Combinatorial

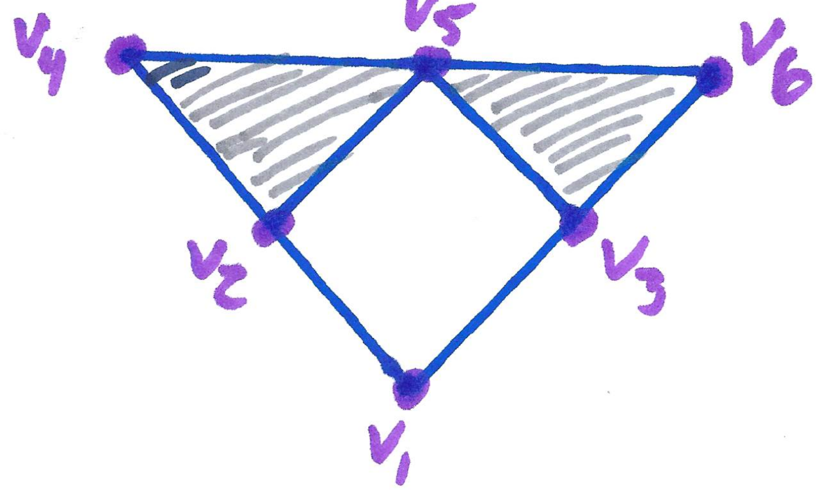
Topology

Def An abstract simplicial complex Σ with underlying vertex set V is a collection of finite subsets of V such that:

If σ is in Σ, then every subset of σ is also in Σ.

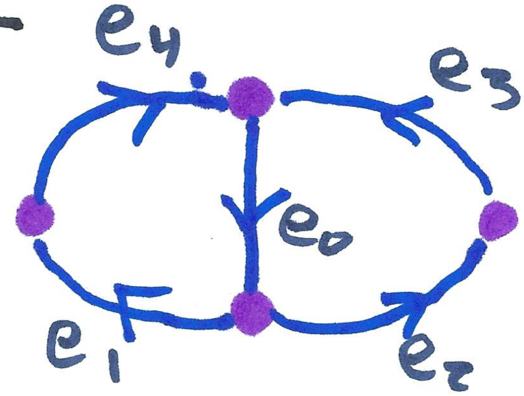
Example

Let Σ consist of all vertices, edges, and triangles in this diagram:



$$\Sigma = \{ \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_2, v_4, v_5\}, \{v_3, v_5, v_6\}, \emptyset \}$$

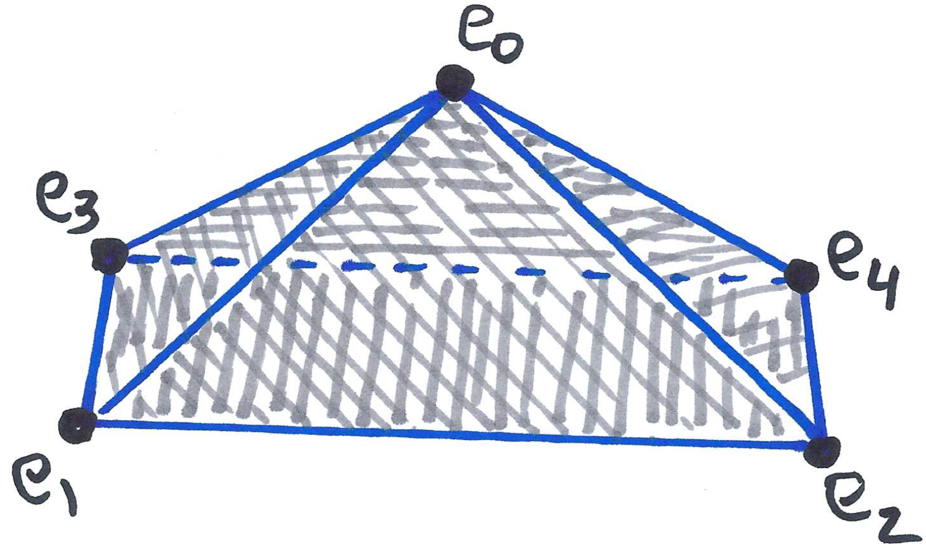
Example

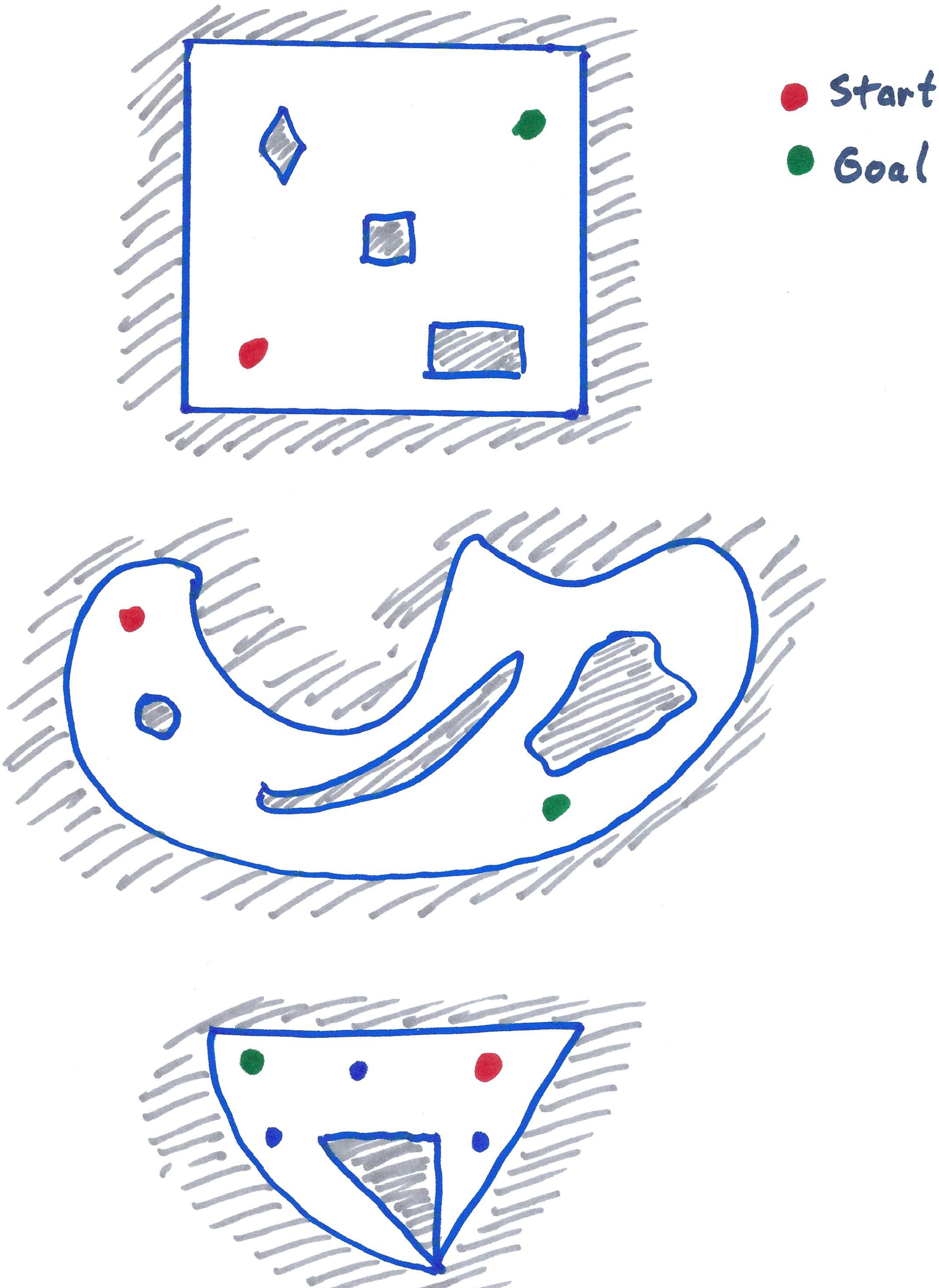


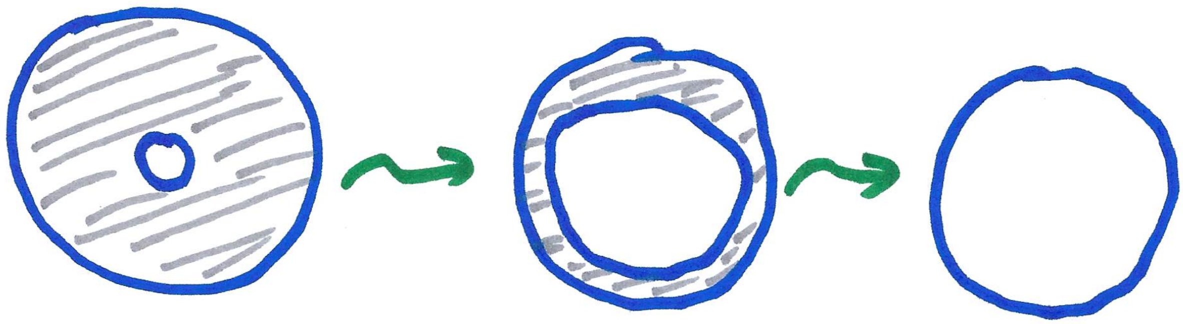
Let Σ's simplices be <u>sets</u> of edges, specifically:

$\sigma \in \Sigma$ iff no subset of σ (incl. σ) forms a directed cycle.

We can visualize Σ as a pointed hat glued to a base. Interior is hollow.




Start
Goal



<u>Def</u> Let $f: X \to Y$ & $g: X \to Y$ be continuous maps.

We say f & g are <u>homotopic</u> and write $f \simeq g$, if there is a continuous map

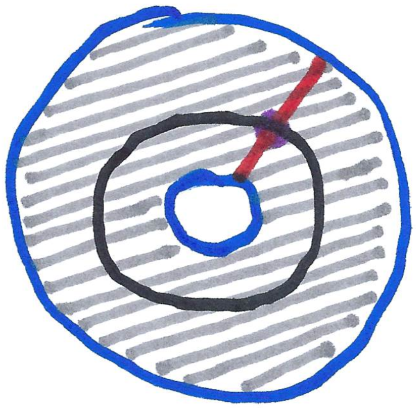
$$F: X \times [0,1] \longrightarrow Y \text{ such that}$$

$$F(x,0) = f(x)$$
$$\text{\& } F(x,1) = g(x) \text{ for all } x \in X.$$

<u>Def</u> We say two spaces X & Y are <u>homotopy equivalent</u> (or of the <u>same homotopy type</u> or even <u>homotopic</u>) and write $X \simeq Y$, if there are continuous functions $f: X \to Y$ & $g: Y \to X$ such that

$$g \circ f \simeq id_X$$

& $$f \circ g \simeq id_Y.$$



X is the wheel

Y is the circle

Define $f: X \to Y$ to collapse ray segments onto the circle.

Define $g: Y \to X$ to be inclusion.

Then $f \circ g \simeq id_Y$ since $f \circ g = id_Y$.

To see that $g \circ f \simeq id_X$, use

$$F: X \times [0,1] \longrightarrow X$$

$$F(x,t) = \frac{x}{\|x\|}\left(tr + (1-t)\|x\|\right),$$

with r the radius of the circle.

(We place the origin at the center of both circle & wheel.)

<u>Def</u> Suppose $A \subseteq X$, X a top. space.

A <u>deformation retraction</u> of X onto A is a continuous map

$$F : X \times [0,1] \longrightarrow X$$ such that

$$F(x,0) = x \text{ for all } x \in X,$$

$$F(x,1) \in A \text{ for all } x \in X,$$

& $$F(a,t) = a \text{ for all } a \in A$$
& $t \in [0,1]$.

One says that

A is a <u>deformation retract</u> of X.

Theorem

If A is a deformation retract of X, then $A \simeq X$.

Theorem

Let X & Y be topological spaces.

$X \simeq Y$ iff there exists a topological space Z along with imbeddings $h: X \to Z$ & $k: Y \to Z$ such that $h(X)$ & $k(Y)$ are both deformation retracts of Z.

Theorem

Homotopic spaces have isomorphic homology groups.

Corollary

If $X \simeq Y$ (& if both X & Y have finite triangulations) then $\chi(X) = \chi(Y)$.

Def A space which is homotopy equivalent to a point is called <u>contractible</u>.

Corollary

A contractible space has Euler characteristic 1.

Def

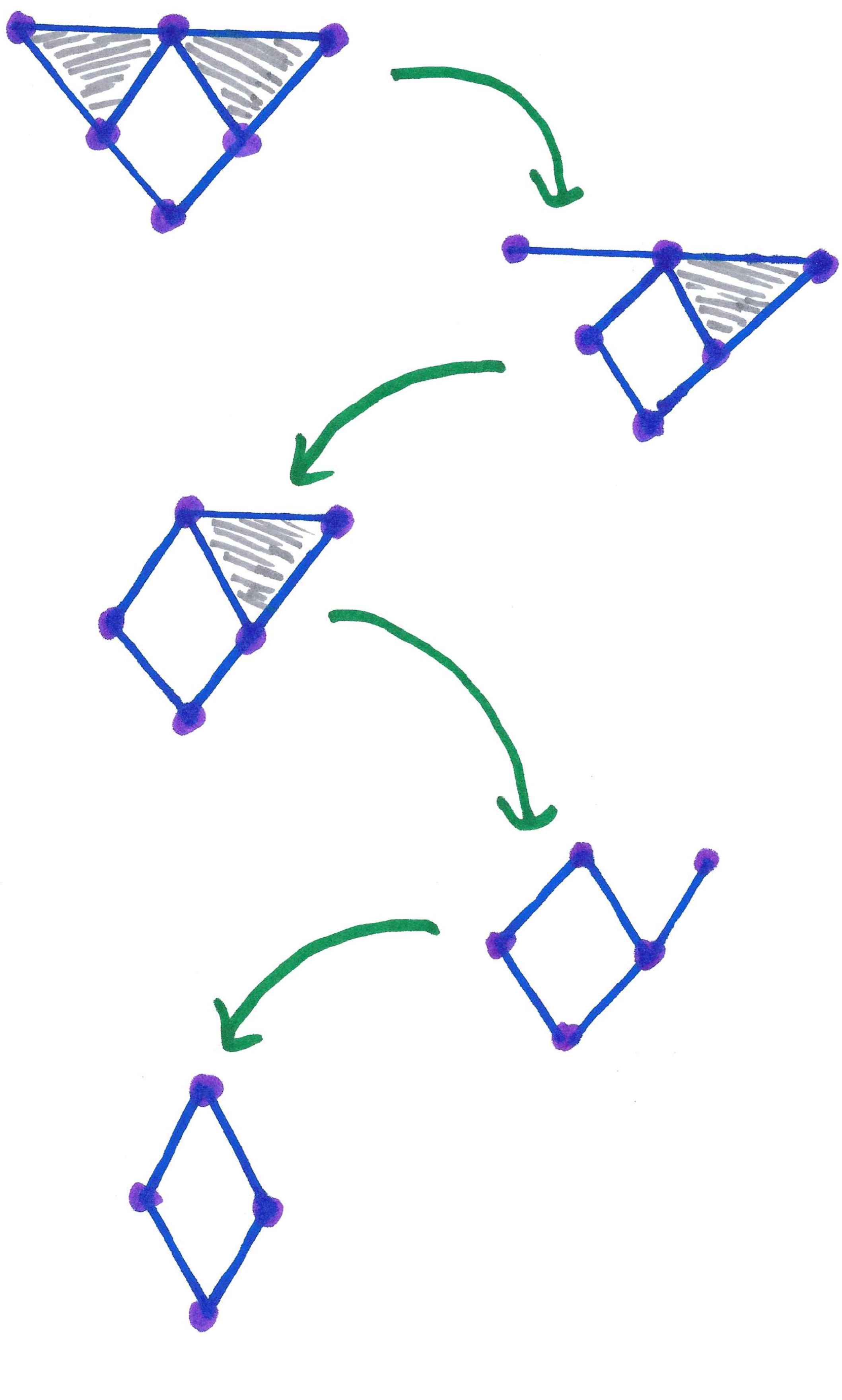
Suppose Σ is a simplicial complex and suppose σ & τ are simplices in Σ such that σ is a proper subset of τ but of no other simplex in Σ.

Let Σ' be the simplicial complex $\Sigma \setminus \{\sigma, \tau\}$.

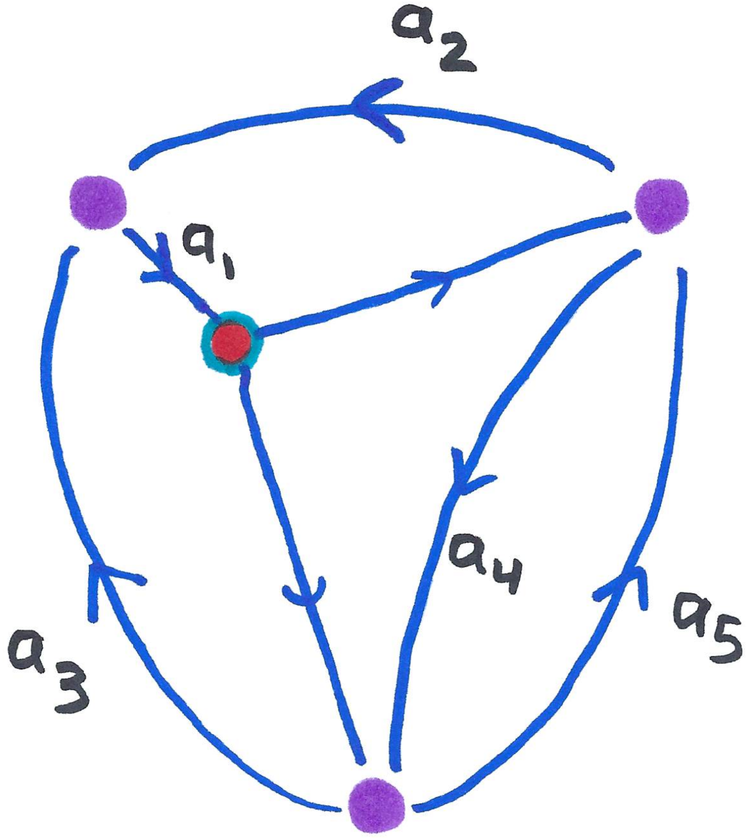
Removing σ & τ from Σ to form Σ' is called an <u>elementary collapse</u>.

<u>Observe:</u> $\Sigma' \simeq \Sigma$.

Example(s)

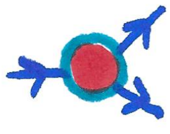


Nondeterministic Graphs



 represents a state/vertex

represents an action

 represents nondeterminism in an action

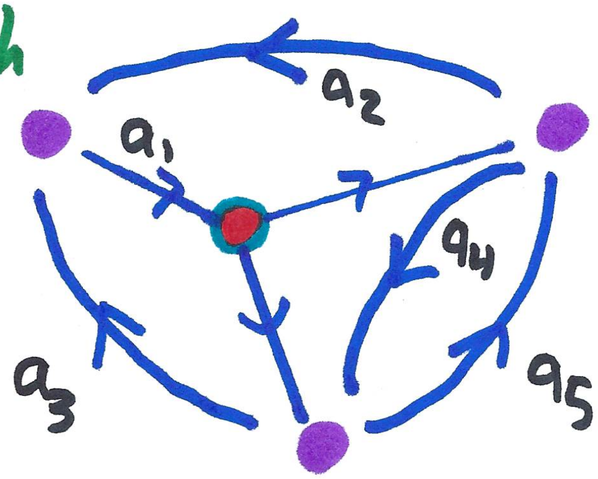
Def

Let G be a nondeterministic graph.

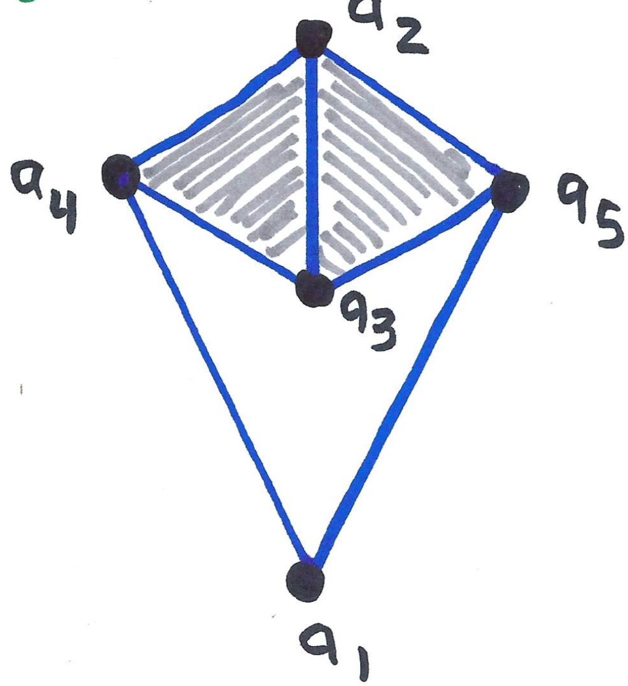
G's strategy complex has as simplices all sets of actions whose underlying arrows (i.e., possible transitions) do not form a cycle.

(Can generalize this definition to stochastic actions by considering recurrent classes in place of cycles.)

Graph



Strategy Complex



Defs Let G be a nondeterministic graph.

- The vertices of G are called **states**.
- The transitions possible out of a given state s are called the **actions at state s**. An action may have either a deterministic or nondeterministic outcome.
- An action at state s is said to have **source s**.
- **Δ_G** denotes G's strategy complex. Every $\sigma \in \Delta_G$ is called a **strategy**.
- Let $\sigma \in \Delta_G$. Every $a \in \sigma$ is called an **action**. **$src(\sigma)$** denotes the set of all states that are sources of some action in σ.
- G is **fully controllable** if for every state s in G there exists a strategy σ such that $src(\sigma) =$ all states of G except s.

Theorem

Let G be a nondeterministic graph with n states.

Then G is fully controllable

iff

$$\Delta_G \simeq S^{n-2}.$$

sphere of dimension $n-2$

Corollary

Suppose G is a nondeterministic graph with n states.

Let g be a desired goal state.

Let G_g consist of G with deterministic actions added from g to each of the other $n-1$ states.

Then there exists a strategy for moving from any state to g iff $\Delta_{G_g} \simeq S^{n-2}$.

Moreover, if no such strategy exists, then Δ_{G_g} is contractible.

Corollary

$\chi(\Delta_{G_g})$ is even

iff

there exists a strategy for attaining g from any state in G.

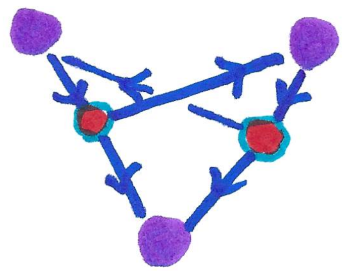
Proof

The Euler characteristic of a sphere is either 0 or 2.

The Euler characteristic of a contractible space is 1.

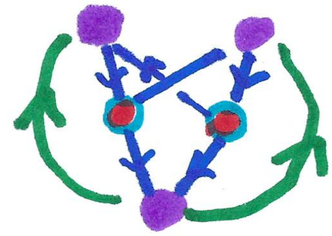
Example(s)

G



g

G_g

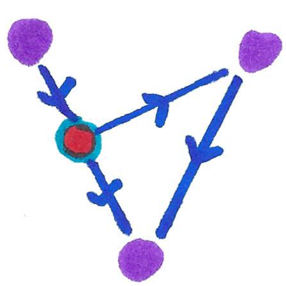


Δ_{G_g}

contractible

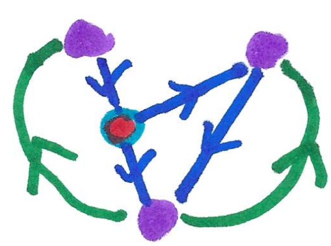
$\chi(\Delta_{G_g}) = 4-3=1$

G

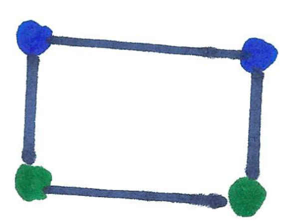


g

G_g



Δ_{G_g}



homotopic to S^1

$\chi(\Delta_{G_g}) = 4-4=0$