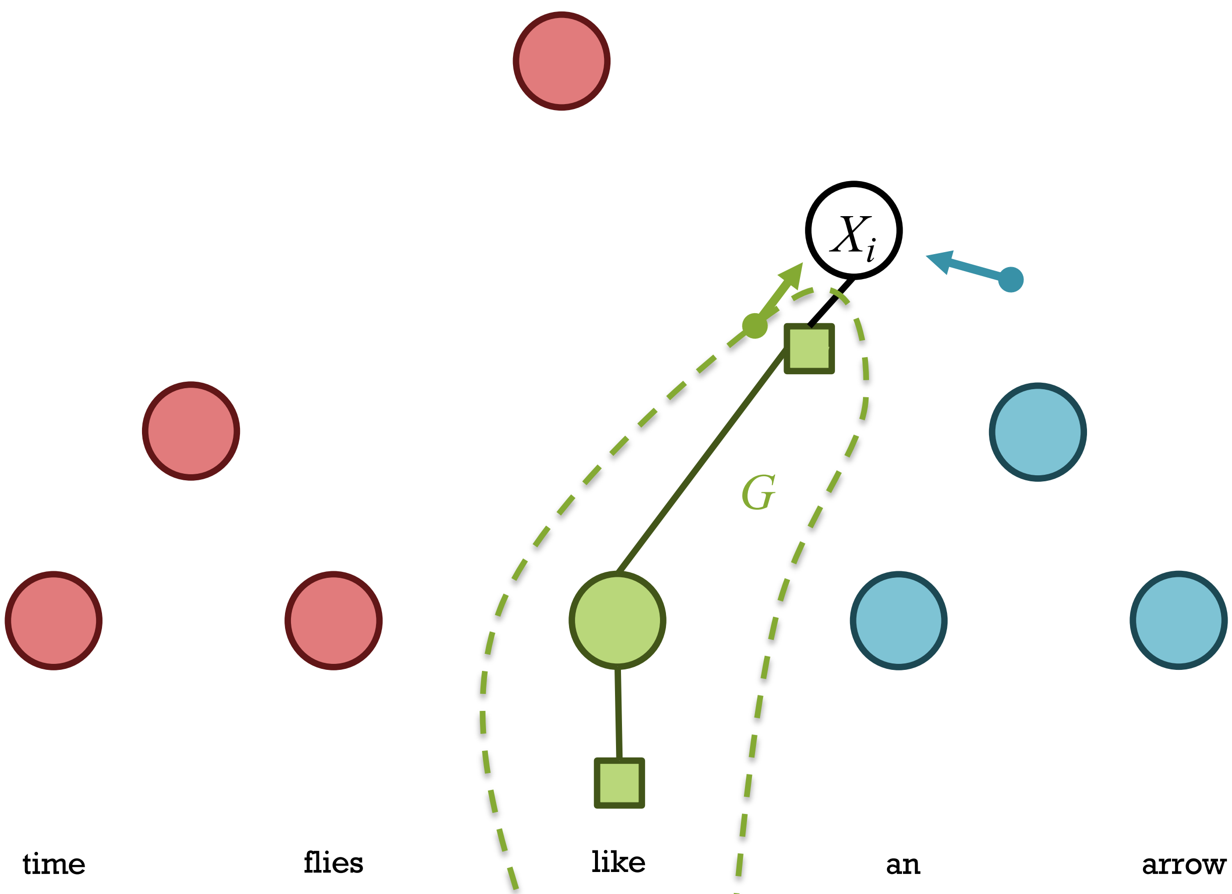

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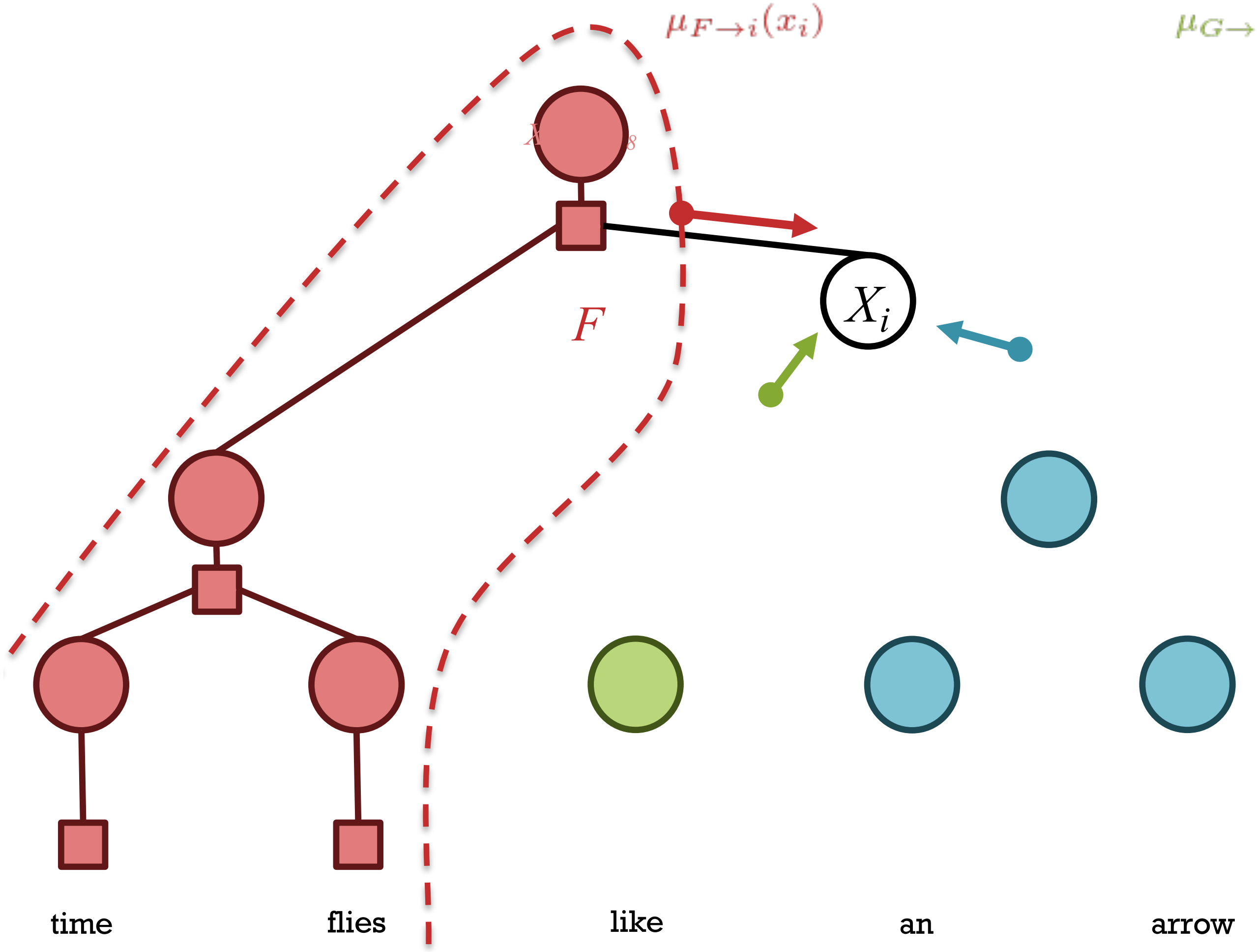

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Inference using just the factors in subgraph H

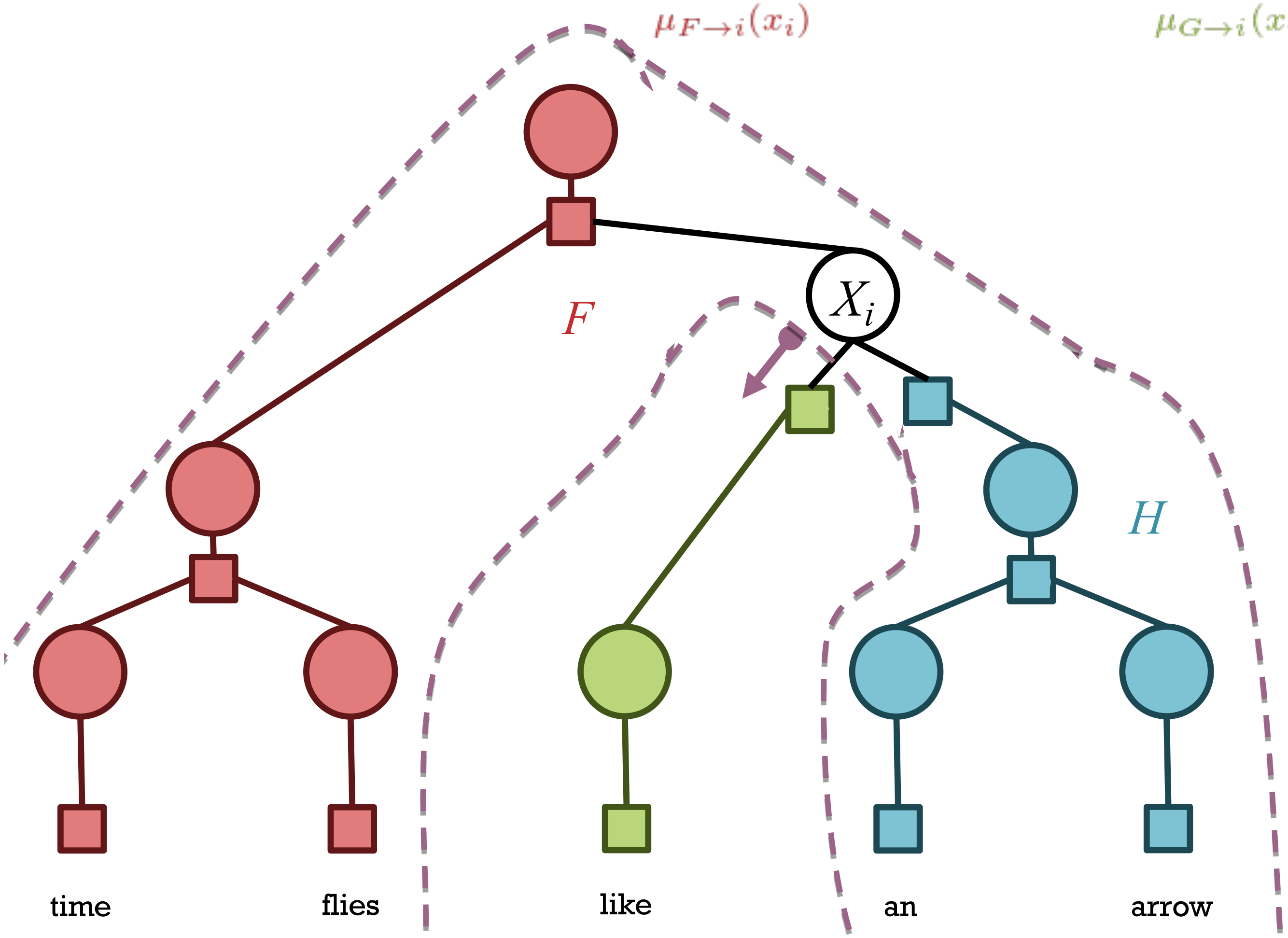
The marginal of X_i in that smaller model is the message sent to X_i from subgraph H

*Message **to** a variable*

Acyclic BP as Dynamic Programming

$$p(X_i = x_i) \propto b_i(x_i) = \sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha} \psi_\alpha(\boldsymbol{x}_\alpha)$$

$$= \underbrace{\left(\sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha \subseteq F} \psi_\alpha(\boldsymbol{x}_\alpha) \right)}_{\mu_{F \to i}(x_i)} \underbrace{\left(\sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha \subseteq G} \psi_\alpha(\boldsymbol{x}_\alpha) \right)}_{\mu_{G \to i}(x_i)} \underbrace{\left(\sum_{\boldsymbol{x}:\boldsymbol{x}[i]=x_i} \prod_{\alpha \subseteq H} \psi_\alpha(\boldsymbol{x}_\alpha) \right)}_{\mu_{H \to i}(x_i)}$$



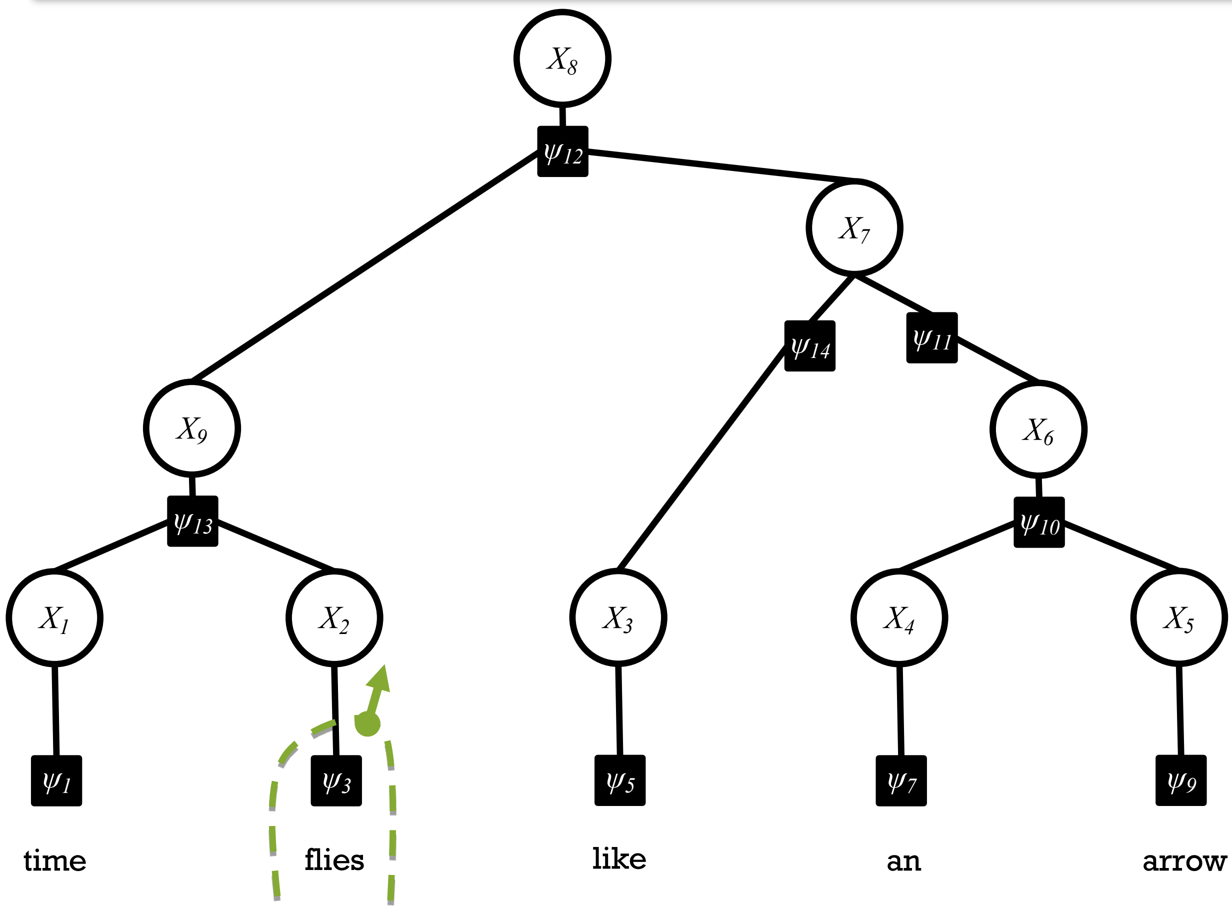
Subproblem:
Inference using just the factors in subgraph $F \cup H$

The marginal of X_i in that smaller model is the message sent by X_i out of subgraph $F \cup H$

Message ***from*** *a variable*

Acyclic BP as Dynamic Programming

- If you want the **marginal** $p_i(x_i)$ where X_i has degree k, you can think of that summation as a **product of k marginals** computed on smaller subgraphs.
- Each subgraph is obtained by **cutting** some edge of the tree.
- The message-passing algorithm uses **dynamic programming** to compute the marginals on all such subgraphs, working from **smaller to bigger**. So you can compute all the marginals.



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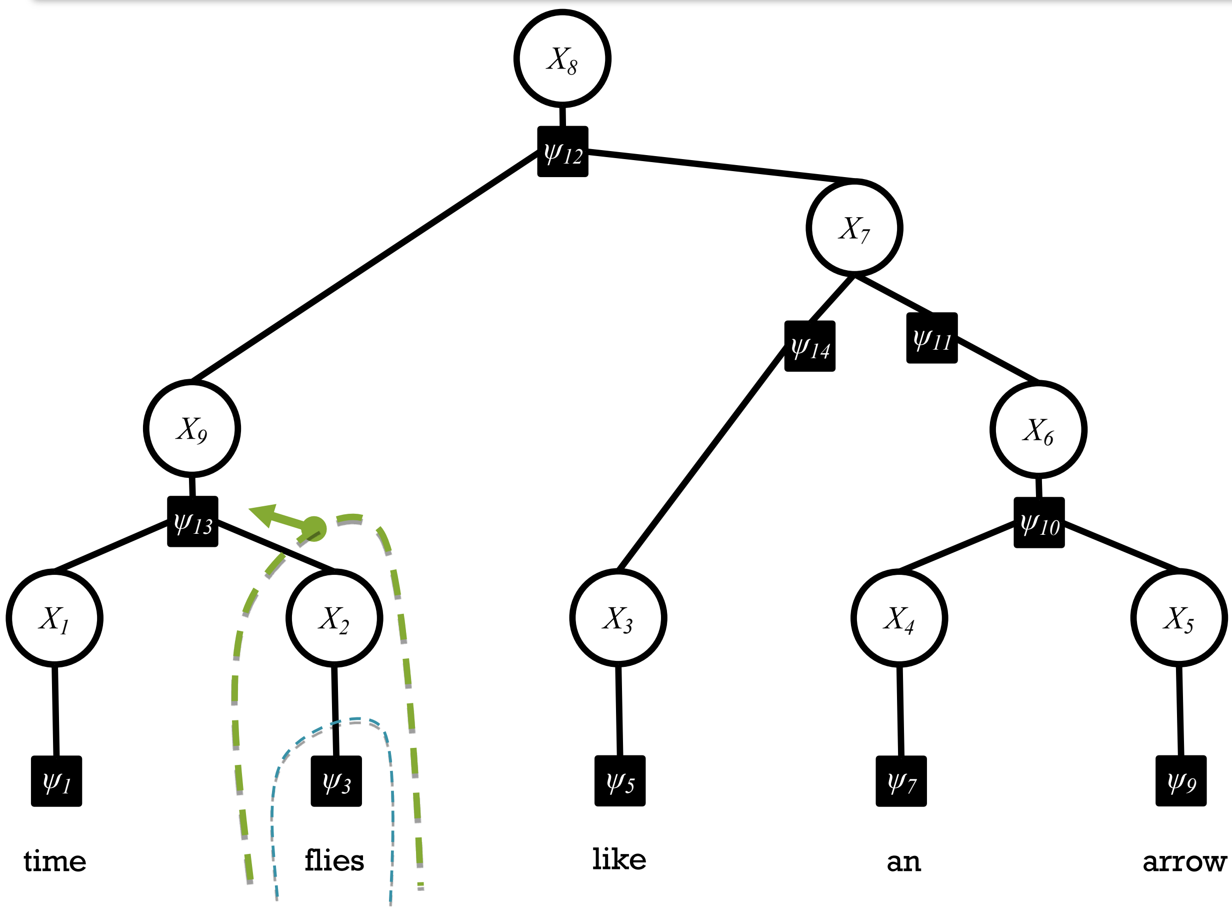

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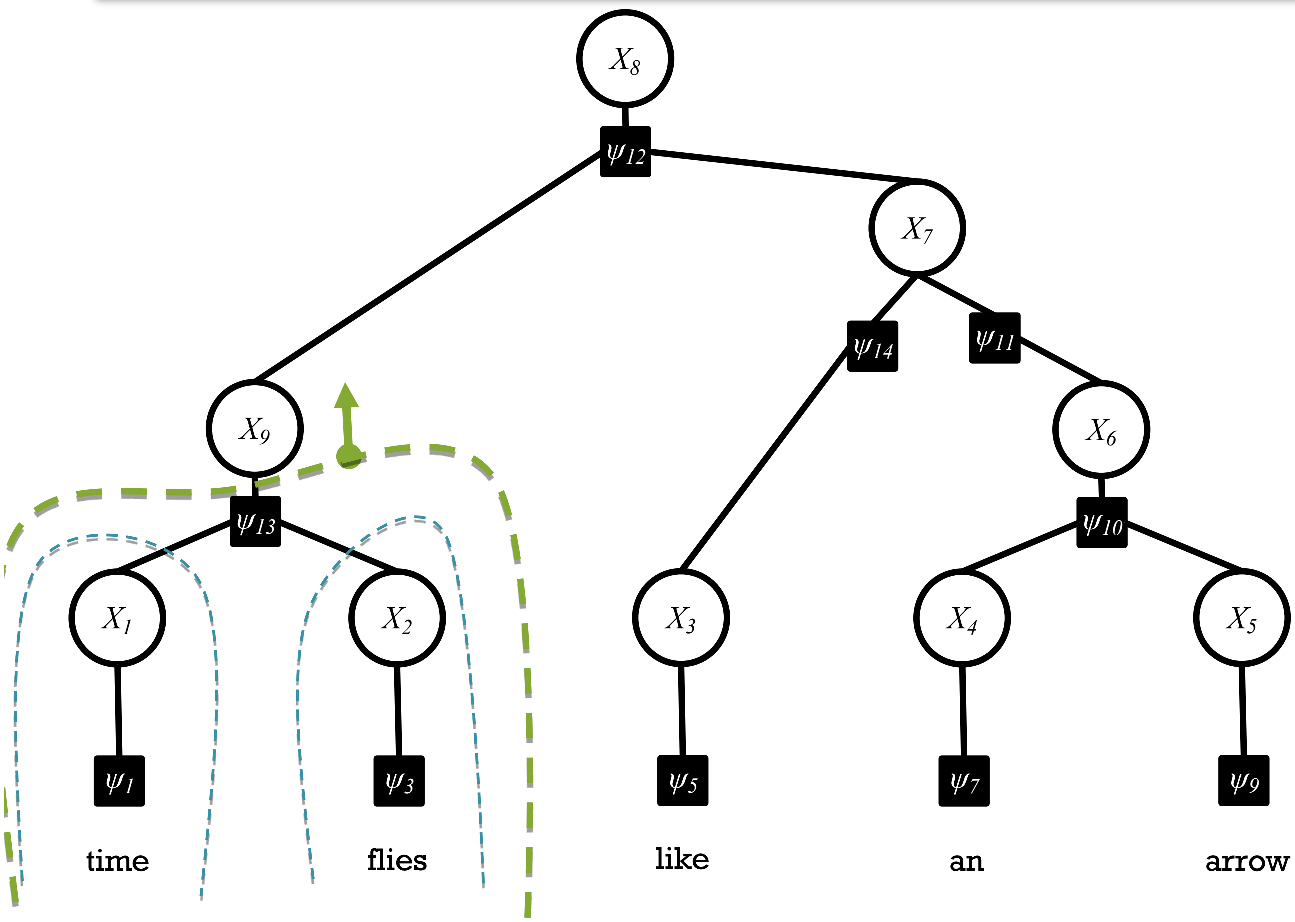

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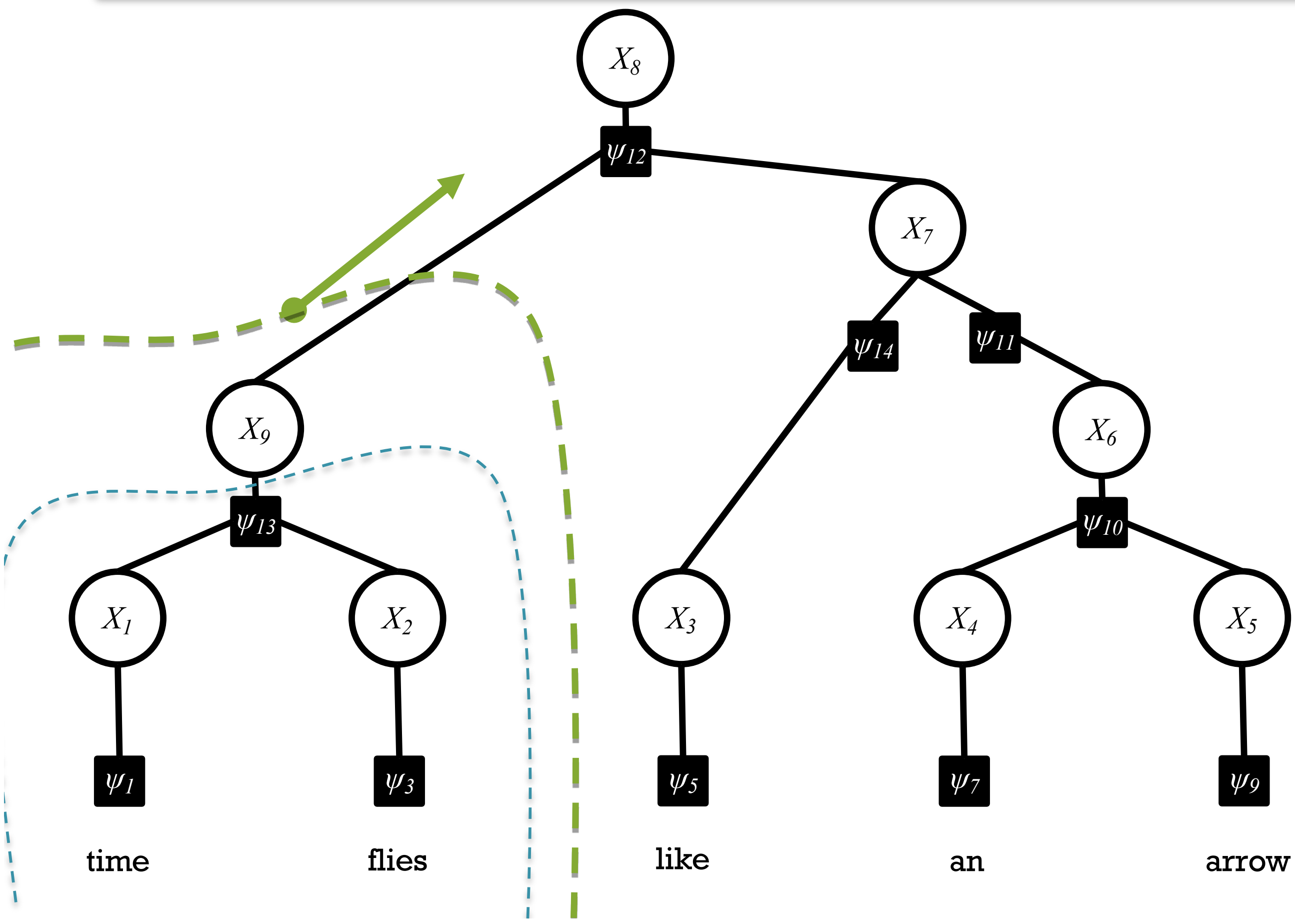

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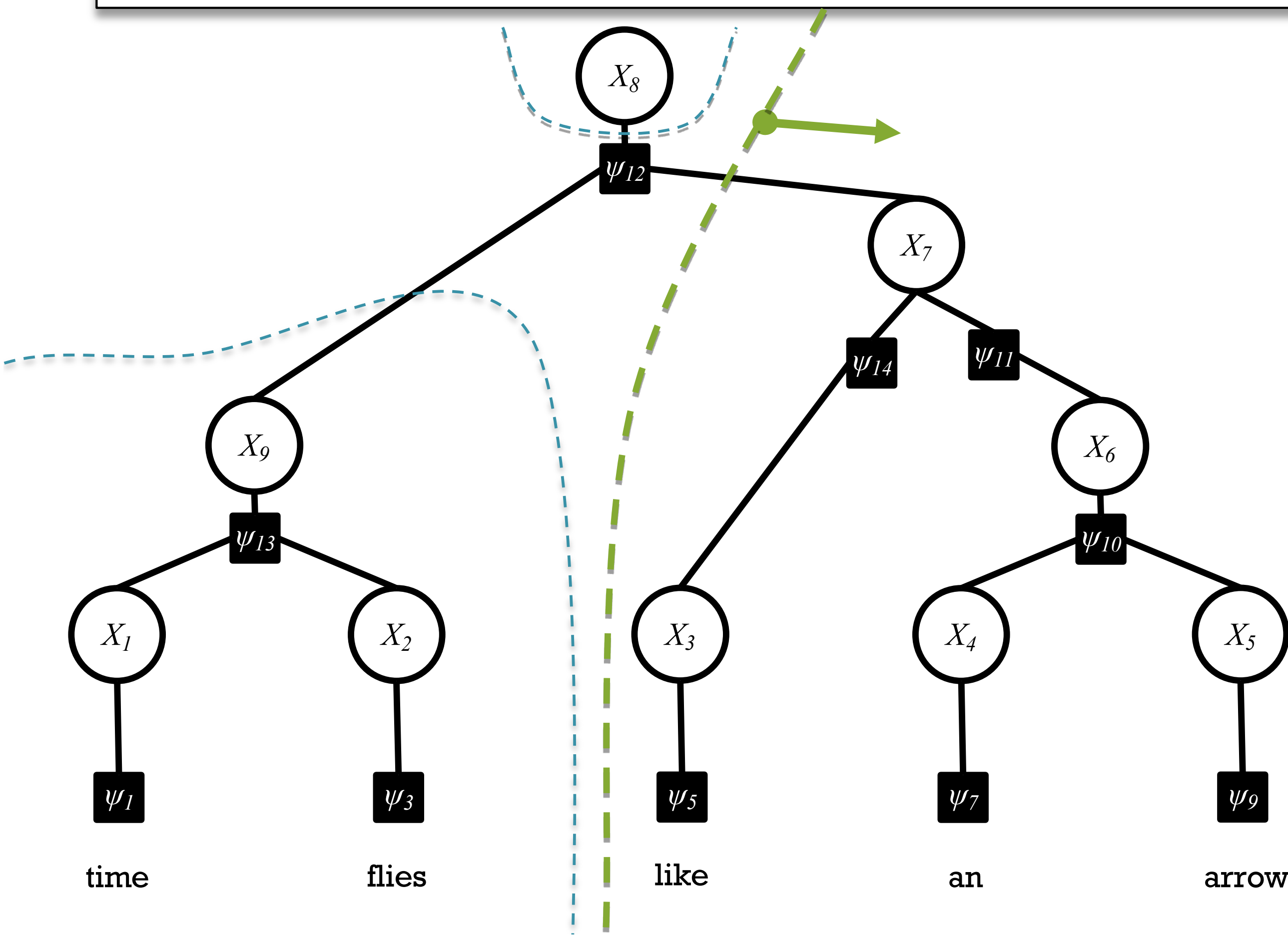

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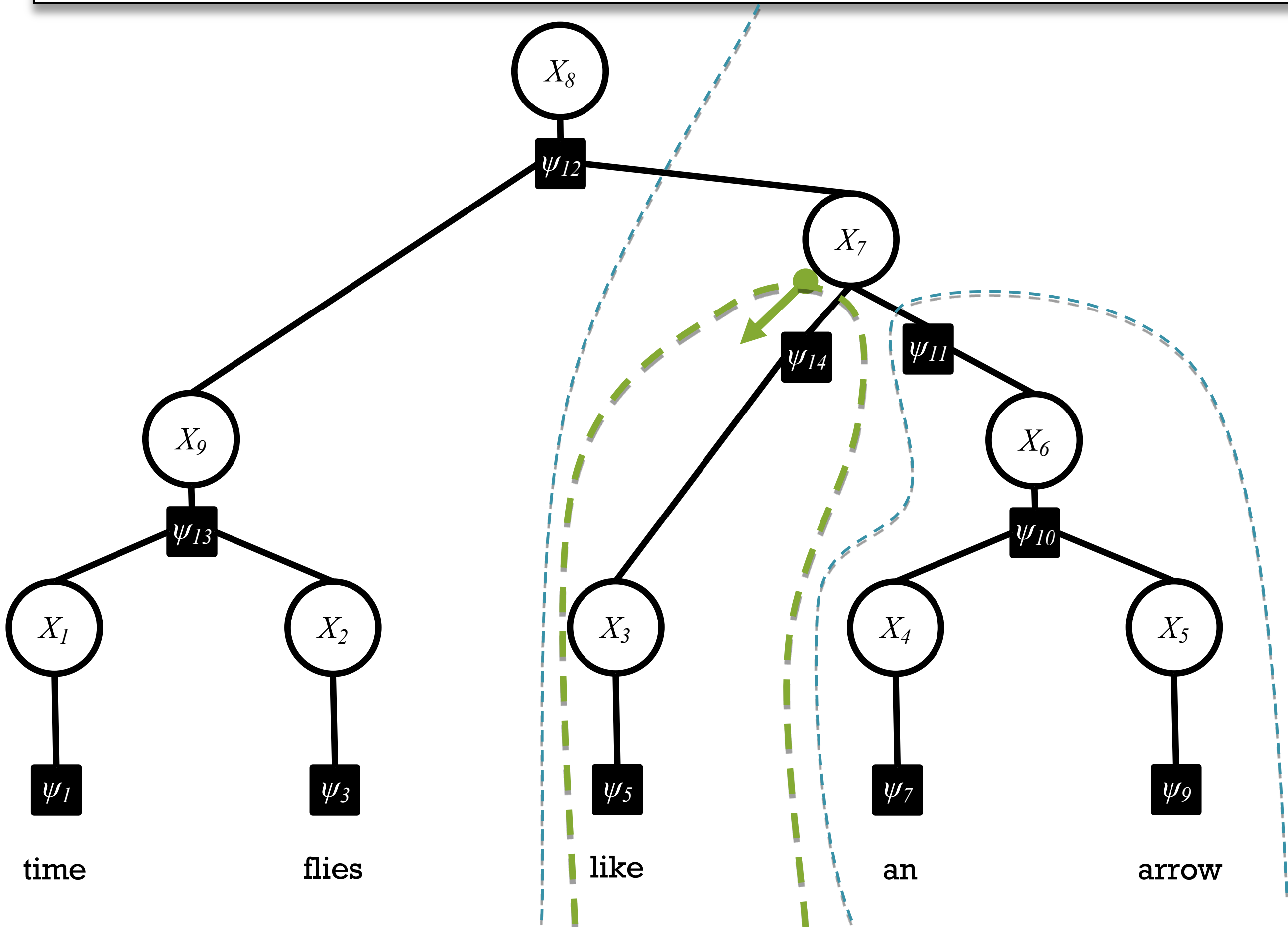

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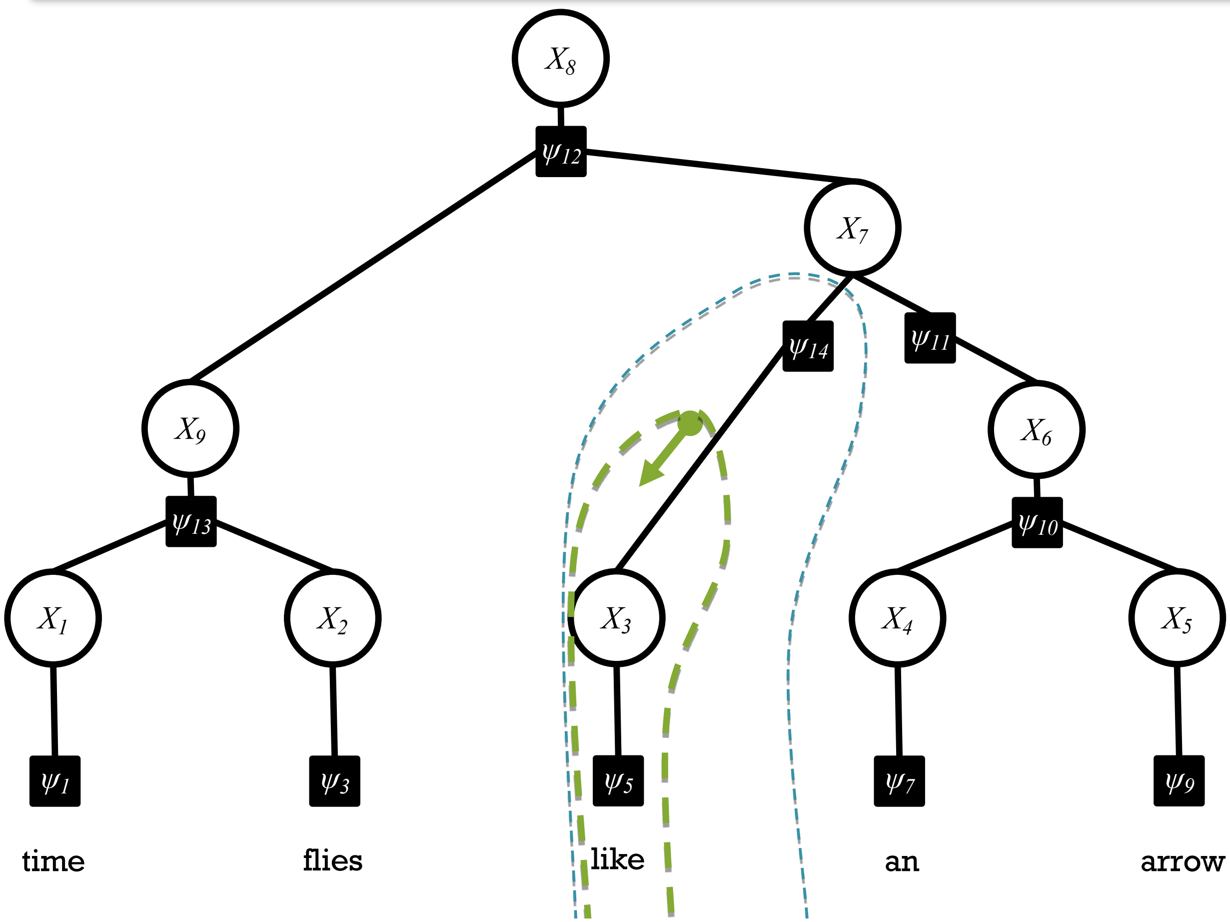

Exact MAP inference for factor trees

MAX-PRODUCT BELIEF PROPAGATION

Max-product Belief Propagation

- **Sum-product BP** can be used to
  compute the marginals, $p_i(X_i)$
  compute the partition function, Z

- **Max-product BP** can be used to
  compute the most likely assignment,
  $\boldsymbol{X}^* = \operatorname{argmax}_{\boldsymbol{X}} p(\boldsymbol{X})$

Max-product Belief Propagation

- Change the sum to a max:

$$\mu_{i \to \alpha}(x_i) = \prod_{\alpha \in \mathcal{N}(i) \setminus \alpha} \mu_{\alpha \to i}(x_i)$$

$$\mu_{\alpha \to i}(x_i) = \sum_{\boldsymbol{x}_{\boldsymbol{\alpha}} : \boldsymbol{x}_{\boldsymbol{\alpha}}[i] = x_i} \psi_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) \prod_{j \in \mathcal{N}(\alpha) \setminus i} \mu_{j \to \alpha}(\boldsymbol{x}_{\boldsymbol{\alpha}}[i])$$

- **Max-product BP** computes **max-marginals**
  - The max-marginal $b_i(x_i)$ is the (unnormalized) probability of the MAP assignment under the constraint $X_i = x_i$.
  - For an acyclic graph, the MAP assignment (assuming there are no ties) is given by:

$$x_i^* = \arg\max_{x_i} b_i(x_i)$$

Max-product Belief Propagation

- Change the sum to a max:

$$\mu_{i \to \alpha}(x_i) = \prod_{\alpha \in \mathcal{N}(i) \setminus \alpha} \mu_{\alpha \to i}(x_i)$$

$$\mu_{\alpha \to i}(x_i) = \max_{\boldsymbol{x}_{\boldsymbol{\alpha}} : \boldsymbol{x}_{\boldsymbol{\alpha}}[i] = x_i} \psi_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}}) \prod_{j \in \mathcal{N}(\alpha) \setminus i} \mu_{j \to \alpha}(\boldsymbol{x}_{\boldsymbol{\alpha}}[i])$$

- **Max-product BP** computes **max-marginals**
  - The max-marginal $b_i(x_i)$ is the (unnormalized) probability of the MAP assignment under the constraint $X_i = x_i$.
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$$x_i^* = \arg\max_{x_i} b_i(x_i)$$

Deterministic Annealing

Motivation: Smoothly transition from sum-product to max-product

1. Incorporate inverse temperature parameter into each factor:

Annealed Joint Distribution

$$p(\boldsymbol{x}) = \frac{1}{Z} \prod_{\alpha} \psi_\alpha(\boldsymbol{x_\alpha})^{\frac{1}{T}}$$

1. Send messages as usual for sum-product BP
2. Anneal T from 1 to 0:

$T = 1$	Sum-product
$T \rightarrow 0$	Max-product

3. Take resulting beliefs to power T

Semirings

- Sum-product +/* and max-product max/* are commutative semirings
- We can run BP with any such commutative semiring

$$\mu_{i\to\alpha}(x_i) = \prod_{\alpha \in \mathcal{N}(i)\setminus\alpha} \mu_{\alpha\to i}(x_i)$$

$$\mu_{\alpha\to i}(x_i) = \sum_{\boldsymbol{x_\alpha}:\boldsymbol{x_\alpha}[i]=x_i} \psi_\alpha(\boldsymbol{x_\alpha}) \prod_{j\in\mathcal{N}(\alpha)\setminus i} \mu_{j\to\alpha}(\boldsymbol{x_\alpha}[i])$$

- In practice, multiplying many small numbers together can yield underflow
  – instead of using +/*, we use log-add/+
  – Instead of using max/*, we use max/+

Exact inference for linear chain models

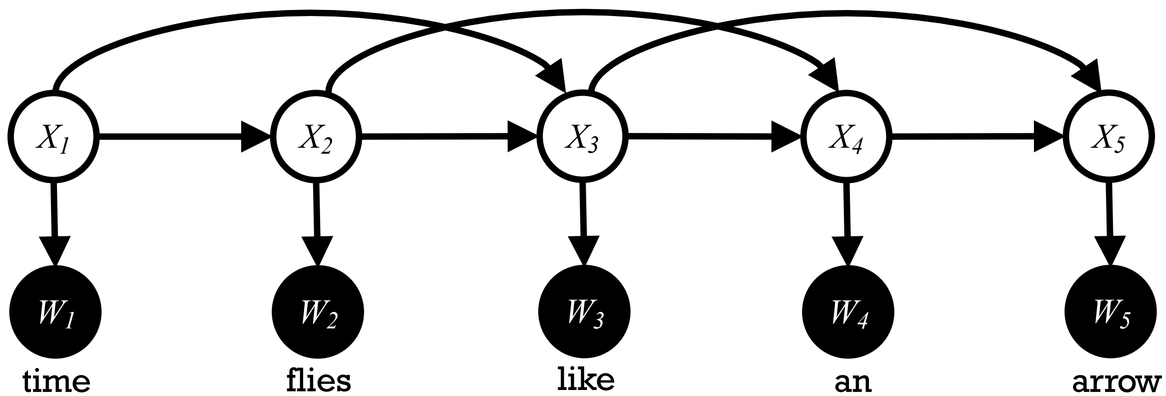
FORWARD-BACKWARD AND VITERBI ALGORITHMS

Forward-Backward Algorithm

- Sum-product BP on an HMM is called the **forward-backward algorithm**
- Max-product BP on an HMM is called the **Viterbi algorithm**

Forward-Backward Algorithm

Trigram HMM is not a tree, even when converted to a factor graph



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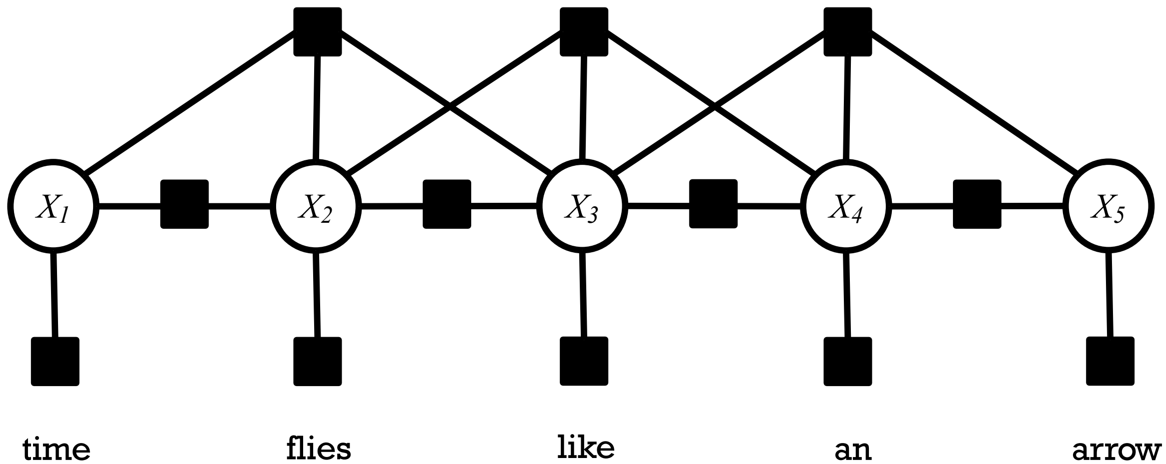

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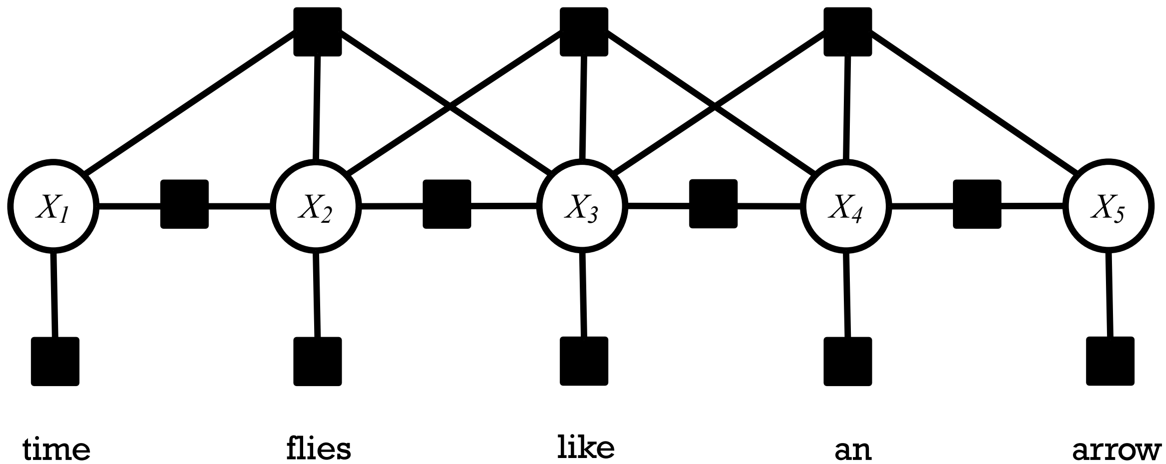


Trick: (See also Sha & Pereira (2003))

- Replace each variable domain with its cross product
  e.g. {B,I,O} → {BB, BI, BO, IB, II, IO, OB, OI, OO}
- Replace each pair of variables with a single one. For all i, $y_{i,i+1} = (x_i, x_{i+1})$
- Add features with weight -∞ that disallow illegal configurations between pairs of the new variables
  e.g. **legal** = BI and IO **illegal** = II and OO
- This is effectively a special case of the junction tree algorithm

Summary

1. **Factor Graphs**
   – Alternative representation of directed / undirected graphical models
   – Make the cliques of an undirected GM explicit
2. **Variable Elimination**
   – Simple and general approach to exact inference
   – Just a matter of being clever when computing sum-products
3. **Sum-product Belief Propagation**
   – Computes all the marginals and the partition function in only twice the work of Variable Elimination
4. **Max-product Belief Propagation**
   – Identical to sum-product BP, but changes the semiring
   – Computes: max-marginals, probability of MAP assignment, and (with backpointers) the MAP assignment itself.

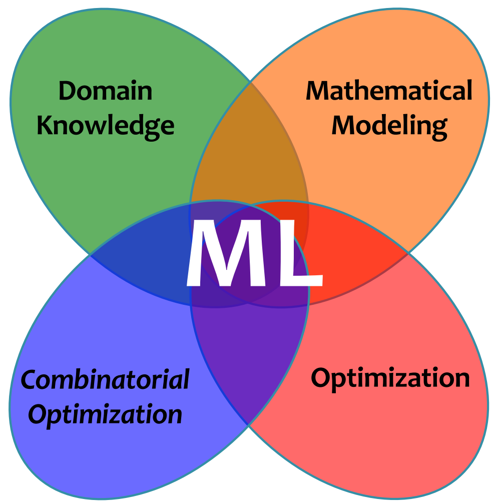
LEARNING FOR MRFS

Machine Learning

The **data** inspires the structures we want to predict

Our **model** defines a score for each structure

It also tells us what to optimize



Learning tunes the parameters of the model

Inference finds {best structure, marginals, partition function} for a new observation

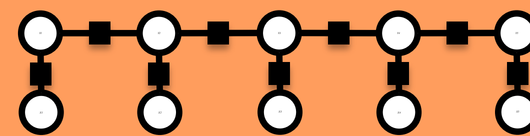
(**Inference** is usually called as a subroutine in learning)

1. Data

$$\mathcal{D} = \{\boldsymbol{x}^{(n)}\}_{n=1}^{N}$$

2. Model

$$p(\boldsymbol{x} \mid \boldsymbol{\theta}) = \frac{1}{Z(\boldsymbol{\theta})} \prod_{C \in \mathcal{C}} \psi_C(\boldsymbol{x}_C)$$



3. Objective

$$\ell(\theta; \mathcal{D}) = \sum_{n=1}^{N} \log p(\boldsymbol{x}^{(n)} \mid \boldsymbol{\theta})$$

5. Inference

1. Marginal Inference

$$p(\boldsymbol{x}_C) = \sum_{\boldsymbol{x}' : \boldsymbol{x}'_C = \boldsymbol{x}_C} p(\boldsymbol{x}' \mid \boldsymbol{\theta})$$

2. Partition Function

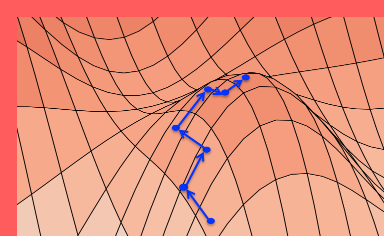
$$Z(\boldsymbol{\theta}) = \sum_{\boldsymbol{x}} \prod_{C \in \mathcal{C}} \psi_C(\boldsymbol{x}_C)$$

3. MAP Inference

$$\hat{\boldsymbol{x}} = \operatorname*{argmax}_{\boldsymbol{x}} \; p(\boldsymbol{x} \mid \boldsymbol{\theta})$$

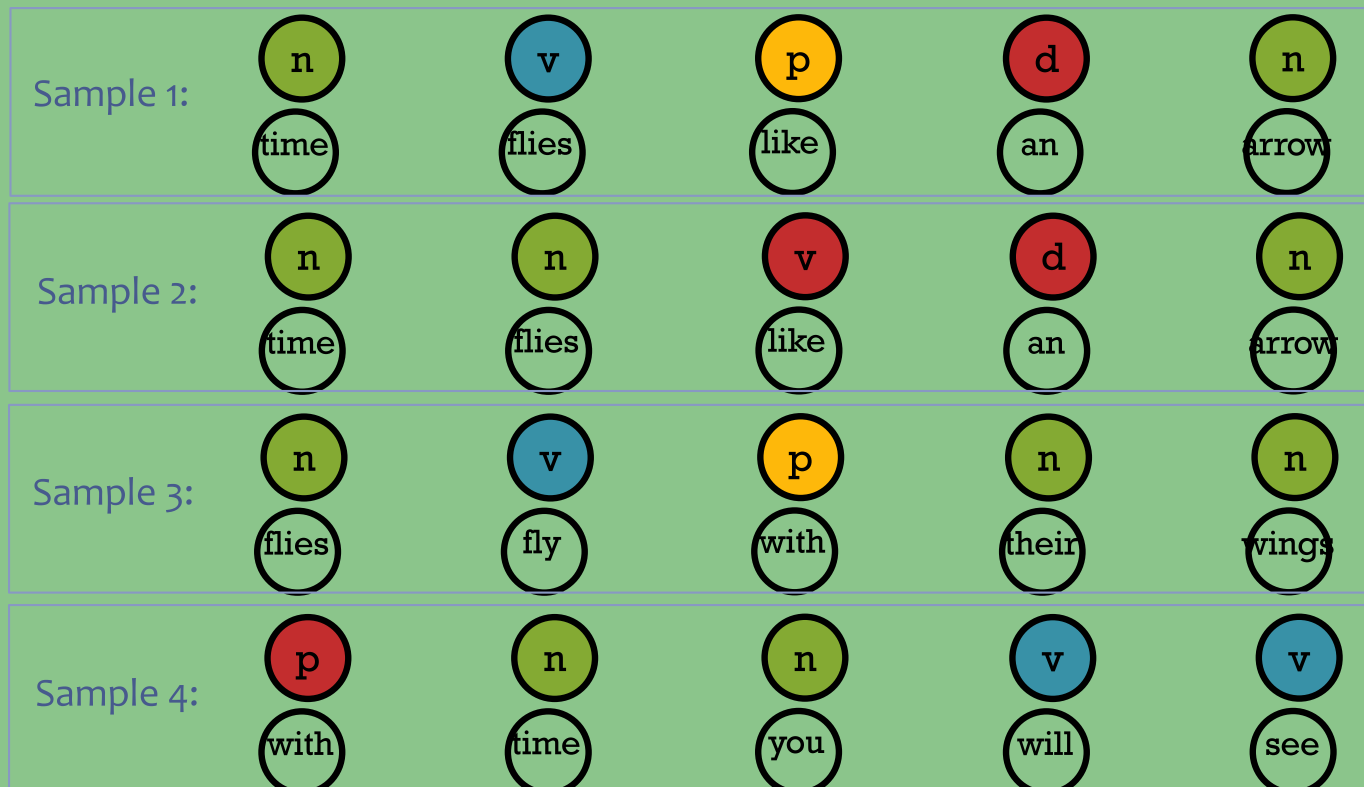
4. Learning

$$\boldsymbol{\theta}^* = \operatorname*{argmax}_{\boldsymbol{\theta}} \ell(\boldsymbol{\theta}; \mathcal{D})$$

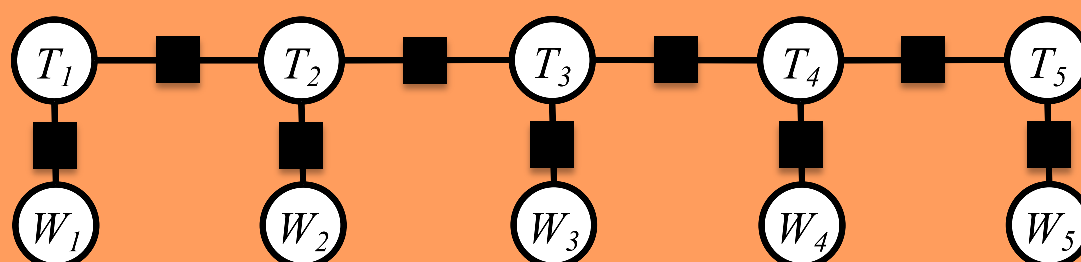


1. Data

Given training examples: $\mathcal{D} = \{\boldsymbol{x}^{(n)}\}_{n=1}^{N}$



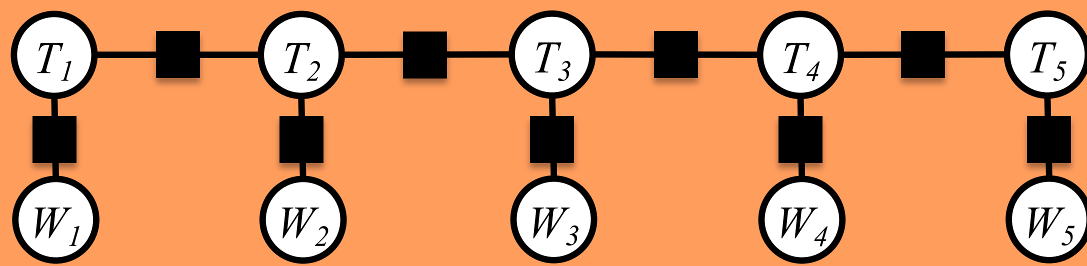
2. Model



2. Model

Define the model to be an MRF:

$$p(\boldsymbol{x} \mid \boldsymbol{\theta}) = \frac{1}{Z(\boldsymbol{\theta})} \prod_{C \in \mathcal{C}} \psi_C(\boldsymbol{x}_C)$$



3. Objective

Choose the objective to be log-likelihood:

(Assign high probability to the things we observe and low probability to everything else)

$$\ell(\theta; \mathcal{D}) = \sum_{n=1}^{N} \log p(\boldsymbol{x}^{(n)} \mid \boldsymbol{\theta})$$

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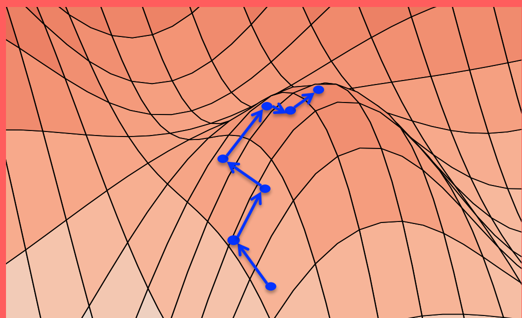
4. Learning

Tune the parameters to maximize the objective function

$$\boldsymbol{\theta}^* = \operatorname*{argmax}_{\boldsymbol{\theta}} \ell(\boldsymbol{\theta}; \mathcal{D})$$



3. Objective

Choose the objective to be log-likelihood:

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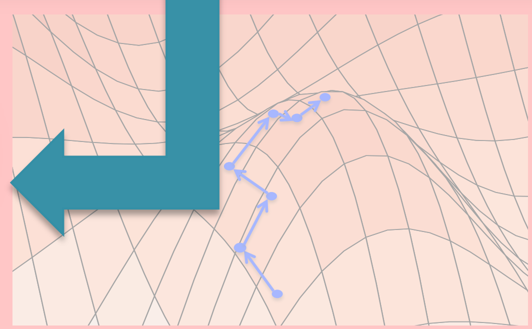
Goals for Today's Lecture

1. Consider different parameterizations
2. Optimize this objective function

Tune the parameters to optimize the objective function

$$\theta^* = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, \ell(\boldsymbol{\theta}; \mathcal{D})$$



5. Inference

Three Tasks:

1. Marginal Inference

Compute marginals of variables and cliques

$$p(x_i) = \sum_{\boldsymbol{x}' : x'_i = x_i} p(\boldsymbol{x}' \mid \boldsymbol{\theta}) \quad \Bigg| \quad p(\boldsymbol{x}_C) = \sum_{\boldsymbol{x}' : \boldsymbol{x}'_C = \boldsymbol{x}_C} p(\boldsymbol{x}' \mid \boldsymbol{\theta})$$

2. Partition Function

Compute the normalization constant

$$Z(\boldsymbol{\theta}) = \sum_{\boldsymbol{x}} \prod_{C \in \mathcal{C}} \psi_C(\boldsymbol{x}_C)$$

3. MAP Inference

Compute variable assignment with highest probability

$$\hat{\boldsymbol{x}} = \operatorname*{argmax}_{\boldsymbol{x}} \; p(\boldsymbol{x} \mid \boldsymbol{\theta})$$

1. Data

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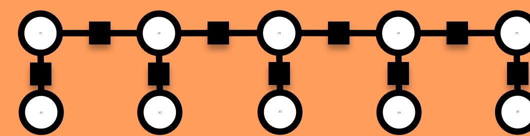

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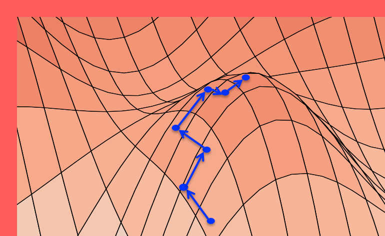


MLE for Undirected GMs

- Today's parameter estimation assumptions:
  1. The graphical model structure is given
  2. Every variable appears in the training examples

Questions

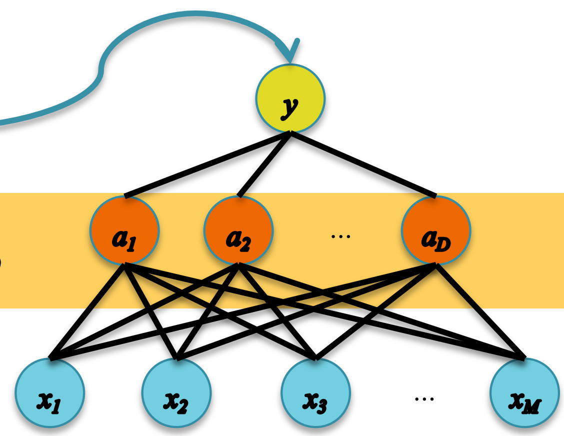
1. What does the **likelihood objective** accomplish?
2. Is likelihood the ***right*** **objective** function?
3. **How do we optimize** the objective function (i.e. learn)?
4. What **guarantees** does the optimizer provide?
5. (What is the **mapping from data → model**? In what ways can we incorporate our domain knowledge? How does this impact learning?)

Options for MLE of MRFs

- **Setting I:** $\psi_C(\boldsymbol{x}_C) = \theta_{C,\boldsymbol{x}_C}$
  A. MLE by inspection (Decomposable Models)
  B. Iterative Proportional Fitting (IPF)

- **Setting II:** $\psi_C(\boldsymbol{x}_C) = \exp(\boldsymbol{\theta} \cdot \boldsymbol{f}(\boldsymbol{x}_C))$
  C. Generalized Iterative Scaling
  D. Gradient-based Methods

- **Setting III:** $\psi_C(\boldsymbol{x}_C) =$
  E. Gradient-based Methods



LOG-LINEAR PARAMETERIZATION OF CONDITIONAL RANDOM FIELD

Options for MLE of MRFs

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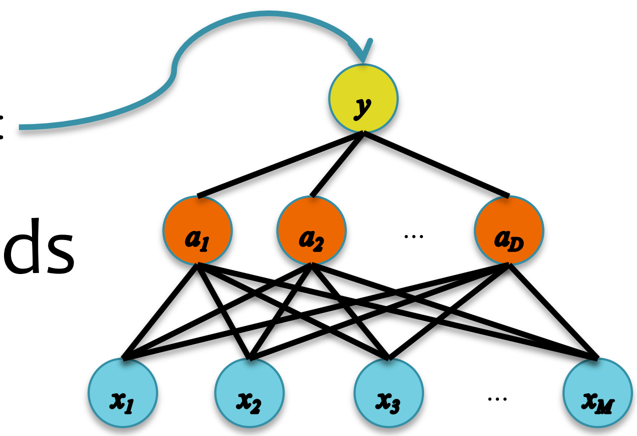
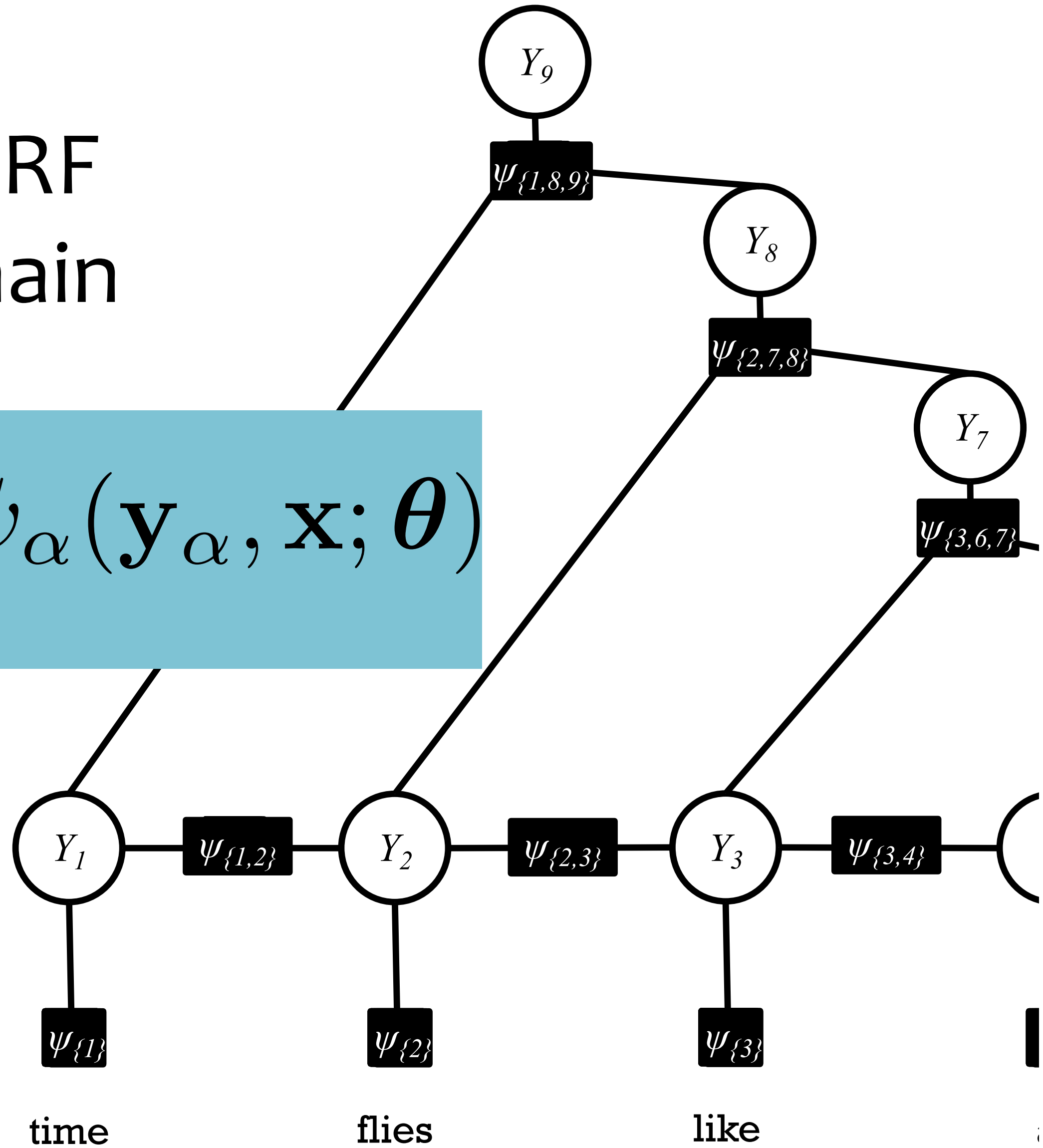


General CRF

The topology of the graphical model for a CRF doesn't have to be a chain

$$p_{\boldsymbol{\theta}}(\mathbf{y}|\mathbf{x}) = \frac{1}{Z(\mathbf{x})} \prod_{\alpha} \psi_{\alpha}(\mathbf{y}_{\alpha}, \mathbf{x}; \boldsymbol{\theta})$$



Log-linear CRF Parameterization

$$p_{\boldsymbol{\theta}}(\mathbf{y}|\mathbf{x}) = \frac{1}{Z(\mathbf{x})} \prod_{\alpha} \psi_{\alpha}(\mathbf{y}_{\alpha}, \mathbf{x}; \boldsymbol{\theta})$$

Define each potential function in terms of a fixed set of feature functions:

$$\psi_{\alpha}(\mathbf{y}_{\alpha}, \mathbf{x}; \boldsymbol{\theta}) = \exp(\boldsymbol{\theta} \cdot \mathbf{f}_{\alpha}(\mathbf{y}_{\alpha}, \mathbf{x}))$$

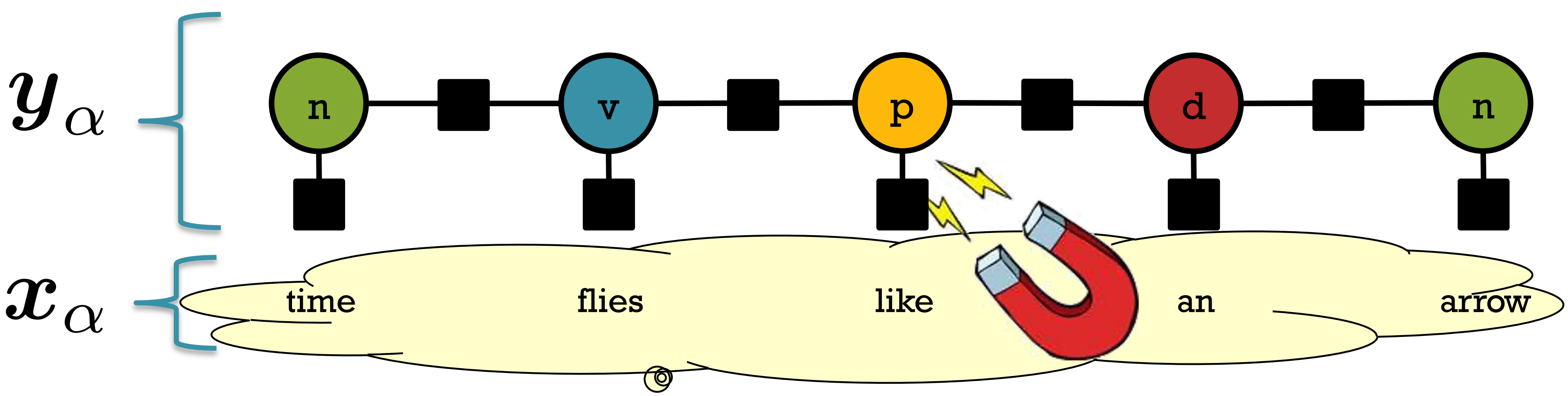
↑ Predicted variables

↑ Observed variables

Log-linear CRF Parameterization

Define each potential function in terms of a fixed set of feature functions:

$$\psi_\alpha(\mathbf{y}_\alpha, \mathbf{x}; \boldsymbol{\theta}) = \exp(\boldsymbol{\theta} \cdot \mathbf{f}_\alpha(\mathbf{y}_\alpha, \mathbf{x}))$$



Log-linear CRF Parameterization

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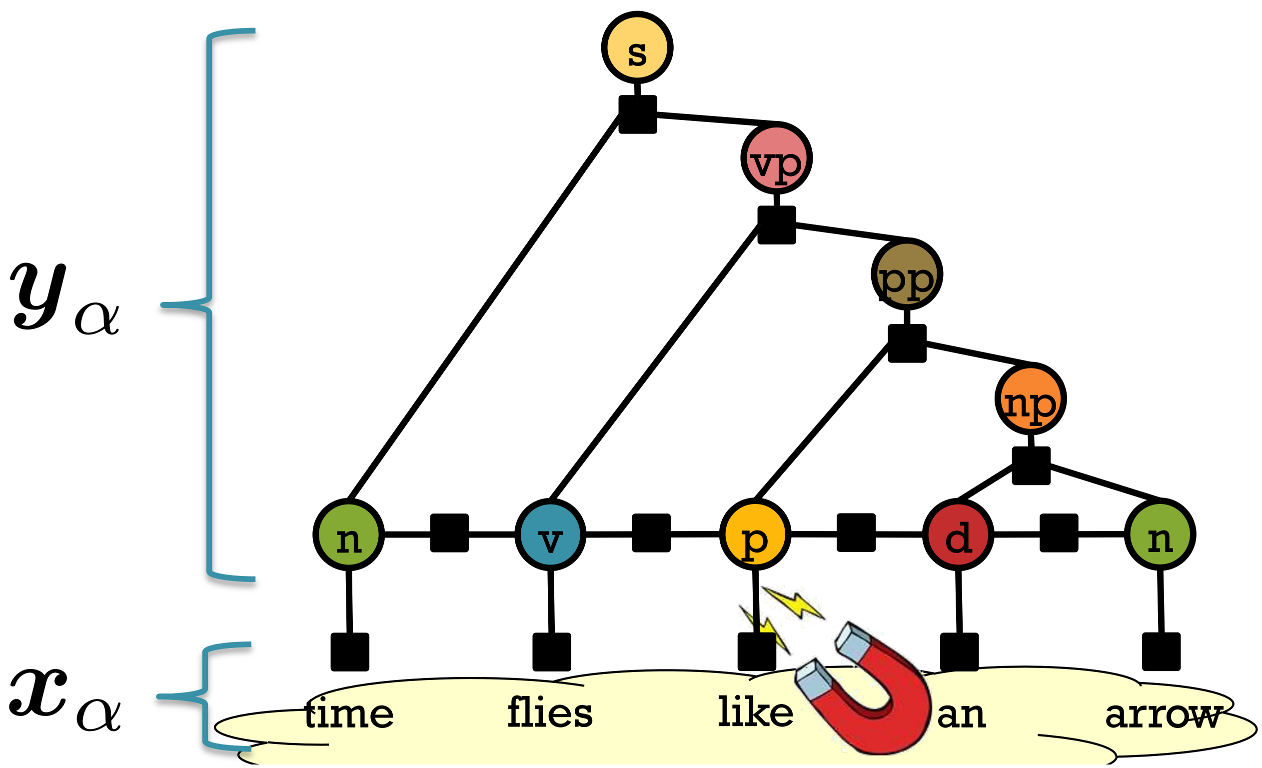


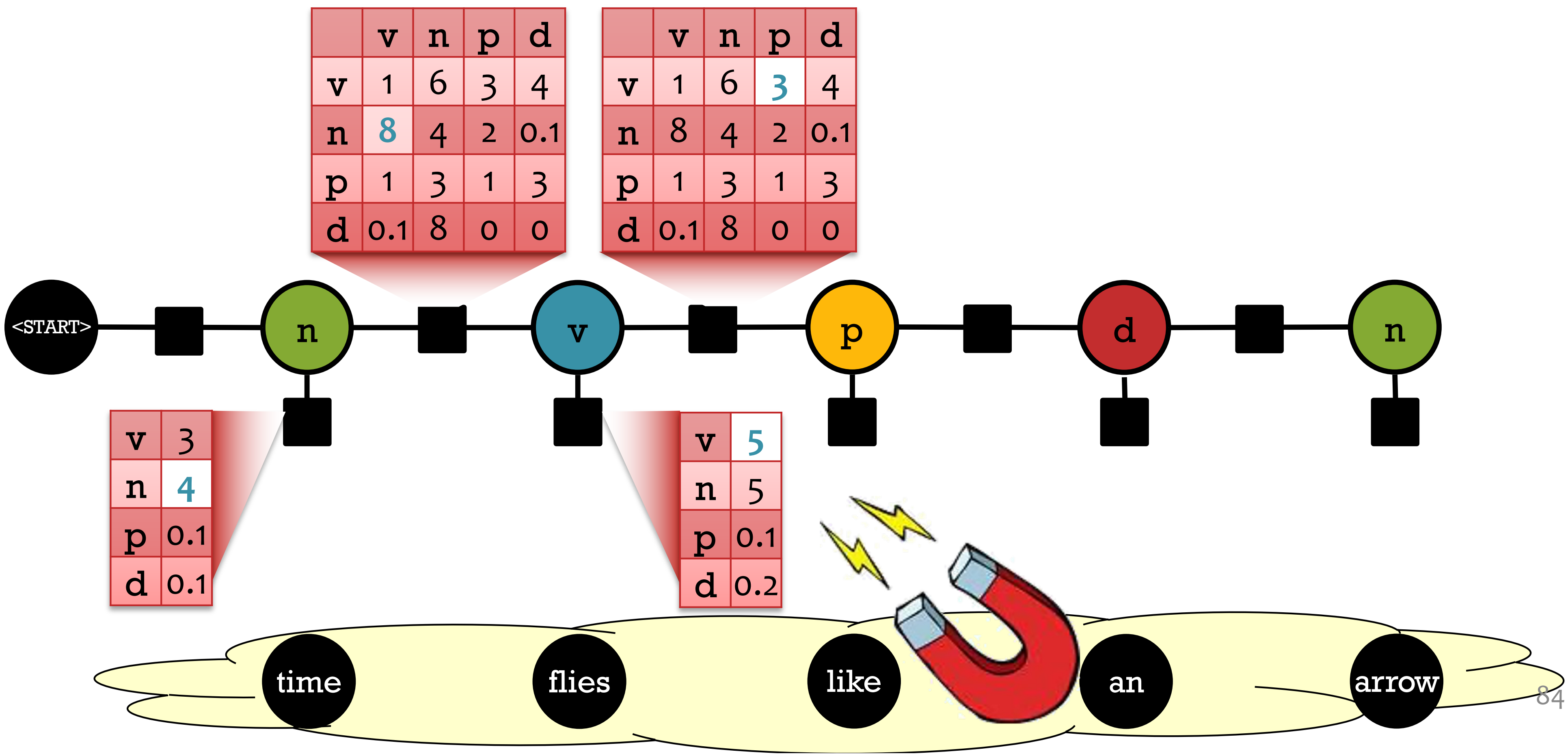
Conditional Random Fields (CRFs) for time series data

LINEAR-CHAIN CRFS (LOG-LINEAR PARAMETERIZATION)

Conditional Random Field (CRF)

Conditional distribution over tags X_i given words w_i.
The factors and Z are now specific to the sentence w.

$$p(\text{n, v, p, d, n} \mid \text{time, flies, like, an, arrow}) = \frac{1}{Z}(4 * 8 * 5 * 3 * \ldots)$$



	v	n	p	d
v	1	6	3	4
n	8	4	2	0.1
p	1	3	1	3
d	0.1	8	0	0

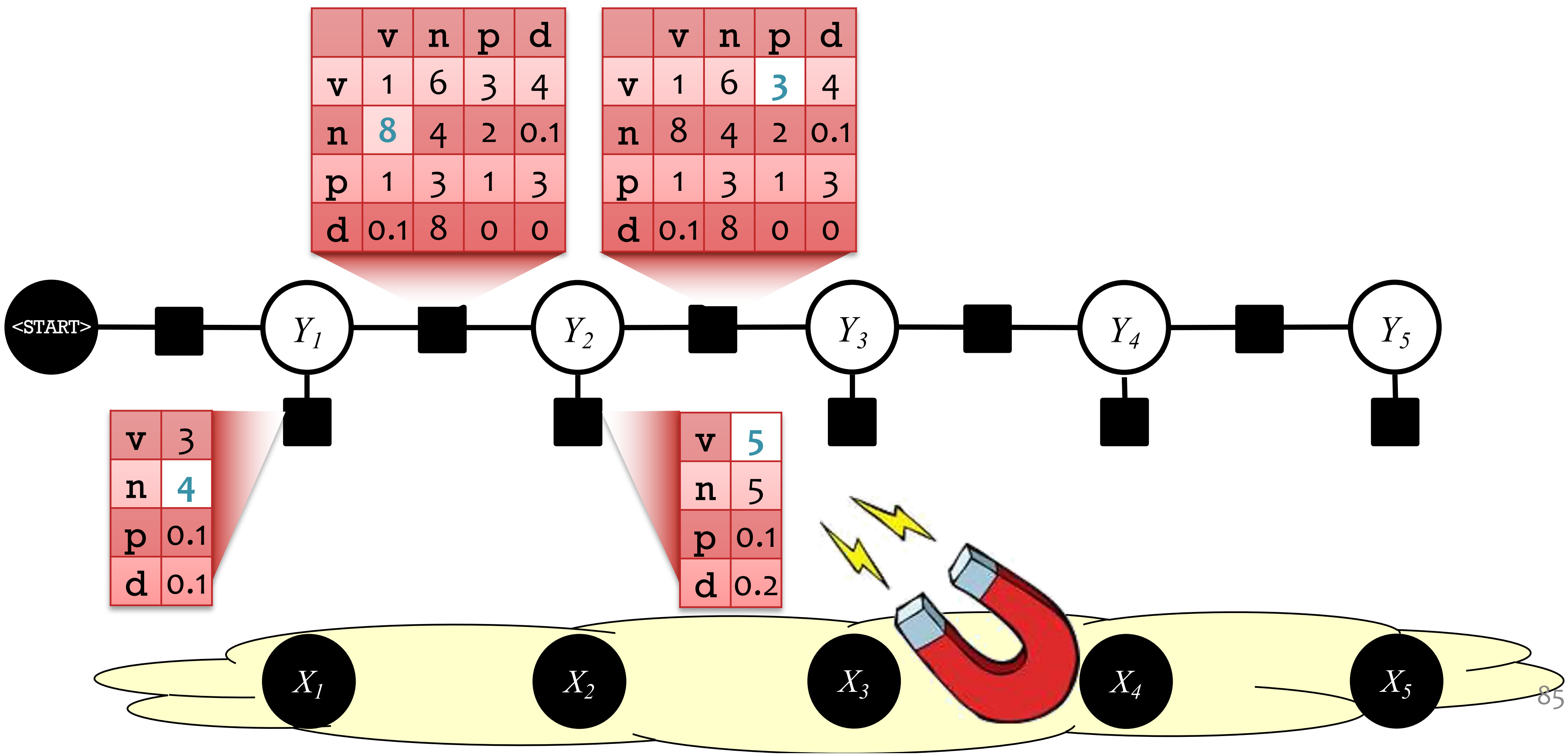
	v	n	p	d
v	1	6	3	4
n	8	4	2	0.1
p	1	3	1	3
d	0.1	8	0	0

v	3
n	4
p	0.1
d	0.1

v	5
n	5
p	0.1
d	0.2

Conditional Random Field (CRF)

Recall: Shaded nodes in a graphical model are **observed**



	v	n	p	d
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	v	n	p	d
v	1	6	3	4
n	8	4	2	0.1
p	1	3	1	3
d	0.1	8	0	0

v	3
n	4
p	0.1
d	0.1

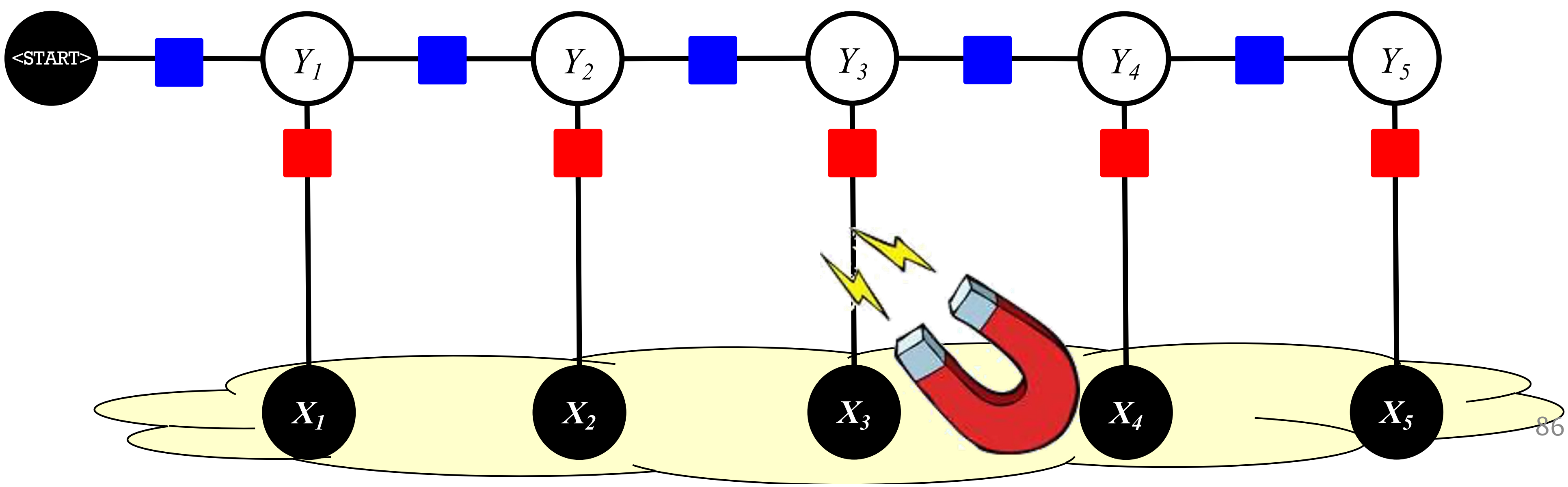
v	5
n	5
p	0.1
d	0.2

Conditional Random Field (CRF)

This **linear-chain CRF** is just **like an HMM,** except that its factors are **not** necessarily probability distributions

$$p(\mathbf{y}|\mathbf{x}) = \frac{1}{Z(\mathbf{x})} \prod_{k=1}^{K} \psi_{\text{em}}(y_k, x_k) \psi_{\text{tr}}(y_k, y_{k-1})$$

$$= \frac{1}{Z(\mathbf{x})} \prod_{k=1}^{K} \exp(\boldsymbol{\theta} \cdot \mathbf{f}_{\text{em}}(y_k, x_k)) \exp(\boldsymbol{\theta} \cdot \mathbf{f}_{\text{tr}}(y_k, y_{k-1}))$$



Exercise

$$p(\mathbf{y}|\mathbf{x}) = \frac{1}{Z(\mathbf{x})} \prod_{k=1}^{K} \psi_{\text{em}}(y_k, x_k) \psi_{\text{tr}}(y_k, y_{k-1})$$

$$= \frac{1}{Z(\mathbf{x})} \prod_{k=1}^{K} \exp(\boldsymbol{\theta} \cdot \mathbf{f}_{\text{em}}(y_k, x_k)) \exp(\boldsymbol{\theta} \cdot \mathbf{f}_{\text{tr}}(y_k, y_{k-1}))$$

Select All That Apply: Which model does the above distribution share the most in common with?

A. Hidden Markov Model
B. Bernoulli Naïve Bayes
C. Gaussian Naïve Bayes
D. Logistic Regression

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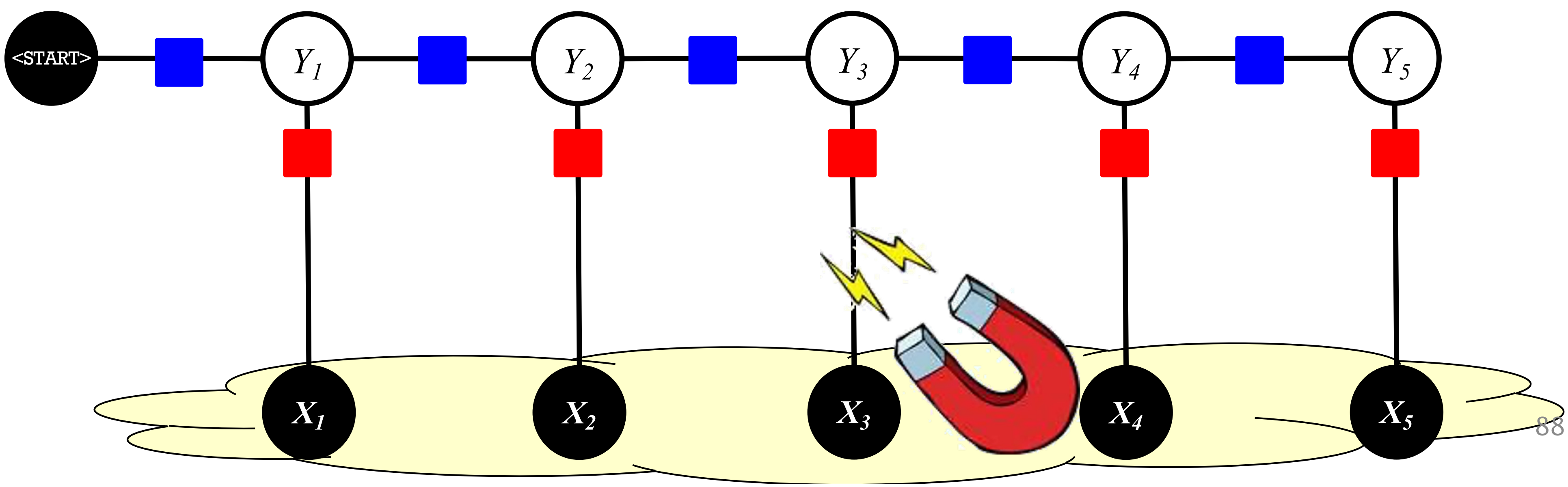
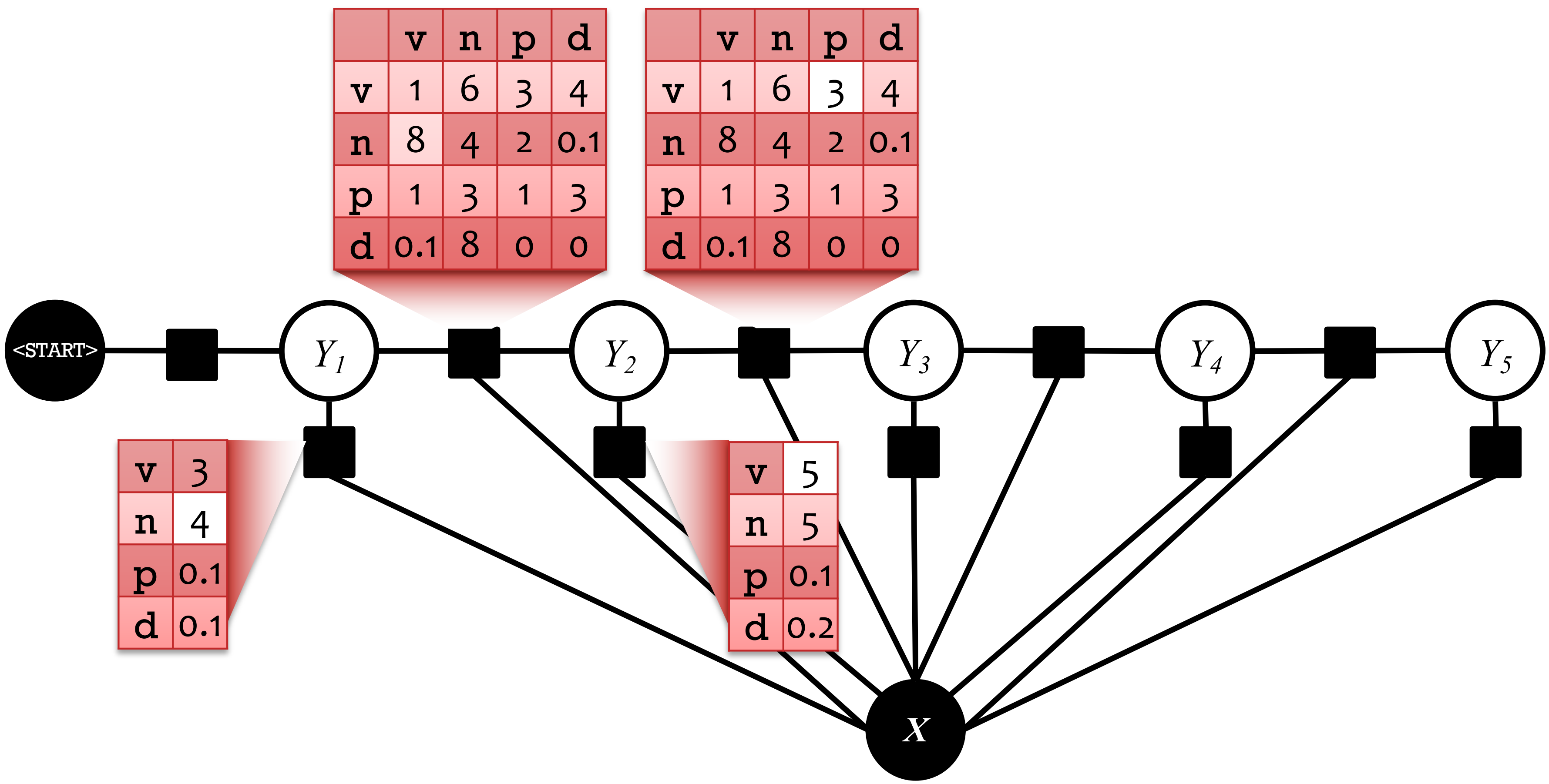


Conditional Random Field (CRF)

- That is the **vector** X
- Because it's observed, we can condition on it for free
- Conditioning is how we converted from the MRF to the CRF (i.e. when taking a slice of the emission factors)



	v	n	p	d
v	1	6	3	4
n	8	4	2	0.1
p	1	3	1	3
d	0.1	8	0	0

	v	n	p	d
v	1	6	3	4
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v	3
n	4
p	0.1
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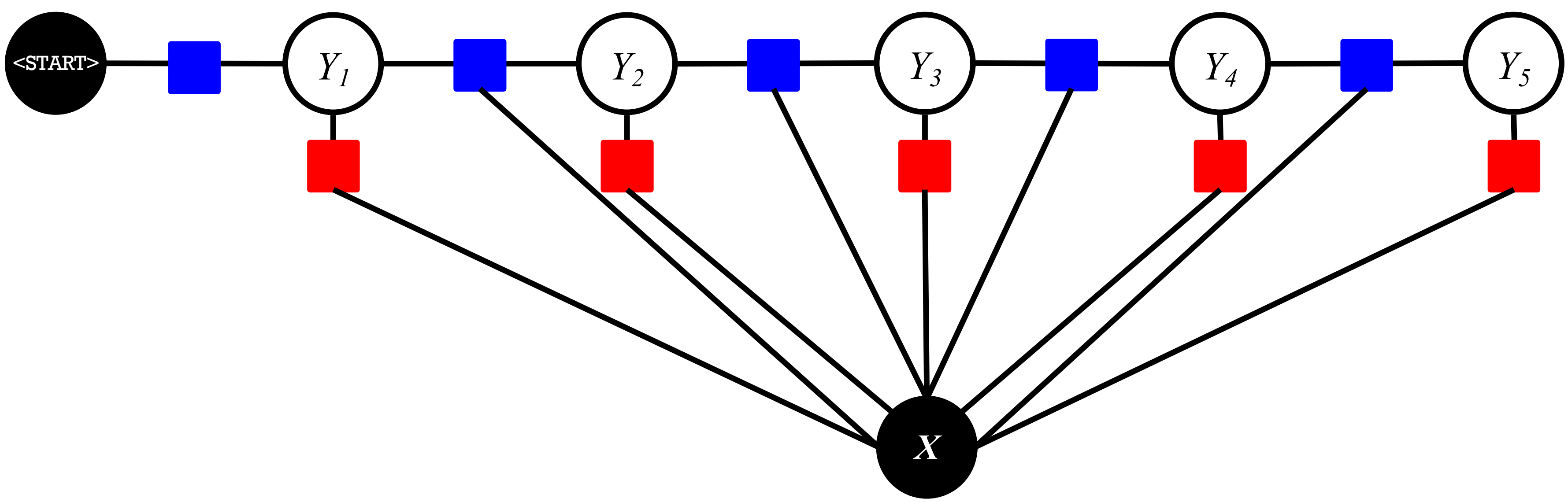
n	5
p	0.1
d	0.2

Conditional Random Field (CRF)

- This is the **standard** linear-chain CRF definition
- It permits rich, overlapping features of the vector $\boldsymbol{X}$

$$p(\mathbf{y}|\mathbf{x}) = \frac{1}{Z(\mathbf{x})} \prod_{k=1}^{K} \psi_{\text{em}}(y_k, \mathbf{x}) \psi_{\text{tr}}(y_k, y_{k-1}, \mathbf{x})$$

$$= \frac{1}{Z(\mathbf{x})} \prod_{k=1}^{K} \exp(\boldsymbol{\theta} \cdot \mathbf{f}_{\text{em}}(y_k, \mathbf{x})) \exp(\boldsymbol{\theta} \cdot \mathbf{f}_{\text{tr}}(y_k, y_{k-1}, \mathbf{x}))$$



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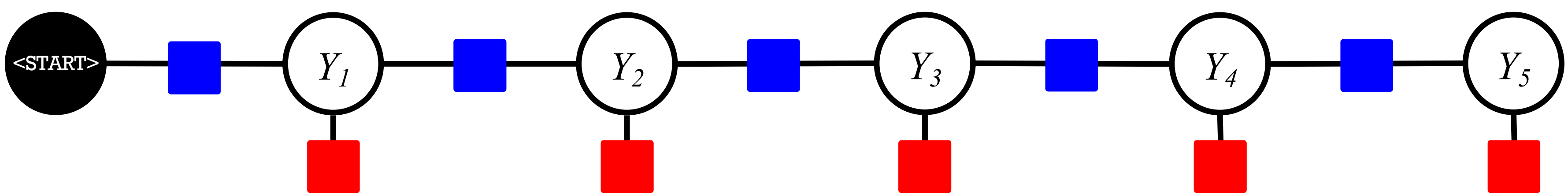


Visual Notation: Usually we draw a CRF **without** showing the variable corresponding to $\boldsymbol{X}$

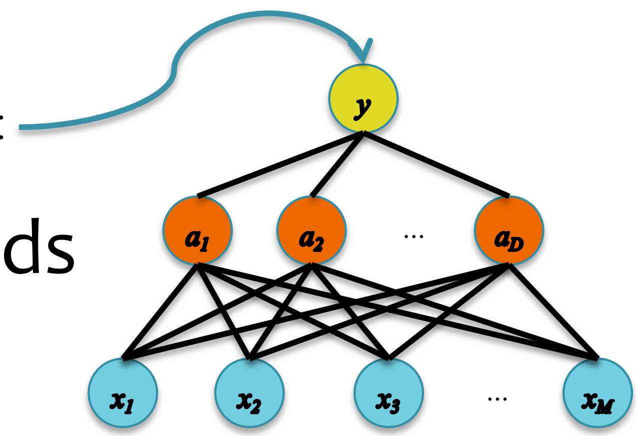
MRF AND CRF LEARNING (LOG-LINEAR PARAMETERIZATION)

Options for MLE of MRFs

- **Setting I:** $\psi_C(\boldsymbol{x}_C) = \theta_{C,\boldsymbol{x}_C}$
  A. MLE by inspection (Decomposable Models)
  B. Iterative Proportional Fitting (IPF)

- **Setting II:** $\psi_C(\boldsymbol{x}_C) = \exp(\boldsymbol{\theta} \cdot \boldsymbol{f}(\boldsymbol{x}_C))$
  C. Generalized Iterative Scaling
  D. Gradient-based Methods

- **Setting III:** $\psi_C(\boldsymbol{x}_C) =$
  E. Gradient-based Methods



MRF and CRF Learning

Whiteboard

- log-linear MRF model (i.e. with feature based potentials)
- log-linear MRF derivatives
- log-linear MRF training with SGD
- log-linear CRF model (i.e. with feature based potentials)
- log-linear CRF derivatives
- log-linear CRF training with SGD