"is in"

Let's verify that $Ax = b$ when $b \in \text{colspace}(A)$

& $x = V \frac{1}{\Sigma} U^T b$.

First, notice that $U^T b$ is computing the coordinates of b expressed relative to a basis given by the columns of U: $U^T b = \begin{pmatrix} u_1 \cdot b \\ \vdots \\ u_k \cdot b \\ \vdots \\ u_m \cdot b \end{pmatrix}$

Since the vectors in this basis are orthonormal, we also know that $b = \sum_{i=1}^{m} (u_i \cdot b) u_i$.

Since b is in the column space of A, we actually have $b = \sum_{i=1}^{k} (u_i \cdot b) u_i$, with $k = \text{rank}(A)$.

So:

$$
\begin{aligned}
Ax &= U \Sigma V^T V \frac{1}{\Sigma} U^T b \\
&= U \Sigma \frac{1}{\Sigma} U^T b \\
&= U \begin{pmatrix} \underbrace{1 & & & & \\ & \ddots & & & \\ & & 1}_{k \text{ 1s}} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} U^T b \\
&= \sum_{i=1}^{k} u_i (u_i \cdot b) \\
&= b.
\end{aligned}
$$

Let's verify that $x = V\frac{1}{\Sigma}U^Tb$ is the least-norm solution to $Ax=b$ when $b \in \text{colspace}(A)$.

(More generally it is the least-norm least-squares solution.)

Recall that the set of all solutions is of the form $x + x_N$, with $x = V\frac{1}{\Sigma}U^Tb$ and x_N varying over the null space of A.

$$\begin{aligned}
|x + x_N| &= \left|V\tfrac{1}{\Sigma}U^Tb + x_N\right| \\
&= \left|V\left(\tfrac{1}{\Sigma}U^Tb + V^Tx_N\right)\right| \\
&= \left|\tfrac{1}{\Sigma}U^Tb + V^Tx_N\right|
\end{aligned}$$

(since V is orthogonal).

By our previous reasoning,

$$\frac{1}{\Sigma}U^Tb = \begin{pmatrix} c_1 \\ \vdots \\ c_K \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ for some } c_1, \dots, c_K.$$

Since x_N is in A's null space,

$$V^Tx_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d_{K+1} \\ \vdots \\ d_n \end{pmatrix}, \text{ for some } d_{K+1}, \dots, d_n.$$

So $|x + x_N|$ is minimal when $x_N = \underline{0}$.