1. Recap of Gibbs Sampling and MH algorithm

**MH algorithm summary**
- Draws a sample $x'$ from $Q(x'|x)$, where $x$ is the previous sample.
- The new sample $x'$ is accepted or rejected with some probability $A(x'|x) = \min(1, \frac{P(x')Q(x|x')}{P(x)Q(x'|x)})$
- In case that $Q$ is symmetric, i.e. $Q(x|x') = Q(x'|x)$ (Gaussian, etc.), the acceptance probability simplifies to $\min(1, \frac{P(x')}{P(x)})$

**pseudo-code for M-H algorithm**
1. Initialize starting state $x^{(0)}$, set $t = 0$
2. Burn-in: while samples have “not converged”:
   - $x = x^{(t)}$
   - $t = t+1$
   - sample $x^* \sim Q(x^*|x)$ (draw proposal)
   - sample $u \sim Uniform(0, 1)$ (draw acceptance threshold)
   - if $u < A(x^*|x)$: $x^{(t)} = x^*$ (accept, make state transition)
   - else: $x^{(t)} = x$ (reject, stay in current state)
3. Takes samples from $P(x)$: after observing convergence, do the same as 2 to sample from the distribution.

**Gibbs sampling**
- Let $x^{(1)}$ be the initial assignment to variables.
- Set $t = 1$
- while true:
  - for $i = 1...J$:
    * sample $x_i^{(t+1)} \sim p(x_i|x\{x_j^{(t)}(j \neq i)})$
    * set $x_i^{(t+1)}$ to $x_i^{(t)}$
    * $t = t+1$
2. Consider $X_1, \ldots, X_n$ being i.i.d. Poisson($\lambda$). Show that a Gamma($\alpha, \beta$) prior on $\lambda$ is a conjugate prior, and find the posterior distribution.

Likelihood:

$$L(\lambda) = \prod_{i=1}^{n} \exp(-\lambda)\frac{\lambda^{x_i}}{x_i!} = \frac{\exp(-n\lambda)\lambda^{\sum x_i}}{\prod_{i} x_i!}$$

Prior:

$$p(\lambda) \sim Gamma(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}\exp(-\beta \lambda)$$

Posterior:

$$p(\lambda) \propto L(\lambda)p(\lambda) \propto \lambda^{\sum x_i + \alpha - 1}$$

So $p(\lambda)$ is Gamma($\sum_i x_i + \alpha, n + \beta$)

3. Gibbs sampling can proceed either rotationally (sweeping through indices $i$) or randomly (by sampling $i$). For the purposes of this problem consider the version where $i$ is sampled randomly with probability $\pi_i$. Show that Gibbs sampling satisfies detailed balance.

Detailed balance means that for each pair of states $x$ and $x'$, (1) arriving at $x$ then $x'$ and (2) arriving at $x'$ then $x$ are equiprobable. That is,

$$S(x' \leftarrow x)p(x) = S(x \leftarrow x')p(x').$$

First, let’s consider the transition probability $S(x' \leftarrow x)$. Since Gibbs sampling samples from the full conditionals, this probability is given by:

$$S(x' \leftarrow x) = \pi_i p(x'_i | x_{\setminus i})$$

Next, let’s compute the left hand side and right hand sides of the detailed balance equation separately.

LHS:

$$S(x' \leftarrow x)p(x) = \pi_i p(x'_i | x_{\setminus i})p(x)$$

$$= \pi_i p(x'_i | x_{\setminus i})p(x_i | x_{\setminus i})p(x_{\setminus i})$$

RHS:

$$S(x \leftarrow x')p(x') = \pi_i p(x_i | x'_i)p(x')$$

$$= \pi_i p(x_i | x'_i)p(x_i | x_{\setminus i})p(x_{\setminus i})$$

$$= \pi_i p(x_i | x_{\setminus i})p(x'_i | x_{\setminus i})p(x_{\setminus i})$$

$$= S(x' \leftarrow x)p(x)$$
where the second to last step follows from the observation that $x'_{i \setminus i} = x_{i \setminus i}$ because Gibbs sampling holds the other variables constant when updating the $i$th variable. Thus, detailed balance holds.

Note: to prove detailed balance for the version of Gibbs sampling where we sweep through indices $i$, we would consider the update after a full sweep.