Complexity of Discrete Energy Minimization Problems

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Abstract

- Energy minimization is NP-hard ☹
- Is it approximable? Not yet resolved
- Sometimes yes: Potts, Metric, Logic MRF ☺
- **We prove that QPBO, planar energy with 3+ labels, and general energy minimization are all inapproximable**
- Useful for algorithm design — finding "good" subclasses
- In practice, useful for model selection

Energy Minimization Formulation

- Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with label space $\mathcal{L}$
- Pairwise energy

$$\min_{x \in \mathcal{L}^{\mathcal{V}}} \sum_{u \in \mathcal{V}} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v)$$

- **Q**uadratic **P**seudo-**B**oolean **O**ptimization (QPBO)

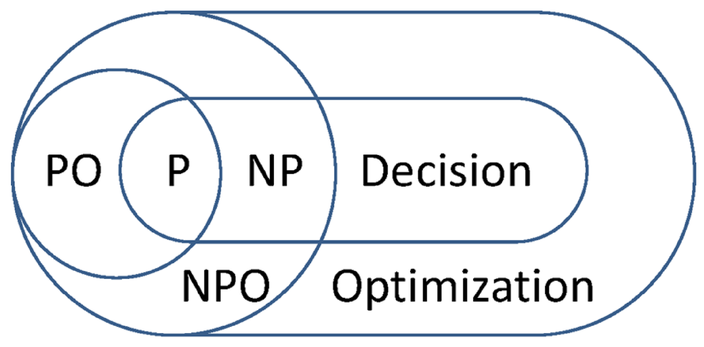
$$\min_{x \in \{0,1\}^{\mathcal{V}}} \sum_{u \in \mathcal{V}} a_u x_u + \sum_{(u,v) \in \mathcal{E}} a_{uv} x_u x_v$$

- General energy minimization

$$\min_{x \in \mathcal{L}^{\mathcal{V}}} \sum_{S \subseteq \mathcal{V}} f_S(x_S)$$

Optimization & Approximation

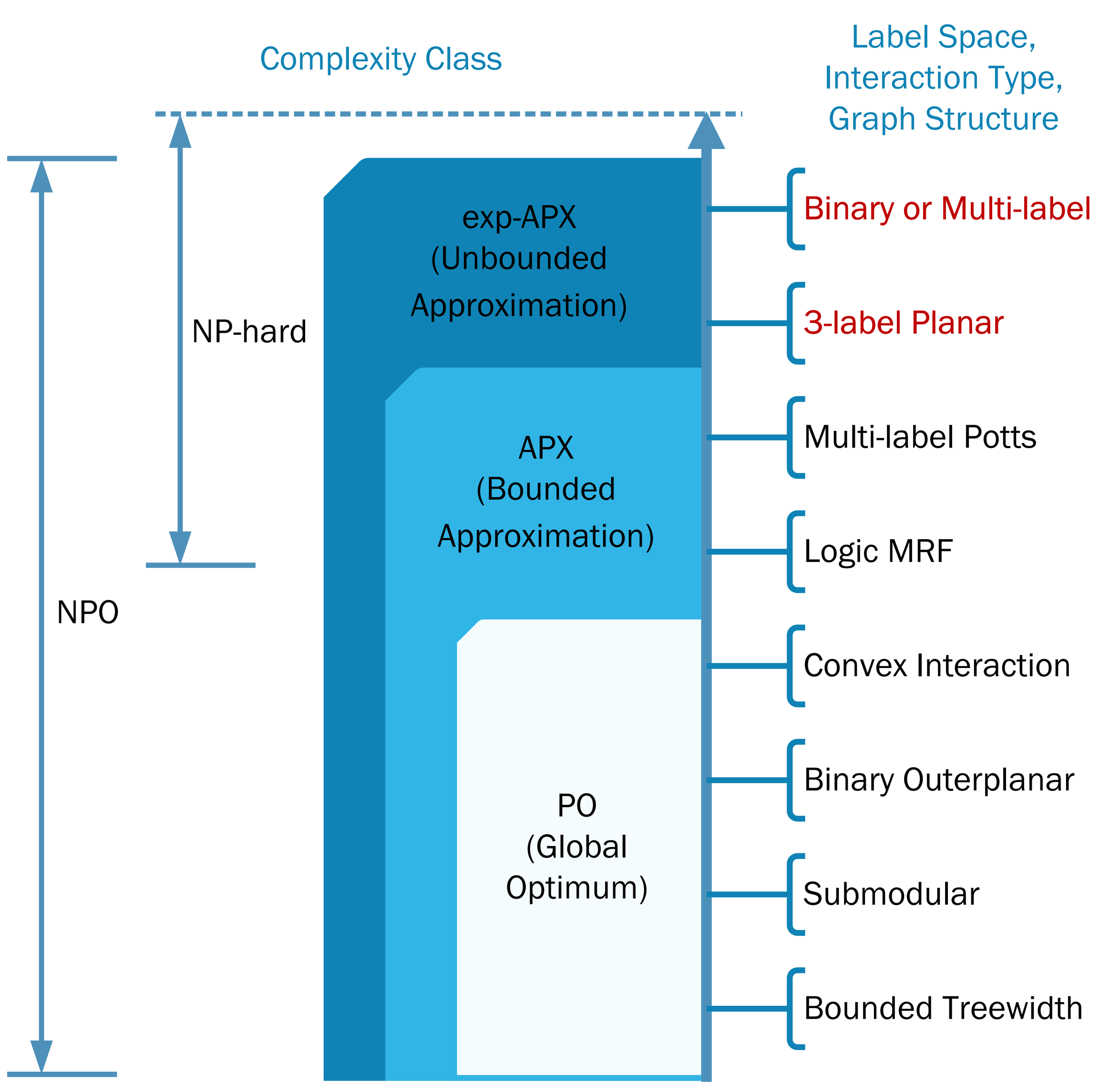
- Optimization problems



- Approximation ratio $f(x)/f(x^*)$, $f(x^*) > 0$
- **APX** — constant ratio approximation
- ***F*-APX** — approximation ratio is a function of class F of the input bit length
- Relations of complexity classes

PO ⊆ APX ⊆ log-APX ⊆ poly-APX ⊆ exp-APX ⊆ NPO

Complexity Axis & Main Results



- Energy minimization problems vary greatly in approximation ratio
- Where do QPBO and general energy minimization fall on this axis?

Theorem:
QPBO (binary labels) is **complete** in exp-APX.

Theorem:
General energy minimization is **complete** in exp-APX.

Theorem:
Planar energy with 3+ labels is **complete** in exp-APX.

- Bounded approximation ratio ☺
  - Indicates a class of practical interest
  - Useful for algorithm design
- **Do not try to prove approximation guarantee if**
  - **Model includes QPBO, planar 3-label, or general energy minimization**
  - Or you can build AP-reduction from them

Details

Non-deterministic **P**olynomial time **O**ptimization (**NPO**)

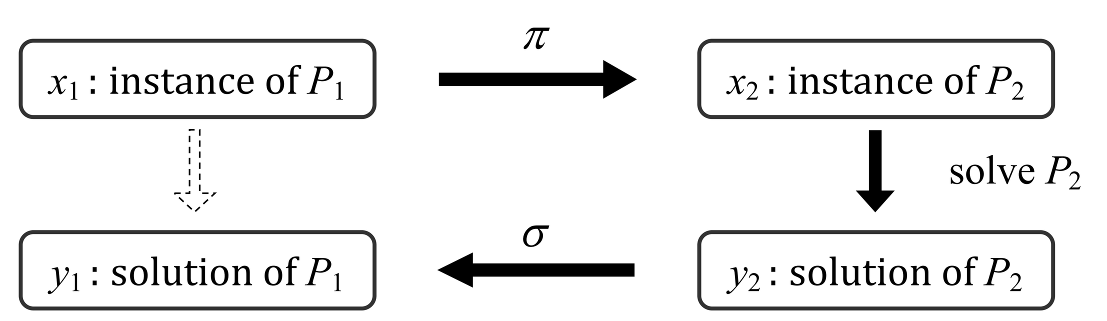
- The set of instances is recognizable in polynomial time
- The solution's feasibility is verifiable in polynomial time
- A positive objective value

Polynomial time **O**ptimization (**PO**)

- The problem is in NPO, and it is solvable in polynomial time

Approximation-**P**reserving reduction (**AP-reduction**)

- Reduce NPO problem P_1 to another NPO problem P_2



- For a given positive constant α, the mappings must satisfy,

$$\frac{f_2(y_2)}{f_2(y_2^*)} \le r \quad \text{implies} \quad \frac{f_1(\sigma(y_2))}{f_1(y_1^*)} \le 1 + \alpha(r-1)$$

$\mathcal{C}$-hard & $\mathcal{C}$-complete

- A problem is **$\mathcal{C}$-hard** if any problem in complexity class $\mathcal{C}$ can be reduced to it
- A $\mathcal{C}$-hard problem is **$\mathcal{C}$-complete** if it belongs to $\mathcal{C}$
- Intuitively, a complexity class $\mathcal{C}$ specifies the **upper bound** on the hardness of the problems within, **$\mathcal{C}$-hard** specifies the **lower bound**, and **$\mathcal{C}$-complete exactly** specifies the hardness

Problem W3SAT-triv

INSTANCE: Boolean CNF formula F with variables $x_1, \ldots, x_n$ and each clause assuming exactly 3 variables; non-negative integer weights $w_1, \ldots, w_n$

SOLUTION: Truth assignment τ to the variables that either satisfies F or assigns the trivial, all-true assignment

MEASURE: $\min \sum_{i=1}^{n} w_i \tau(x_i)$

Proof Scheme

NP Machine

↓ AP-reduction [2, 1]

Weighted 3-SAT

↓ AP-reduction

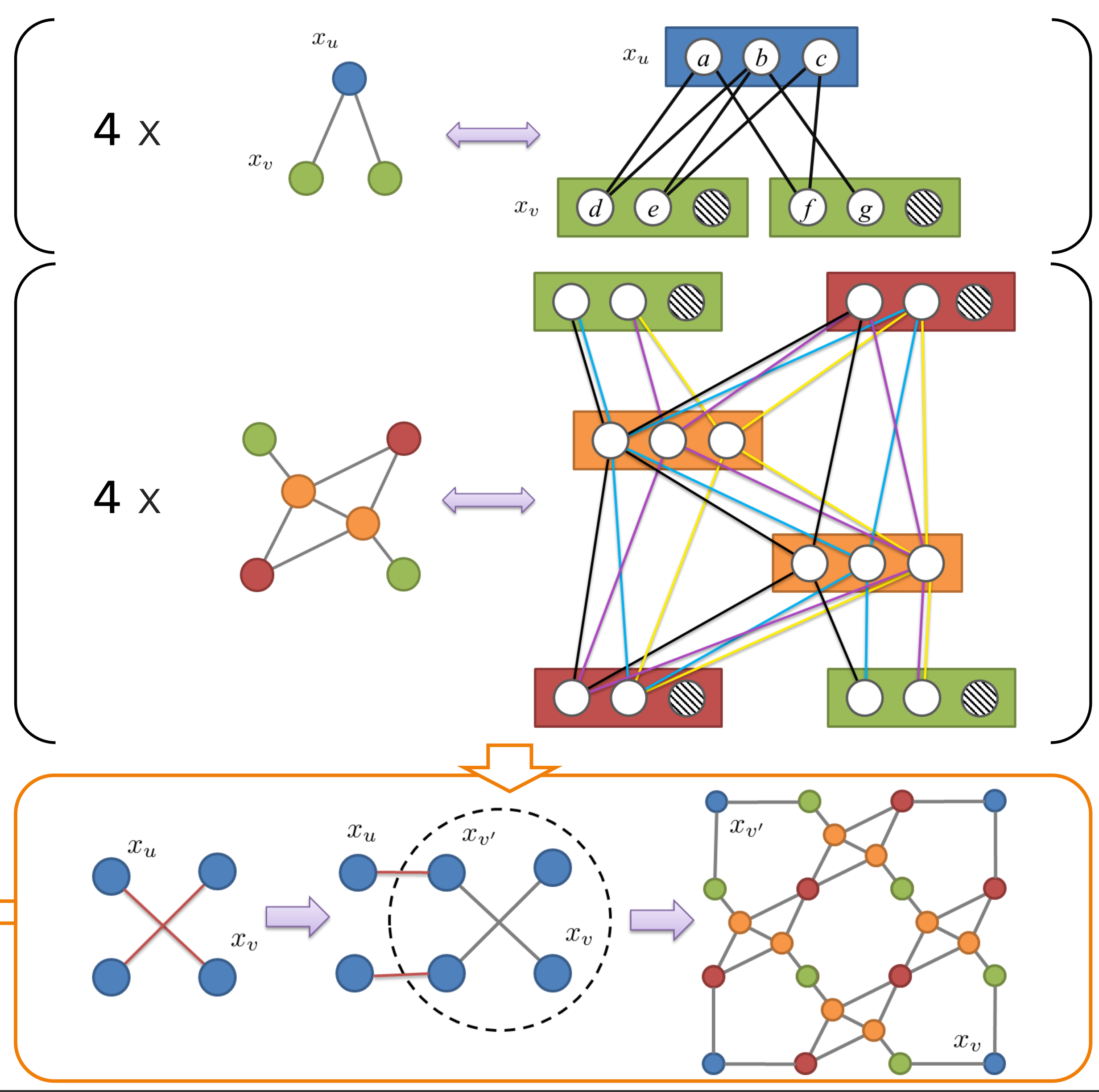
QPBO

↓ AP-reduction

General energy with 3+ labels

↓ AP-reduction / Planar reduction

Planar energy with 3+ labels



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