Let $\displaystyle \Phi(\varepsilon) = \int_{x_0}^{x_1(\varepsilon)} F(x, y^* + \varepsilon\eta, y^{*\prime} + \varepsilon\eta')\,dx,$

with $\eta(x_0) = 0$

& $g(x_1(\varepsilon), y^*(x_1(\varepsilon)) + \varepsilon\,\eta(x_1(\varepsilon))) = 0.$

Recall $\displaystyle \frac{d}{dx}\int^{h(x)} f(y)\,dy = f(h(x))\,h'(x).$

So

$$0 = \frac{d}{d\varepsilon}\Phi'(\varepsilon)\Big|_{\varepsilon=0}$$

$$= \frac{dx_1}{d\varepsilon}F\Big|_{x_1(0)} + \int_{x_0}^{x_1(0)} (F_y\eta + F_{y'}\eta')\,dx$$

$$= \Big(\underbrace{\frac{dx_1}{d\varepsilon}F}_{\text{This is new}} + F_{y'}\eta\Big)\Big|_{x_1(0)} + \int_{x_0}^{x_1(0)} \Big(F_y - \frac{d}{dx}F_{y'}\Big)\eta\,dx$$

Also

$$0 = \frac{d}{d\varepsilon} g(x_1(\varepsilon), y^*(x_1(\varepsilon)) + \varepsilon\eta(x_1(\varepsilon)))$$

$$= g_x \frac{dx_1}{d\varepsilon} + g_y \frac{d(y^*(x_1(\varepsilon)) + \varepsilon\eta(x_1(\varepsilon)))}{d\varepsilon}$$

$$= g_x \frac{dx_1}{d\varepsilon} + g_y \Big(y^{*\prime}(x_1(\varepsilon)) \frac{dx_1}{d\varepsilon} + \eta(x_1(\varepsilon)) + \varepsilon\eta'(x_1(\varepsilon)) \frac{dx_1}{d\varepsilon} \Big).$$

So, at $\varepsilon = 0$,

$$\frac{dx_1}{d\varepsilon} = -\frac{g_y \eta}{g_x + g_y y^{*\prime}}.$$

Plug that into the last purple line, and conclude that

$$F_{y'} - \frac{g_y F}{g_x + g_y y^{*\prime}} = 0$$

at the right endpoint (x_1, y_1).