Let α be a unit vector in $\mathbb{R}^n$ and let P be the corresponding Householder reflection, as on p. 22 of the linear algebra notes.

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(1) <u>To see that P is orthogonal:</u>

(Recall $P = I - 2\alpha\alpha^T$, so P is symmetric.)

$$\begin{aligned}
PP^T = P^TP = P^2 &= (I - 2\alpha\alpha^T)^2 \\
&= I - 4\alpha\alpha^T + 4\alpha\underbrace{\alpha^T\alpha}_{\text{this is the number 1}}\alpha^T \\
&= I - 4\alpha\alpha^T + 4\alpha\alpha^T \\
&= I.
\end{aligned}$$

(2) <u>To establish P's reflection property:</u>

(a)
$$\begin{aligned}
P\alpha &= (I - 2\alpha\alpha^T)\alpha \\
&= I\alpha - 2\alpha\underbrace{\alpha^T\alpha}_{\text{this is the number 1}} \\
&= \alpha - 2\alpha \\
&= -\alpha.
\end{aligned}$$

(b) Let v be a vector in $\mathbb{R}^n$ perpendicular to α.

Then
$$\begin{aligned}
Pv &= (I - 2\alpha\alpha^T)v \\
&= Iv - 2\alpha\underbrace{\alpha^Tv}_{\text{this is the number 0}} \\
&= v.
\end{aligned}$$