Let A be an $n \times n$ matrix.

Consider the polynomial $p(\lambda) = \det(A - \lambda I)$ with I the $n \times n$ identity matrix.

An <u>eigenvalue</u> λ corresponding to an <u>eigenvalue</u> v of A are defined to be $\lambda \in \mathbb{R}$ & $v \in \mathbb{R}^n$ such that

$$Av = \lambda v.$$

So λ is an eigenvalue of A

iff

λ is a root of p, i.e., $p(\lambda) = 0$

---

p is a polynomial of degree n.

So A has n eigenvalues (possibly complex, possibly repeated).

A has at most n linearly independent eigenvectors; it could have fewer if some eigenvalues are repeated.

Some types of A have n linearly independent eigenvectors for sure (even if some λ are repeated).

E.g.:

Symmetric A: $A = A^T$ all λ are real

Skew-symmetric: $A = -A^T$ all λ are imaginary

Orthogonal: $A^{-1} = A^T$ $|\lambda| = 1$ for all λ

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Special case: A symmetric & (and) $\underbrace{\text{positive definite}}$

Then all λ are real & positive

$\downarrow$ definition

$x^T A x > 0$
for all vectors $x \neq 0$.

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More general fact:

(Recall A^* means the conjugate transpose of A.)

$AA^* = A^*A$ iff $A = S \Lambda S^*$ with S unitary & Λ diagonal with entries the eigenvalues of A.

(S unitary means $SS^* = I = S^*S$, so then $S^{-1} = S^*$.)

E.g., if $AA^T = A^TA$ & A real, then fact applies & $S^{-1} = S^*$.
If also all λ real, then $S^{-1} = S^T$.

Example

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{(symmetric)}$$

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1$$
$$= (\lambda-5)(\lambda-3).$$

$\lambda = 5$ $\quad Av = 5v$, i.e., $(A-5I)v = \underline{0}$, i.e.,

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $v_1 = v_2$, i.e., eigenvector spans 45° line in $\mathbb{R}^2$.

$\lambda = 3$ $\quad Av = 3v$, i.e., $(A-3I)v = \underline{0}$, i.e.,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $v_1 = -v_2$, i.e., eigenvector spans -45° line in $\mathbb{R}^2$.

Example

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$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1$$

$$= (\lambda-5)(\lambda-3).$$

Eigenvector lines spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

So $A = S \Lambda S^{-1}$

with $\Lambda = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = S^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

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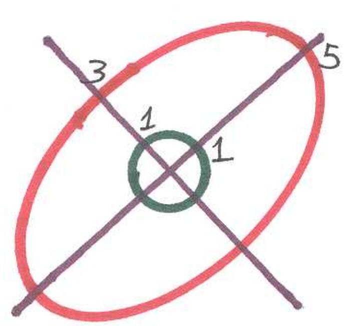
Geometrically, A dilates circles into ellipses:



To solve $Ax = b$, solve $\underbrace{\Lambda y = S^{-1}b}_{\text{easy; decoupled equations}}$.

Then $x = Sy$.

Ex: $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Solve $\Lambda y = S^{-1}b$,

i.e., $\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 11 \\ 1 \end{pmatrix}$.

So $y_1 = \frac{11}{5\sqrt{2}}$

& $y_2 = \frac{1}{3\sqrt{2}}$.

Then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{11}{5} \\ \frac{1}{3} \end{pmatrix}$

$$= \frac{1}{2}\frac{1}{15}\begin{pmatrix} 28 \\ 38 \end{pmatrix},$$

i.e., $x_1 = \frac{14}{15}$

& $x_2 = \frac{19}{15}$.

Equivalently, writing

$$A = S \Lambda S^{-1},$$

we see that

$$A^{-1} = S \Lambda^{-1} S^{-1},$$

which is easy to compute:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{15}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

Solving $Ax = b$ then means $x = A^{-1}b$, so for $b = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ we obtain

$$x = \frac{1}{15}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{14}{15} \\ \frac{19}{15} \end{pmatrix}.$$

Summary

If $A = S \Lambda S^{-1}$,

solve $Ax = b$ by

$$x = S \Lambda^{-1} S^{-1} b.$$

$$\begin{pmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_n \end{pmatrix}$$

That assumes all $\lambda_i \neq 0$, but we will shortly see a more general approach (via SVD).