Example

$$A = \begin{pmatrix} 2 & 5 & 10 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \end{pmatrix}.$$

Row space spanned by $(1\ 0\ 0)$ & $(0\ 1\ 2)$.

Null space spanned by $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.

Column space spanned by $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$.

A has rank 2.

$$A = \underbrace{\begin{pmatrix} -0.847 & -0.341 & 0.408 \\ -0.164 & 0.898 & 0.408 \\ -0.506 & 0.279 & -0.816 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 13.41 & & \\ & 0.408 & \\ & & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} -0.164 & -0.441 & -0.882 \\ -0.986 & -0.073 & 0.147 \\ 0 & 0.894 & -0.447 \end{pmatrix}}_{V^T}.$$

- Last column of V (see last row of V^T) spans A's nullspace.
- First two columns of U form an orthonormal basis for A's column space.
- First two rows of V^T form an orthonormal basis for A's row space.

To solve $Ax = b$, we compute $x = V \frac{1}{\Sigma} U^T b$.

Here, $\frac{1}{\Sigma} = \begin{pmatrix} 0.7457 & & \\ & 2.448 & \\ & & 0 \end{pmatrix}$.

(i) <u>Suppose b is in A's column space.</u>

E.g.: $b = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$.

The "SVD solution" as above is $x = \begin{pmatrix} 0 \\ 0.2 \\ 0.4 \end{pmatrix}$.

Our human eye might give $x' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, since the second column of A is $\begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$.

The difference is $\begin{pmatrix} 0 \\ 0.8 \\ -0.4 \end{pmatrix}$ which lies in A's nullspace.

Indeed, there is a 1D affine line of possible solutions. $\begin{pmatrix} 0 \\ 0.2 \\ 0.4 \end{pmatrix}$ is the closest to the origin; it lies in A's row space.

(ii) Suppose b is not in A's column space.

E.g.: $b = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$.

Then $Ax = b$ does not have an exact solution. However, the "SVD solution" $x = V \frac{1}{\Sigma} U^T b$ will minimize $\| Ax - b \|$, i.e., Ax will be as close to b as possible.

Also known as "Least-squares" solution.

Here, as in (i), we compute $x = \begin{pmatrix} 0 \\ 0.2 \\ 0.4 \end{pmatrix}$.

So $Ax - b = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is perpendicular to A's column space, telling us Ax is as close to b as possible.

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Observe that this vector is parallel to the last column of U.

(That directly tells us the vector is perpendicular to A's column space.)