Consider $A = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$.

We could perform an eigenvalue/eigenvector decomposition $A = S \Lambda S^{-1}$, but it is complicated & produces complex values.

There is a simpler geometric interpretation.

Think of A as ${}^{\mathcal{B}}[f]_{\mathcal{B}}$ for basis $\mathcal{B}$ given by the vectors $\longrightarrow$ & $\uparrow$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now let $\mathcal{B}'$ be $\uparrow$ & $\longleftarrow$, i.e., $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Then ${}^{\mathcal{B}'}[f]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

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Then ${}^{\mathcal{B}'}[f]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Why?

Because we see from A that

$$f(\longrightarrow) = 2 \cdot \uparrow + 0 \cdot \longleftarrow$$

& $$f(\updownarrow) = 0 \cdot \uparrow + 1 \cdot \longleftarrow.$$

That tells us $A = \begin{pmatrix} \text{coordinate change} \\ \text{matrix} \\ \text{from } \mathcal{B}' \text{ to } \mathcal{B} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

And that again means we can solve $Ax = b$ easily.

E.g.,

$$Ax = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{\text{inverse is} \\ \text{transpose} \\ \text{just as with} \\ \text{orthogonal eigenvectors}}} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}.$$

So $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 2 \end{pmatrix}$.

Then $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$.

So $x_1 = \frac{2}{2} = 1$

& $x_2 = \frac{-7}{1} = -7$.

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So $x_1 = \frac{2}{2} = 1$

& $x_2 = \frac{-7}{1} = -7$.

("A" is simple enough here that we could have read off this answer directly, but this process hints at the more general process we will see with SVD.)

Fact:

<u>Not</u> all matrices have nice <u>eigenvector decompositions.</u>

However, if we allow ourselves to choose nice bases for the domain & range of a linear function (possibly different bases even if domain & range are the same), then we can <u>always</u> represent the function using a <u>diagonal matrix</u>.

That's what SVD gives us, as we will see.