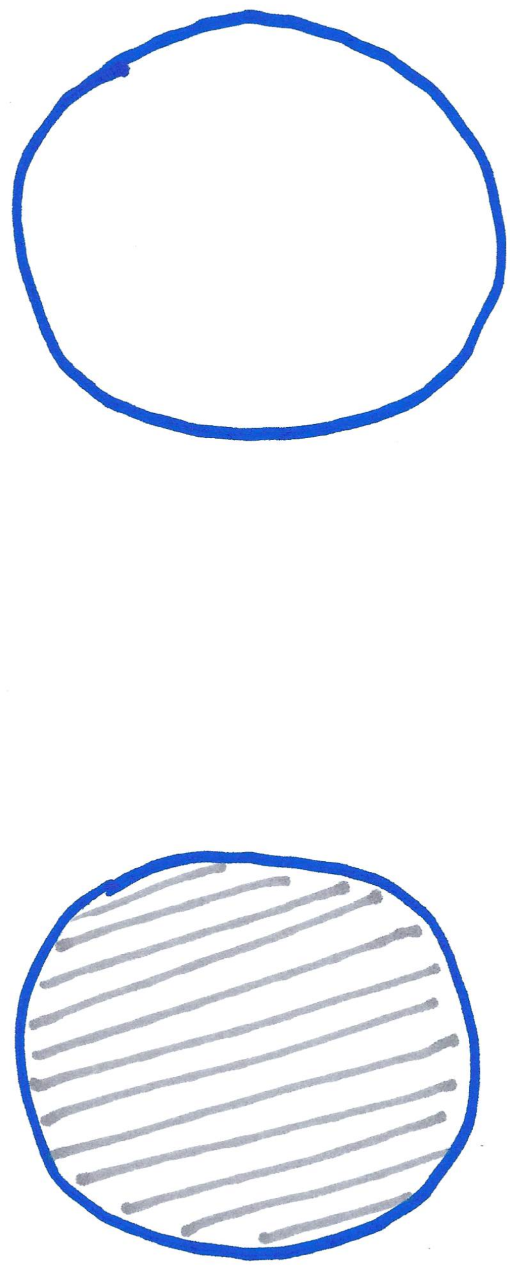
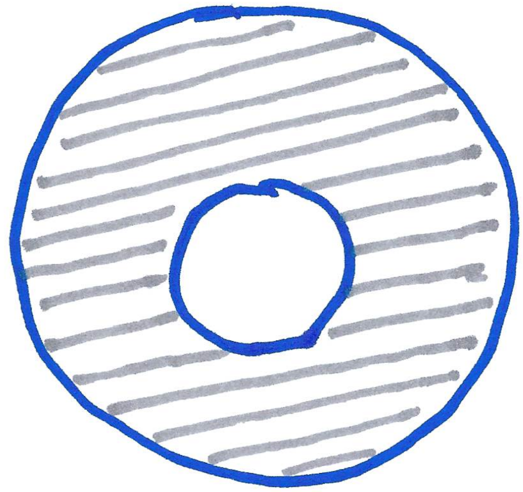
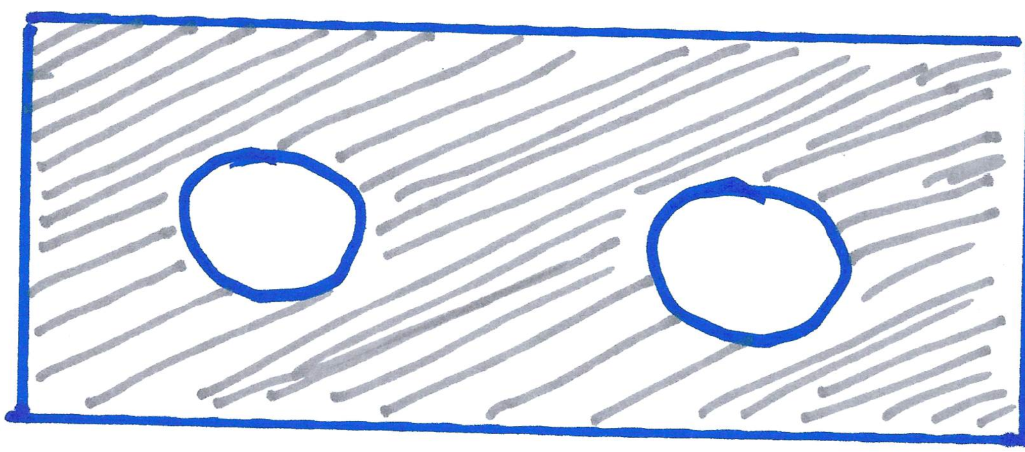
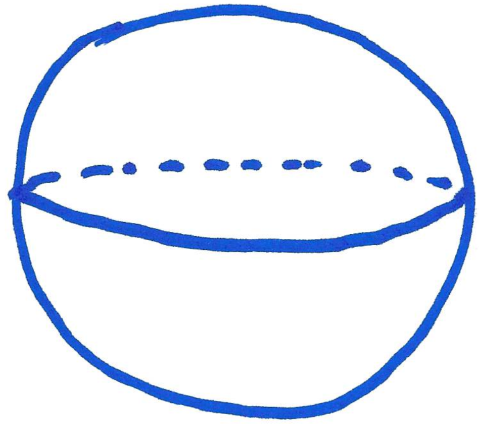
Homology

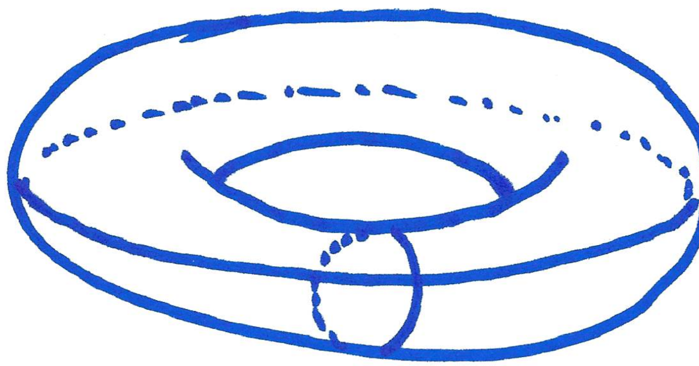
An Introduction
to
Algebraic Topology

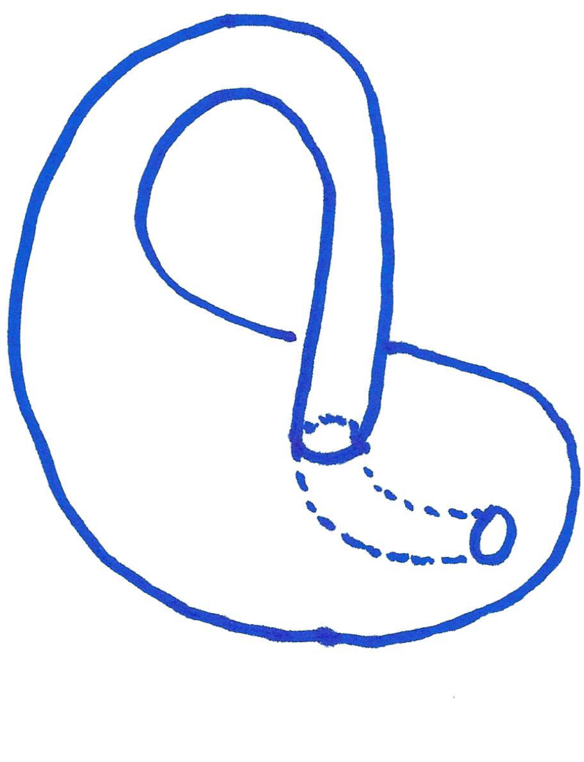


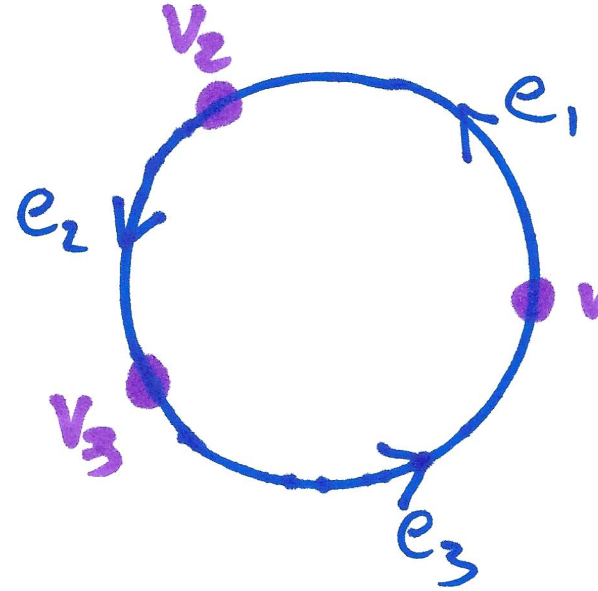












Boundary operator ∂

$$\partial(e_1) = v_2 - v_1$$

$$\partial(e_2) = v_3 - v_2$$

$$\partial(e_3) = v_1 - v_3$$

As matrix

$$\begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$$

$\dim(\text{null space}(\partial)) = 1$,

spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

That vector means

$$e_1 + e_2 + e_3,$$

which is the circle, i.e., a 1D hole.

$$\partial_2(r_1) = e_1 - e_5 + e_4$$

$$\partial_2(r_2) = e_5 + e_2 - e_6$$

$$\partial_2(r_3) = e_6 + e_3 - e_4$$

$$\partial_1(e_1) = v_2 - v_1$$

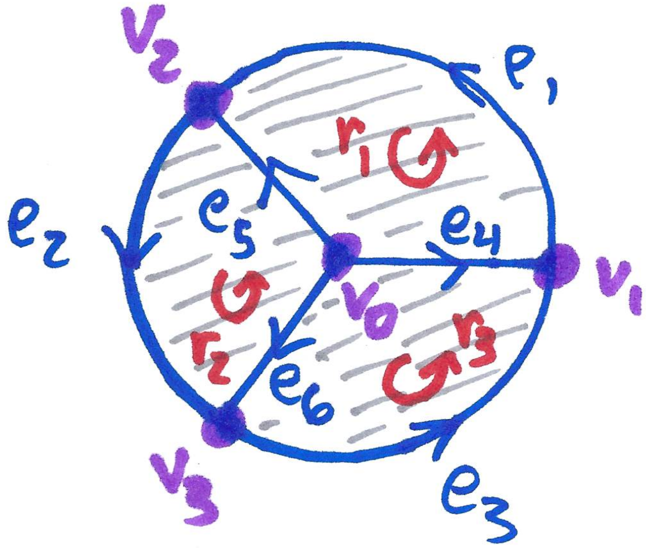
$$\partial_1(e_2) = v_3 - v_2$$

$$\partial_1(e_3) = v_1 - v_3$$

$$\partial_1(e_4) = v_1 - v_0$$

$$\partial_1(e_5) = v_2 - v_0$$

$$\partial_1(e_6) = v_3 - v_0$$



Two boundary operators

$$\partial_2 : \text{Regions} \longrightarrow \text{Edges}$$

$$\partial_1 : \text{Edges} \longrightarrow \text{Vertices}$$

As matrices

$$\partial_1 = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\partial_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

Observe: $\quad \partial_1 \circ \partial_2 = 0$

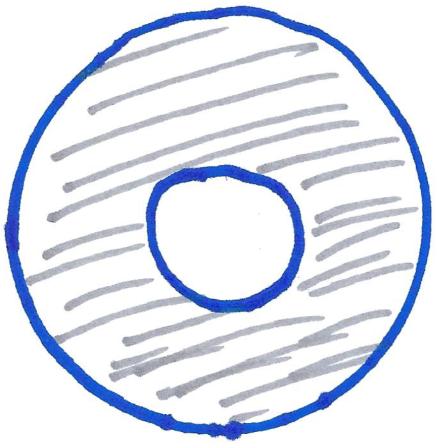
$\dim(\text{null space}(\partial_1)) = 3$,
spanned by

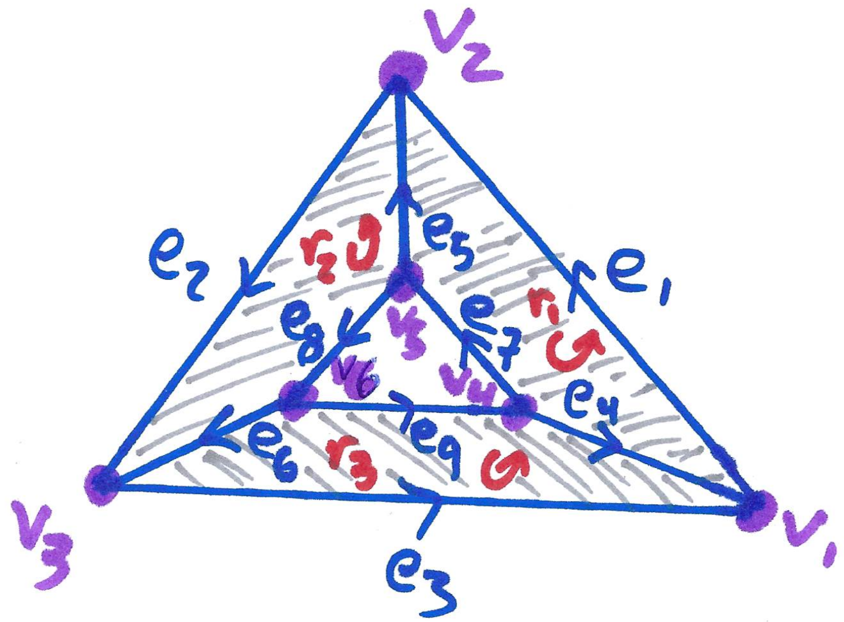
$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.}_{\text{Also the columns of } \partial_2\text{'s matrix.}}$$

Always $\partial_1 \circ \partial_2 = 0$, so $\text{colspace}(\partial_2) \subseteq \text{nullspace}(\partial_1)$.

Extra dimensions in the nullspace describe holes.

Here, $\text{colspace}(\partial_2) = \text{nullspace}(\partial_1)$, so no holes.





$$\underline{\partial_1}$$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\
\begin{pmatrix}
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1
\end{pmatrix}
\end{array}$$

$$\underline{\partial_2}$$

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{array}
\begin{array}{c}
\begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\end{array}$$

$$\partial_1 \circ \partial_2 = 0$$

dim(null space(∂_1)) = 4,

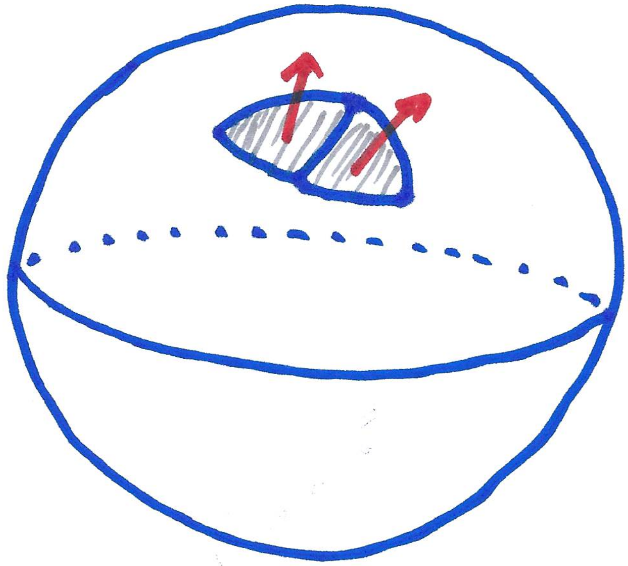
spanned by

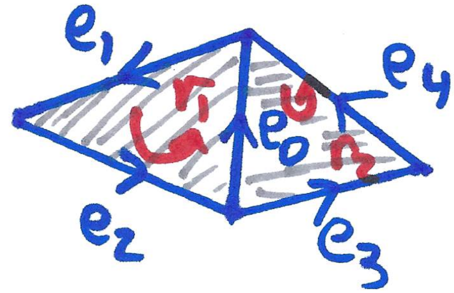
$$\underbrace{\begin{pmatrix}1\\0\\0\\1\\-1\\0\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\\-1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\1\\0\\0\\-1\end{pmatrix}}_{\text{columns of } \partial_2}, \begin{pmatrix}1\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

not in column space of ∂_2.

Therefore there is a 1D hole.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ is in the span of the four null space vectors.





$$\partial_2(r_1) = e_1 + e_2 + e_0$$

$$\partial_2(r_2) = e_3 + e_4 - e_0$$

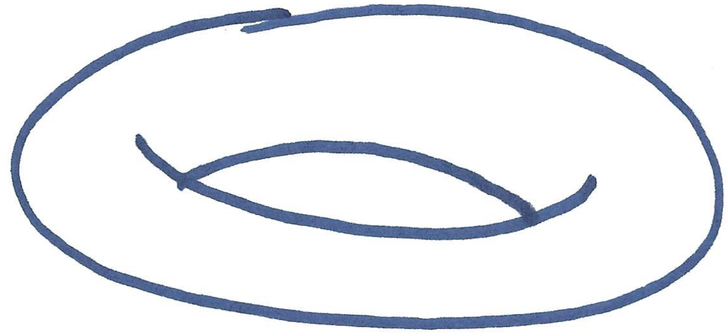
$$\partial_2(r_1 + r_2) = e_1 + e_2 + e_3 + e_4$$

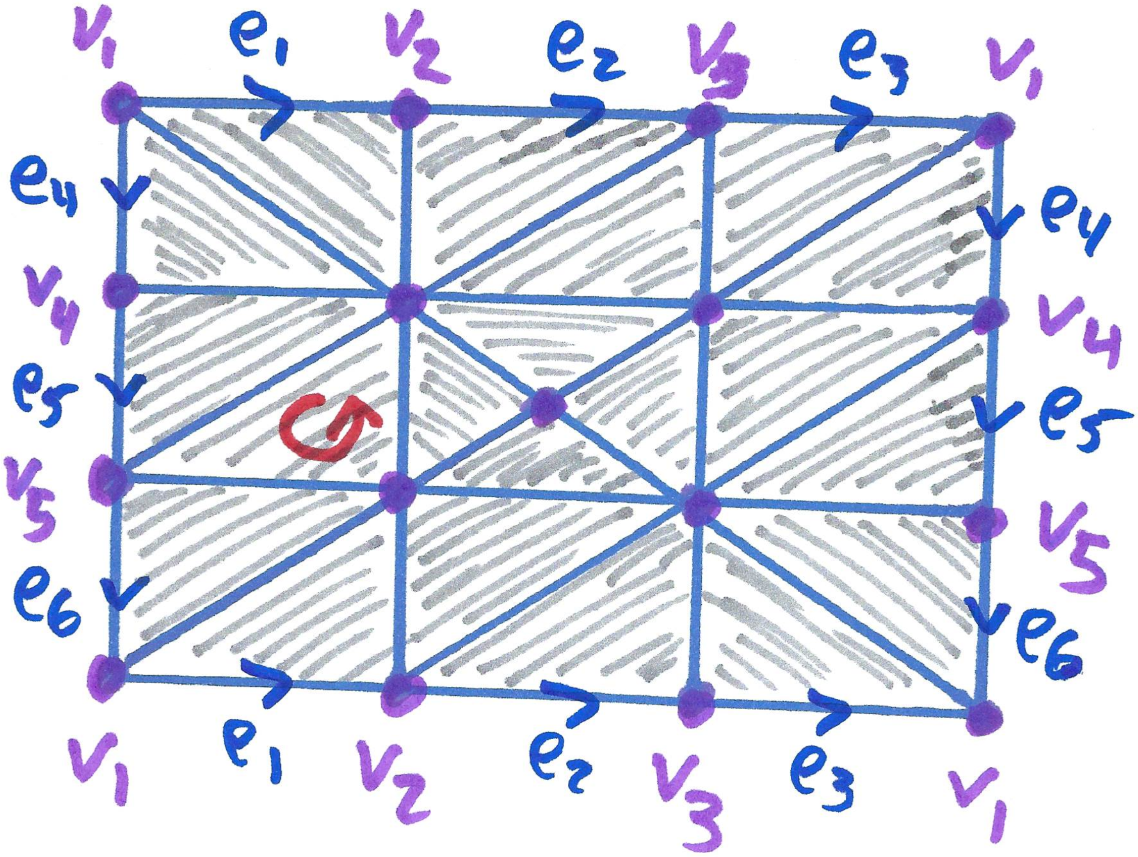
No e_0 !

So $\partial_2\left(\sum_{\text{all } i} r_i\right) = 0$

$\therefore$ There is a 2D hole.

Torus

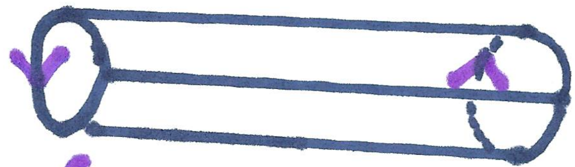




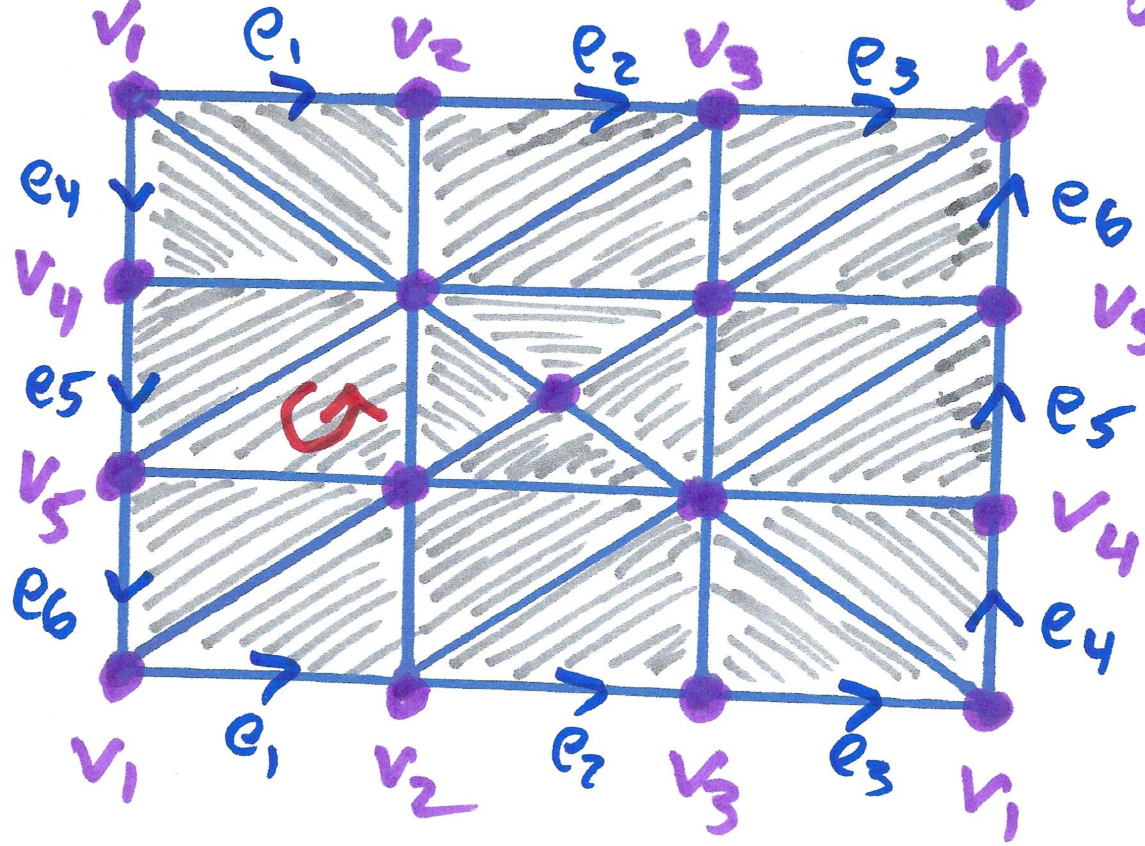
$$\partial_2\left(\sum \text{triangles} \right) = 0$$

The null space of ∂_1 has two dimensions that are not in the column space of ∂_2, spanned by $e_1 + e_2 + e_3$ & $e_4 + e_5 + e_6$.

Klein Bottle



(reverse the gluing orientation)



Let $w_1 = e_1 + e_2 + e_3$ & $z_1 = e_4 + e_5 + e_6$.
Then $\partial_1(w_1) = 0$ & $\partial_1(z_1) = 0$ as before.

However, $\partial_2(\Sigma \text{ triangles}) = 2z_1$.

Def Let X be a topological space.
Let G be an abelian group.

The k^{th} homology group (of X, with coefficients in G) is defined to be the quotient group

$$H_k(X;G) = Z_k(X;G) / B_k(X;G).$$

Here "Z" stands for "cycle" (or "zero").
$Z_k(X;G)$ is the null space of ∂_k, viewed as a group.

"B" stands for "boundary".
$B_k(X;G)$ is the column space of ∂_{k+1}, viewed as a group.

We view H_k as measuring holes in X at dimension k, as detectable by group G.

Theorem

Let X & Y be homeomorphic topological spaces.
Then $H_k(X;G)$ & $H_k(Y;G)$ are isomorphic as groups.

X

	Sphere	Torus	Klein Bottle
$H_1(X;\mathbb{R})$	0	$\mathbb{R}\times\mathbb{R}$	$\mathbb{R}$
$H_2(X;\mathbb{R})$	$\mathbb{R}$	$\mathbb{R}$	0
$H_1(X;\mathbb{Z})$	0	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}/2$
$H_2(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$	0
$H_1(X;\mathbb{Z}/2)$	0	$\mathbb{Z}/2\times\mathbb{Z}/2$	$\mathbb{Z}/2\times\mathbb{Z}/2$
$H_2(X;\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Euler Characteristic

Suppose X is a space with a finite triangulation.

The <u>Euler Characteristic</u> <u>$\chi(X)$</u> is the alternating sum of the number of faces in each dimension:

$$\begin{aligned}\chi(X) = \ &\# \text{ of vertices} \\ &- \# \text{ of edges} \\ &+ \# \text{ of triangles} \\ &- \# \text{ of tetrahedra} \\ &\quad \vdots\end{aligned}$$

Theorem

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \beta_k(X).$$

Here β_k = dimension of the non-torsion part of $H_k(X; \mathbb{Z})$,

β_0 = # of components of X.

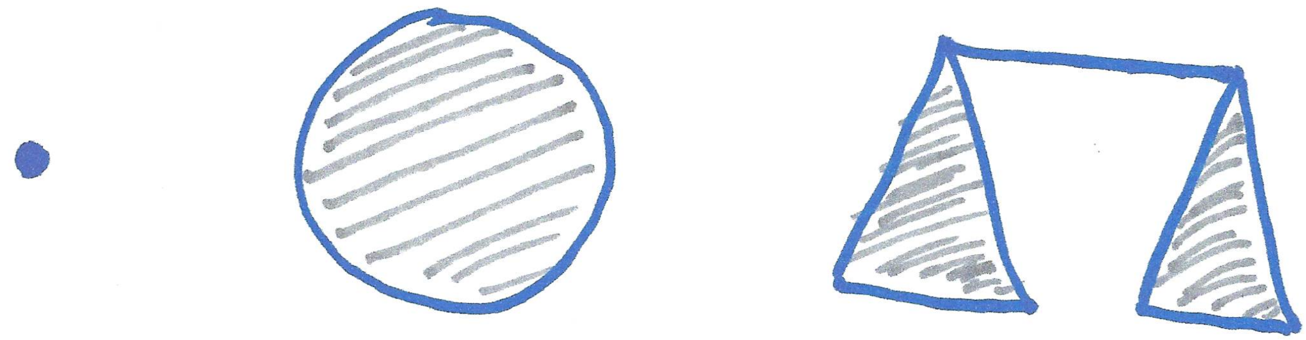
β_k is called the k^{th} betti number.

Corollary

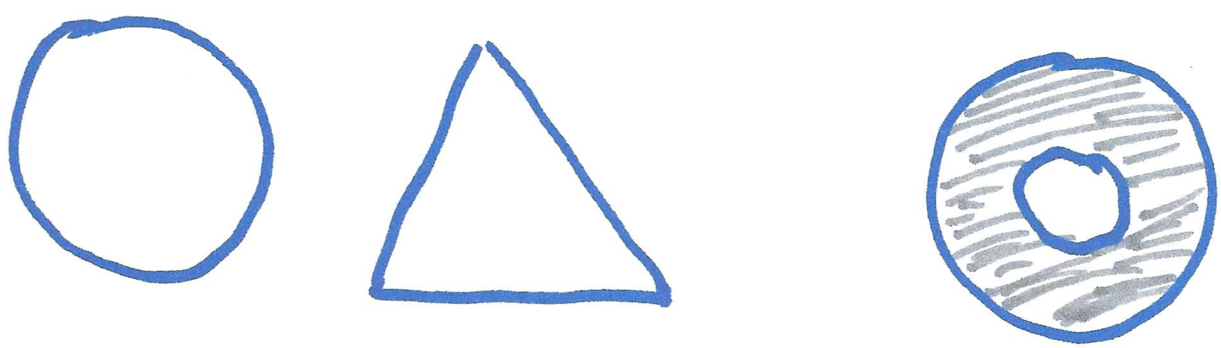
If X & Y have finite triangulations and isomorphic homology groups (with integer coefficients) in each dimension, then

$$\chi(X) = \chi(Y).$$

Euler Characteristic 1

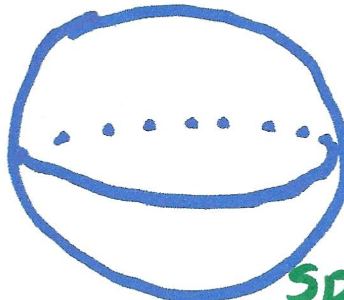
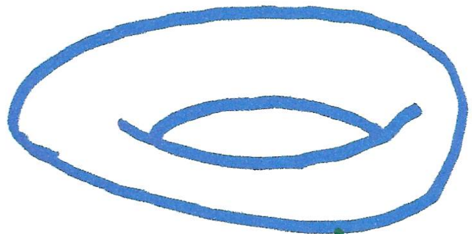
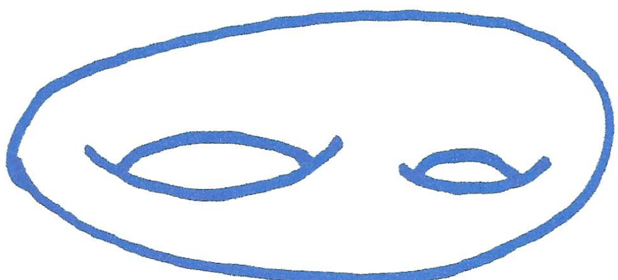
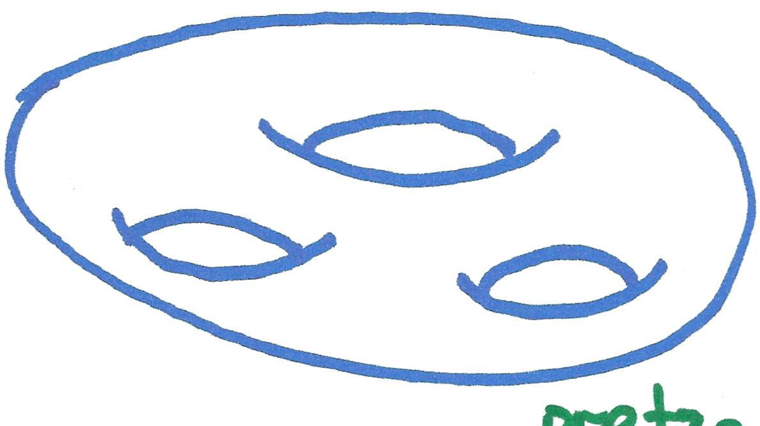


Euler Characteristic 0



2D Surfaces

compact, connected, orientable

χ		genus g
2	 sphere	0
0	 torus	1
-2	 teapot	2
-4	 pretzel	3

$$\chi = 2 - 2g$$

g is the number of "handles".