15-150

Principles of Functional Programming

Slides for Lecture 19

Parallelism, Cost Graphs, Sequences

April 7, 2020

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Lessons:

- Cost Semantics / Cost Graphs
- Brent’s Theorem
- Sequences

Parallelism:

Performing multiple computations simultaneously.

Scheduling:

Telling each processor what to do when.

This course focuses on *deterministic parallelism*:

- We allow *independent expressions* in a program to evaluate in parallel.
- We require parallel evaluation to have *well-defined behavior*.
- We do not worry explicitly about scheduling, but we use ***cost semantics*** to write code that facilitates parallelism.

(Functional programming languages without side-effects facilitate this approach.)

What can a programmer do to facilitate parallelism?

- Write code that does **not** *bake in* a schedule. (Lists bake in sequential evaluation. Trees facilitate parallelism. Today we will introduce an abstract datatype called ***sequences***. Sequences have a linear structure like lists but support the parallelism of trees.)
- Reason about time complexity (Work & Span) to write fast parallel code. (You have been doing that with recurrences. Today we will introduce ***cost graphs*** as another tool.)

Cost Graphs

Cost graphs are a form of *series-parallel* graph.

Such a graph is a directed acyclic graph,
with designated ***source*** and ***sink*** nodes.

(***Source*** means there are no incoming edges.
Sink means there are no outgoing edges.)
We draw graphs with source at top and sink at bottom.
All edges directed downward.)

We will use cost graphs to model computations and to compute Work and Span.

Basic Constructions

Base Case:  (single node, source=sink, modeling no computation)

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Sequential Composition:

$$\begin{array}{c} \mathbf{G_1} \\ \downarrow \\ \mathbf{G_2} \end{array}$$

(Edge from $\mathbf{G_1}$'s sink to $\mathbf{G_2}$'s source, modeling sequential computation: perform $\mathbf{G_1}$'s computation, then $\mathbf{G_2}$'s.

Basic Constructions

Base Case:

(single node, source=sink, modeling no computation)

Sequential Composition:

$$\begin{array}{c} \mathbf{G_1} \\ \downarrow \\ \mathbf{G_2} \end{array}$$

(Edge from $\mathbf{G_1}$'s sink to $\mathbf{G_2}$'s source, modeling sequential computation)

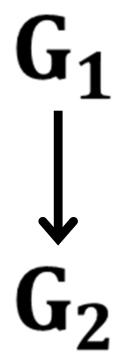
Special case:

(one evaluation step)

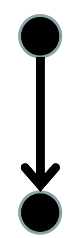
Basic Constructions

Base Case:  (single node, source=sink, modeling no computation)

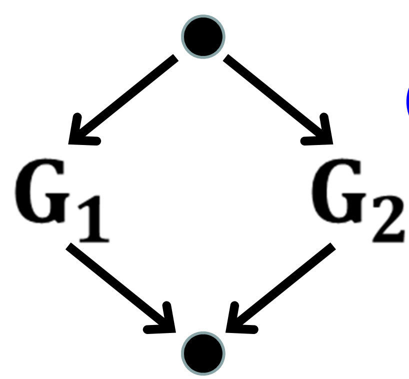
Sequential Composition:



(Edge from $\mathbf{G_1}$'s sink to $\mathbf{G_2}$'s source, modeling sequential computation)

Special case:  (one evaluation step)

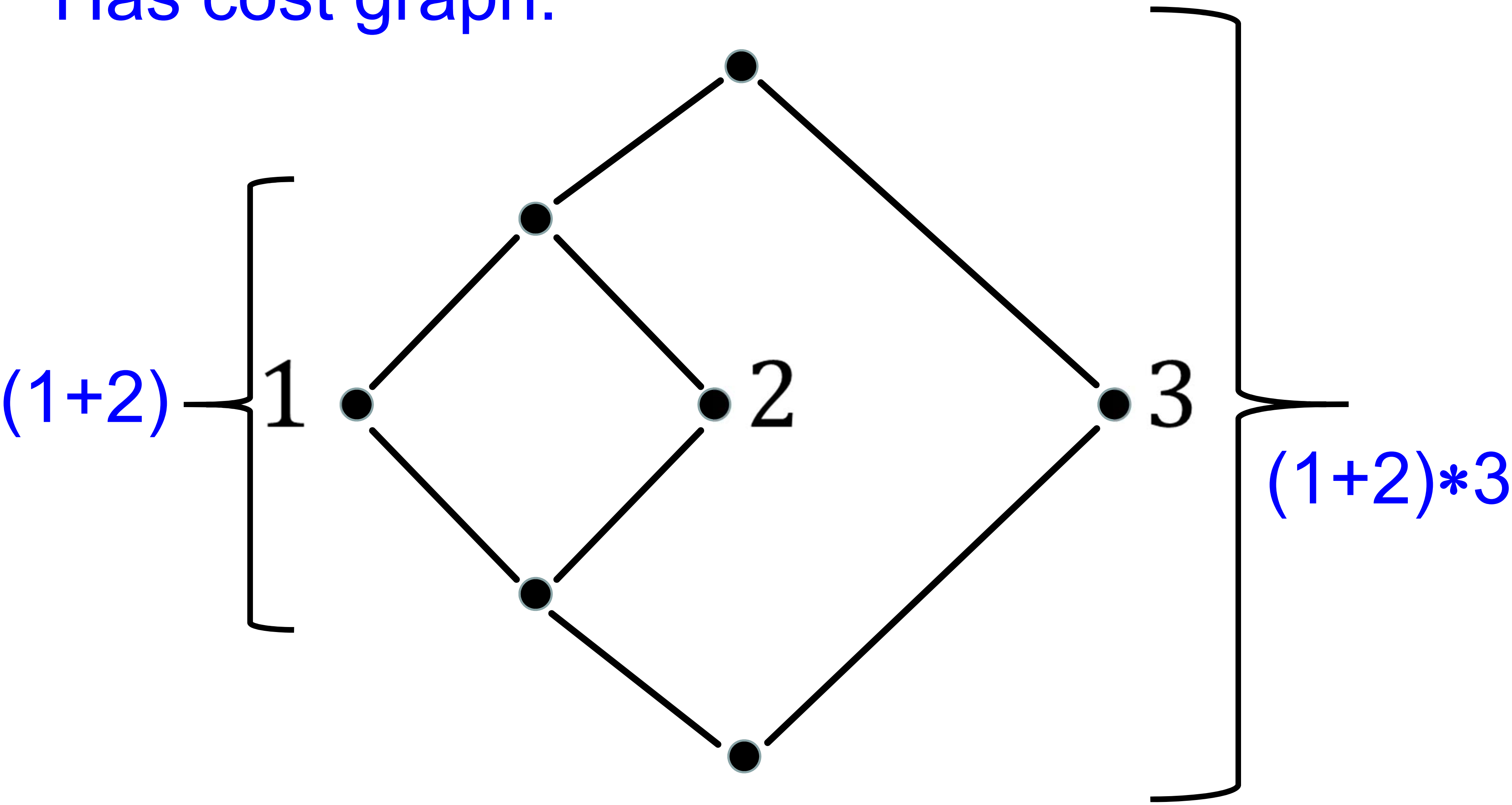
Parallel Composition:



(Fork and Join: new source with edges to original sources of $\mathbf{G_1}$ and $\mathbf{G_2}$, then edges from their sinks to a new sink. Models parallel computation.)

Example: $(1+2)*3$

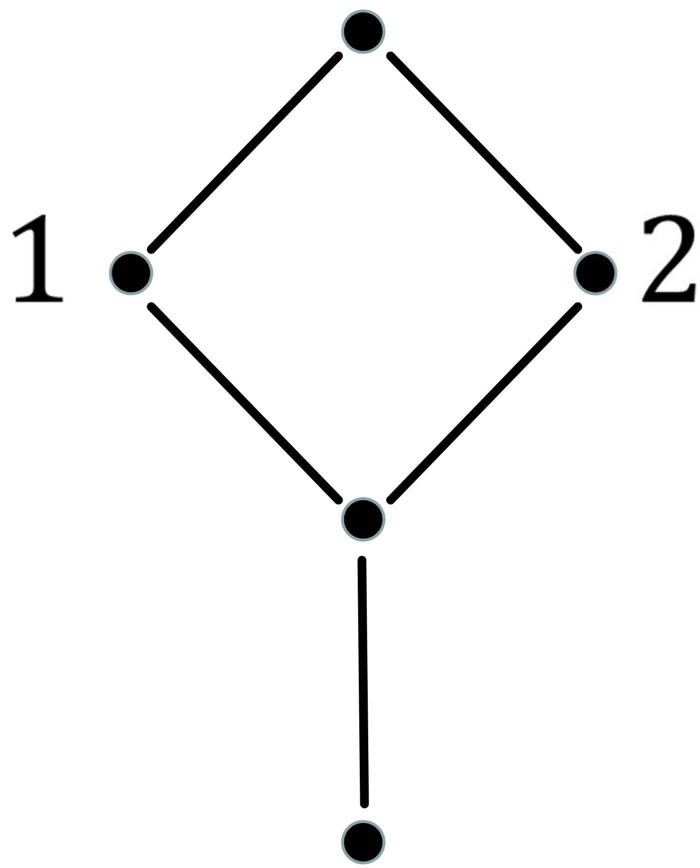
Has cost graph:



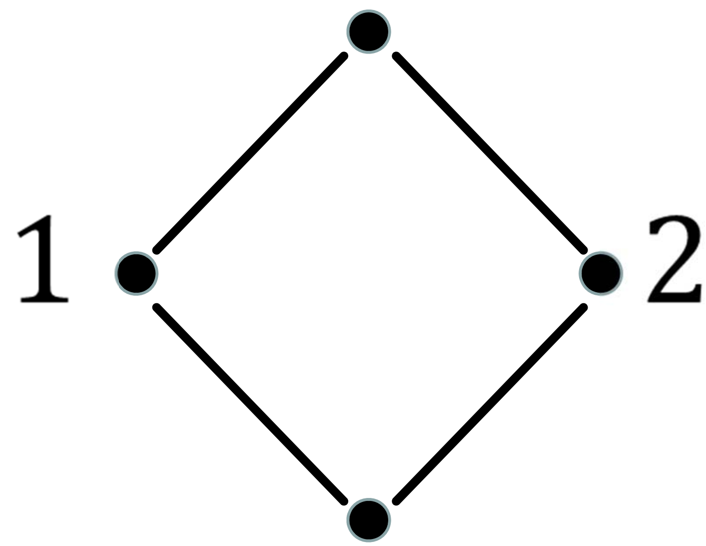
(Edges are implicitly directed downward.)

We are being a little sloppy but it is fine.

Technically, $(1 + 2)$ has cost graph:



We elide that to:

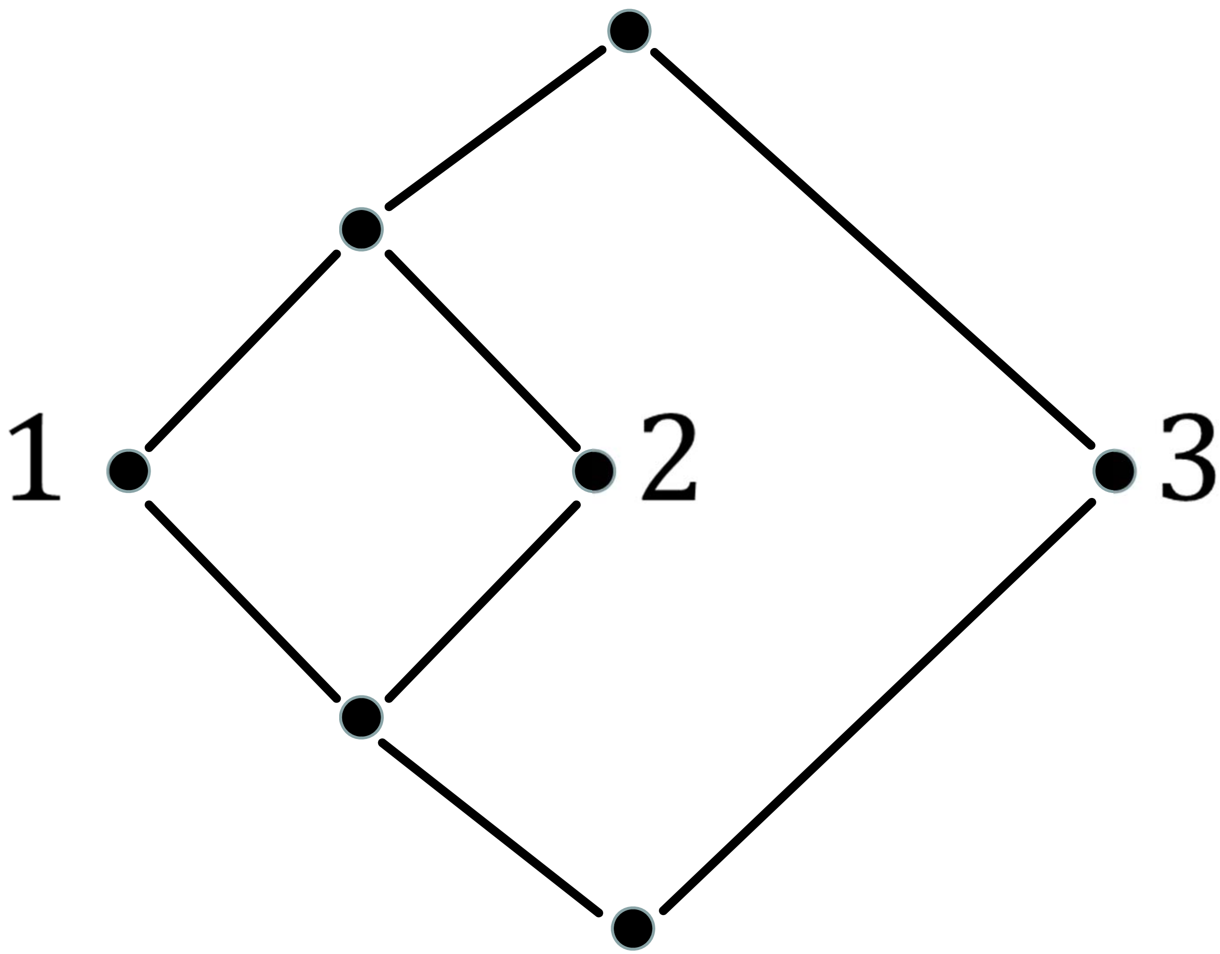


Work and Span

- We define the ***work of a cost graph*** **G** to be the **number of nodes** in **G**.
- We define the ***span of a cost graph*** **G** to be the number of nodes on the **longest path** from **G**’s source to **G**’s sink.
- We now **re-define** the **work** and **span** of an expression **e** to be the work and span of the cost graph **G** representing **e**.

(These numbers differ by constant factors/terms from our earlier definitions, but will be the same asymptotically.)

Example: $(1 + 2) * 3$



Work = 7

Span = 5

Brent's Theorem

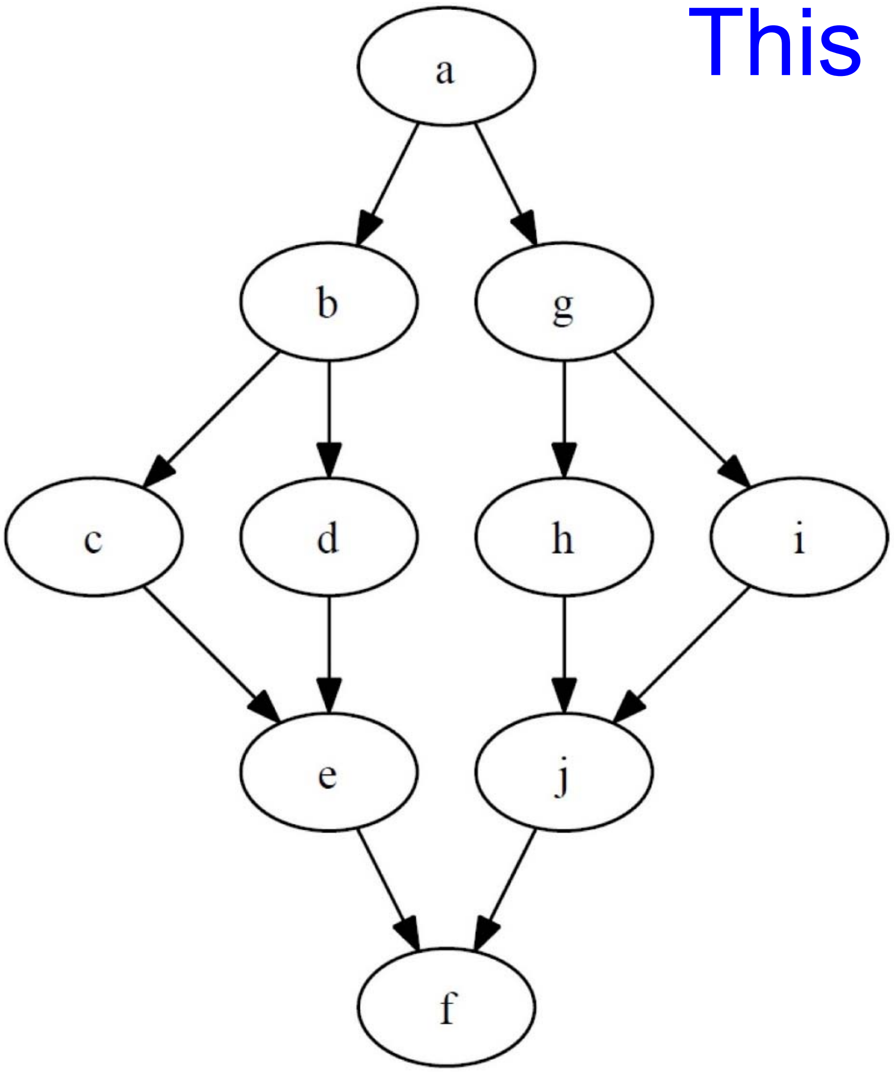
An expression **e** with work W and span S can be evaluated on a p-processor machine in time $O(\max(W/p, S))$.

Scheduling

- (This is a bit of side-topic, just to show you how one might use cost graphs to schedule.)
- We will use *pebbling*:
  - p pebbles, with p the number of processors.
  - Start with one pebble on cost graph $\mathbf{G}$'s source.
  - Putting a pebble on a node *visits* the node.
  - At each time step, pick up all pebbles and put at most p on the graph, no more than one per node. Can only put a pebble on an unvisited node all of whose ancestors have been visited.

(There are various kinds of pebbling strategies.)

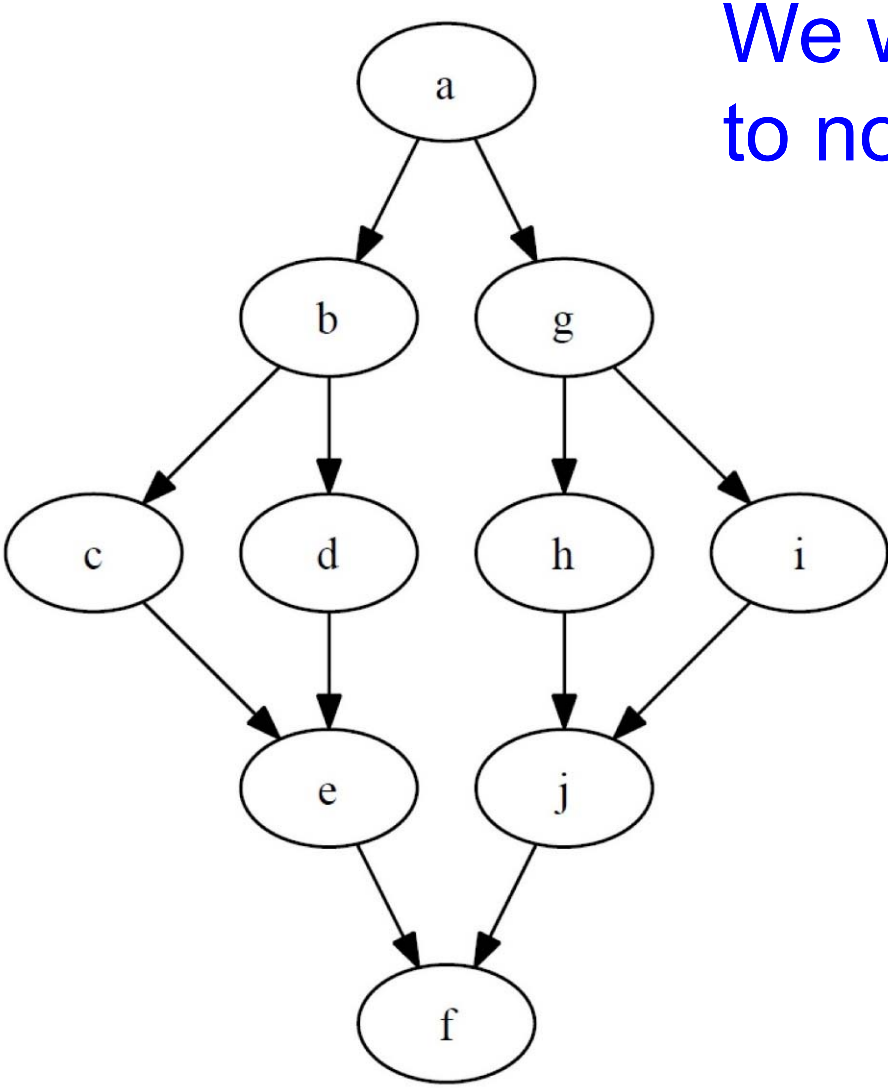
Breadth-First Pebbling Algorithm



This might be a cost graph for

$$(1 + 2) * (3 + 4)$$

Breadth-First Pebbling Algorithm

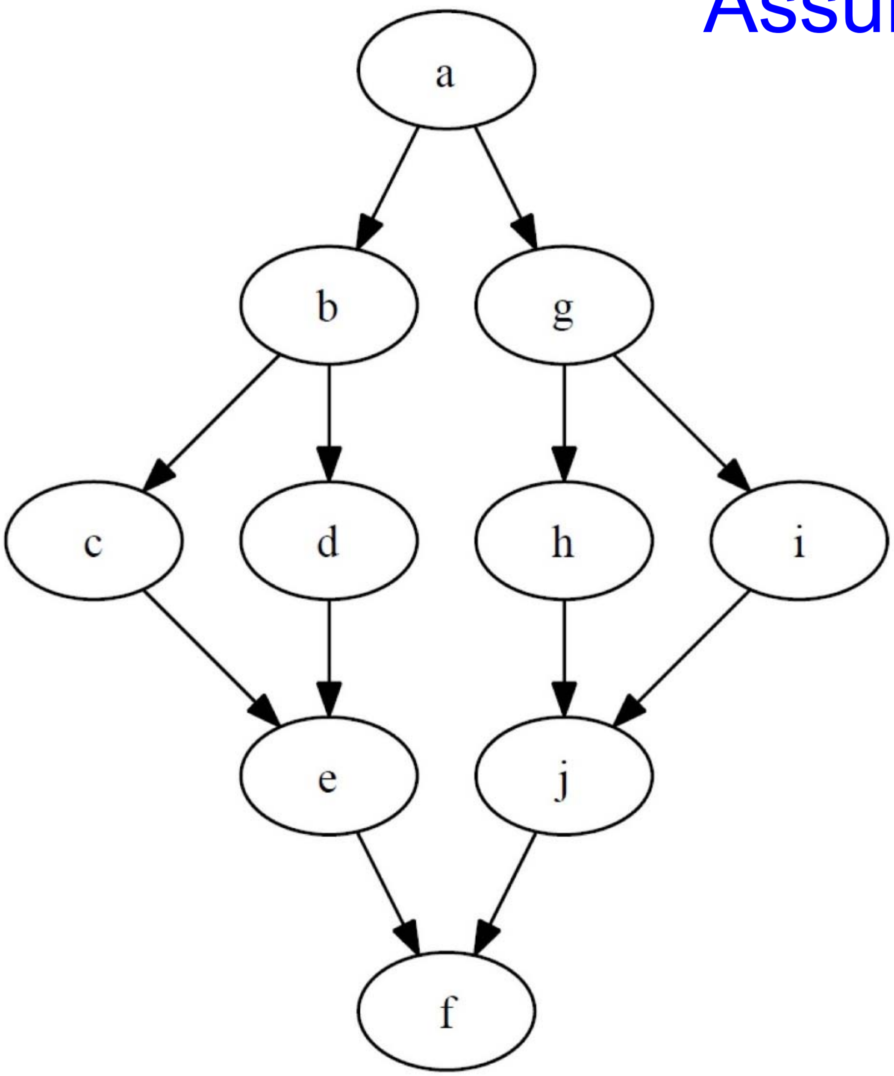


We wish to assign processors to nodes at successive time steps.

[At each time step, the processor assigned to a node will perform the computation represented by the node and its incident edges
(e.g., fork, join, arithmetic).]

Breadth-First Pebbling Algorithm

Assume we have 2 processors.

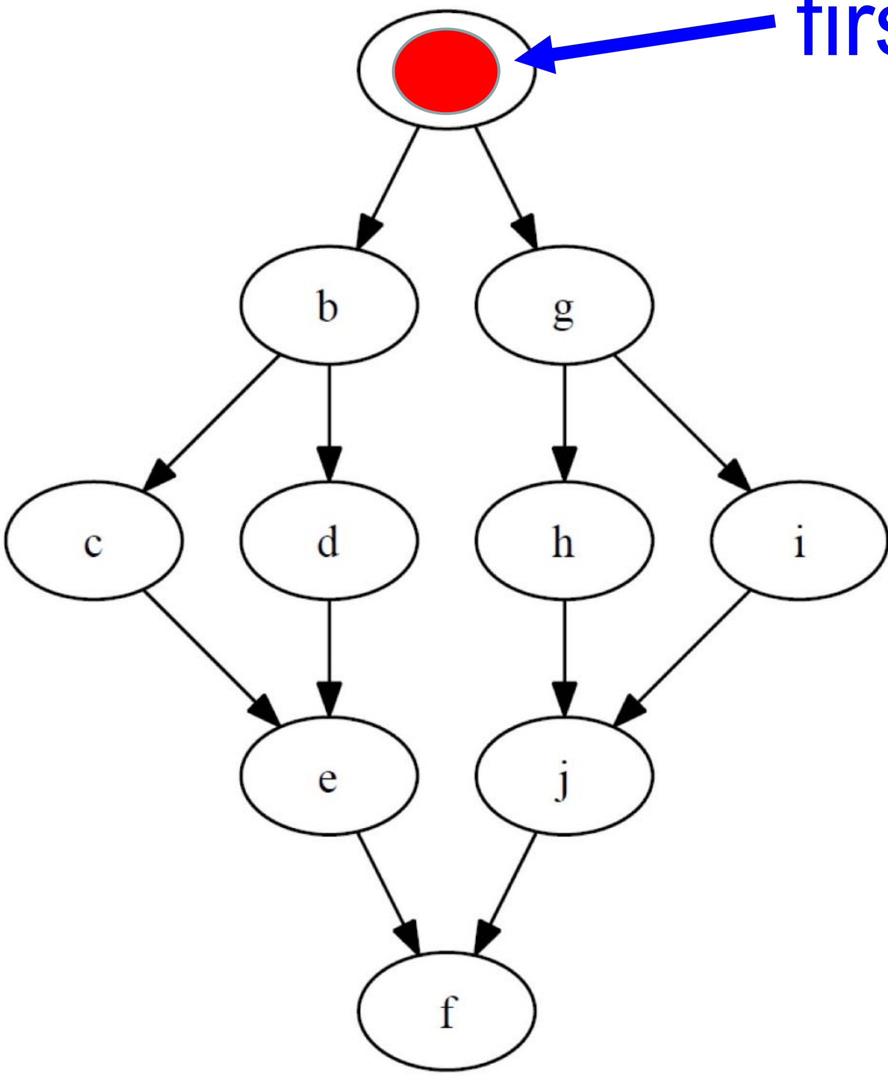


processors

time

	1	2
1		
2		
3		
4		
5		
6		

Breadth-First Pebbling Algorithm

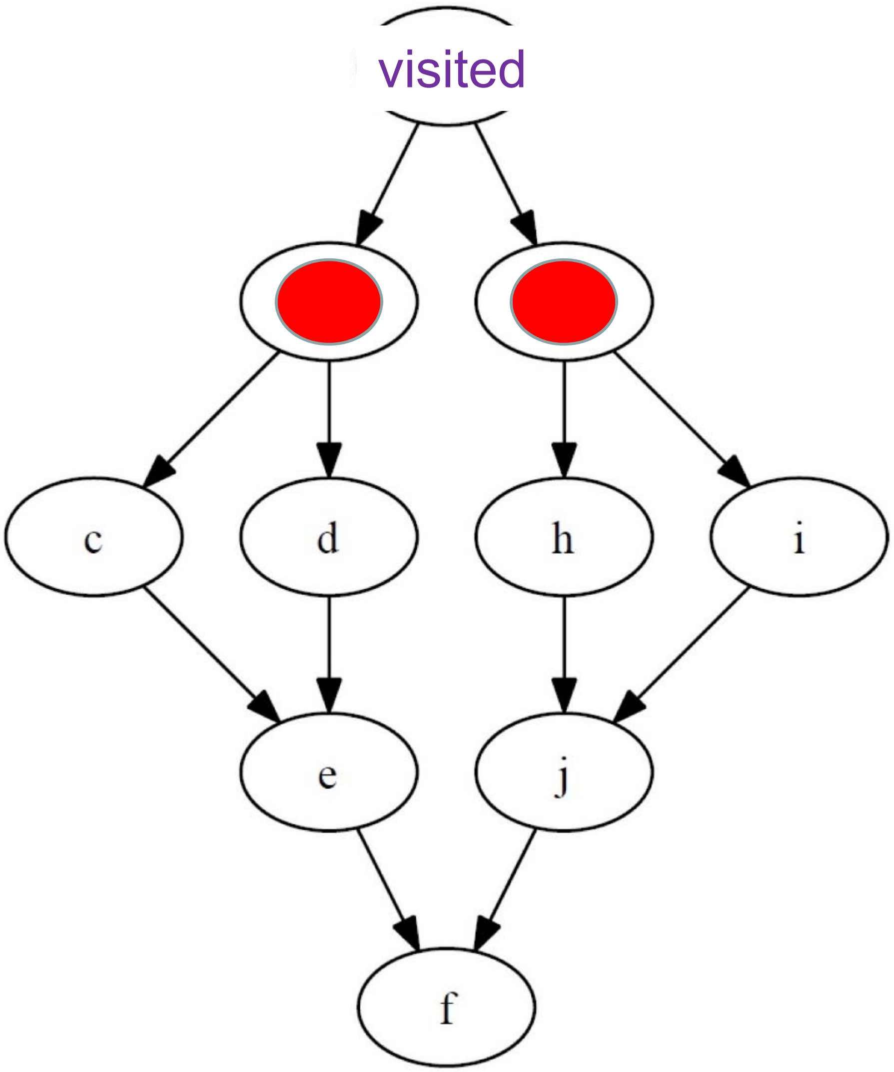


first pebble on node "a"

processors

time	1	2
1	a	(idle)
2		
3		
4		
5		
6		

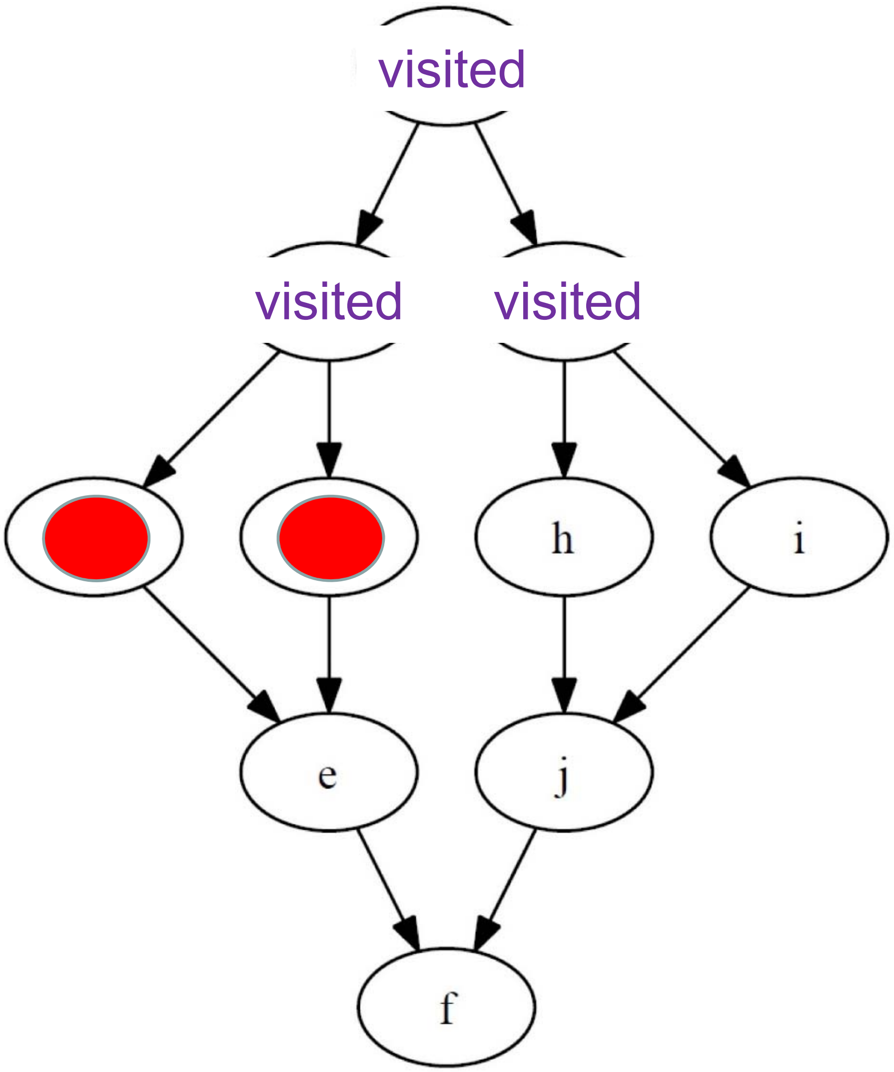
Breadth-First Pebbling Algorithm



processors

time	1	2
1	a	
2	b	g
3		
4		
5		
6		

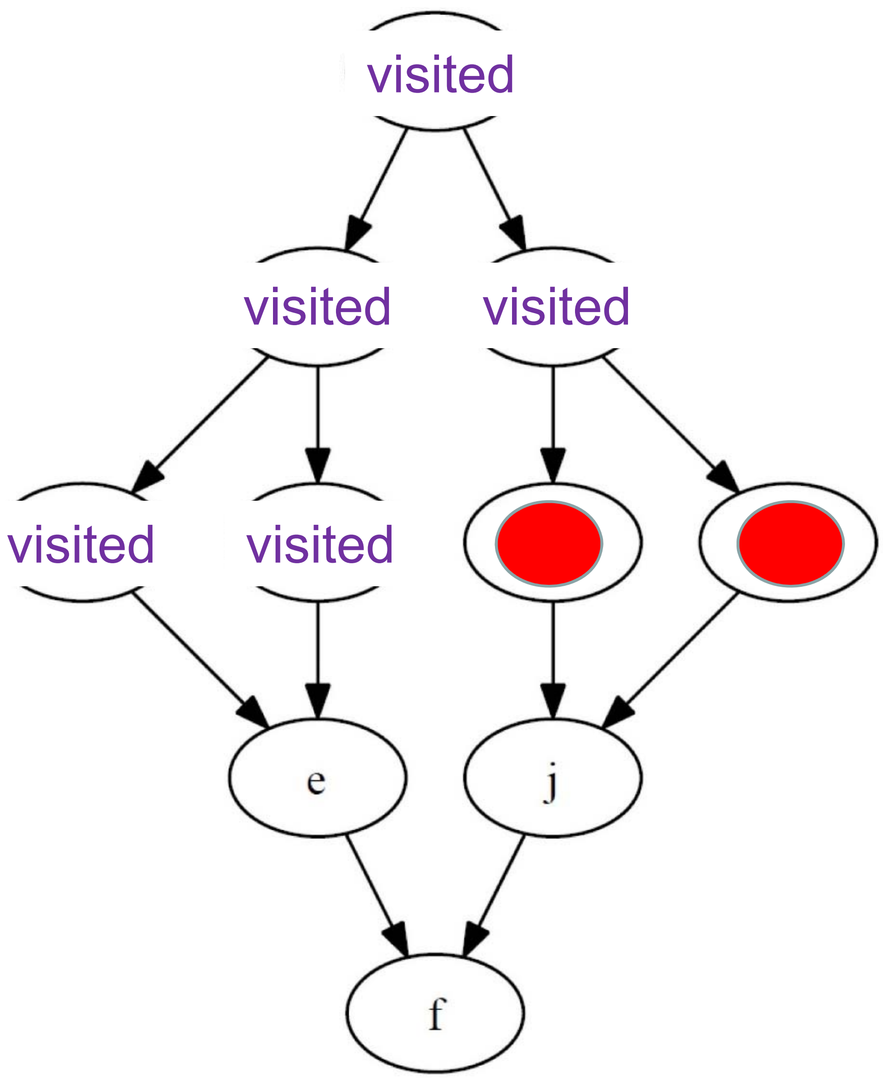
Breadth-First Pebbling Algorithm



processors

time	1	2
1	a	
2	b	g
3	c	d
4		
5		
6		

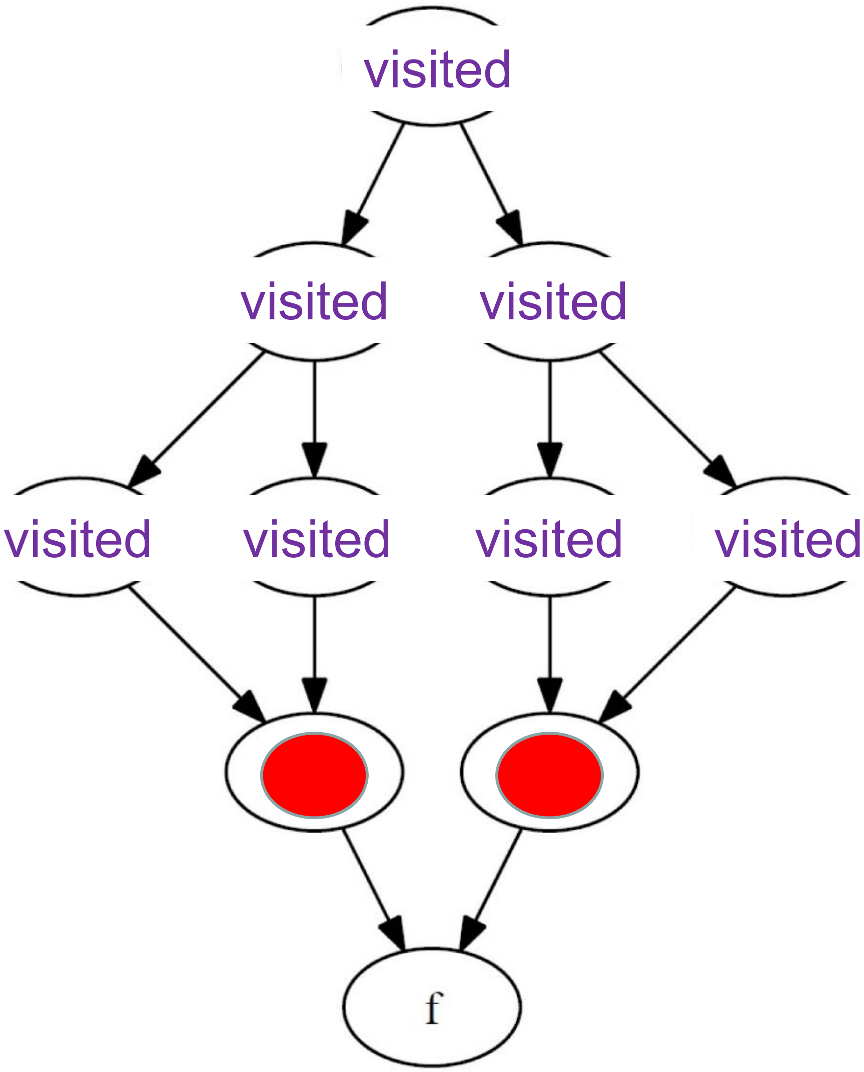
Breadth-First Pebbling Algorithm



processors

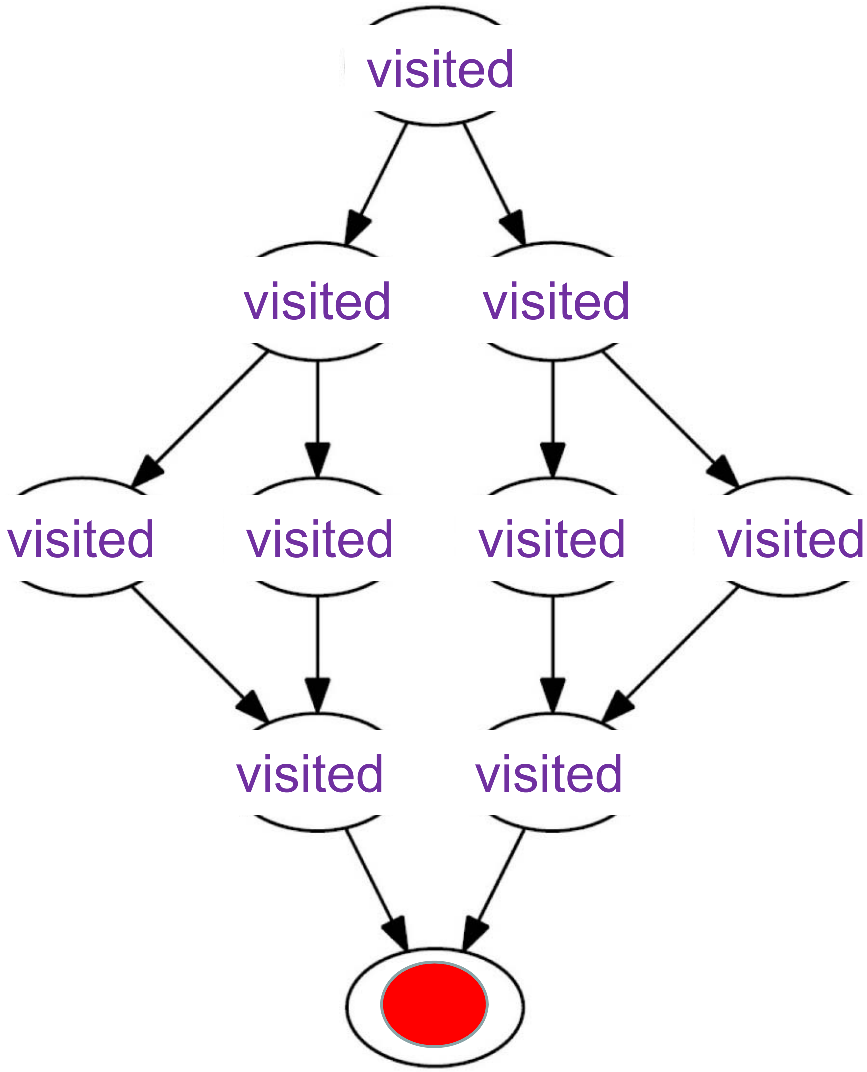
time	1	2
1	a	
2	b	g
3	c	d
4	h	i
5		
6		

Breadth-First Pebbling Algorithm



	processors	
time	1	2
1	a	
2	b	g
3	c	d
4	h	i
5	e	j
6		

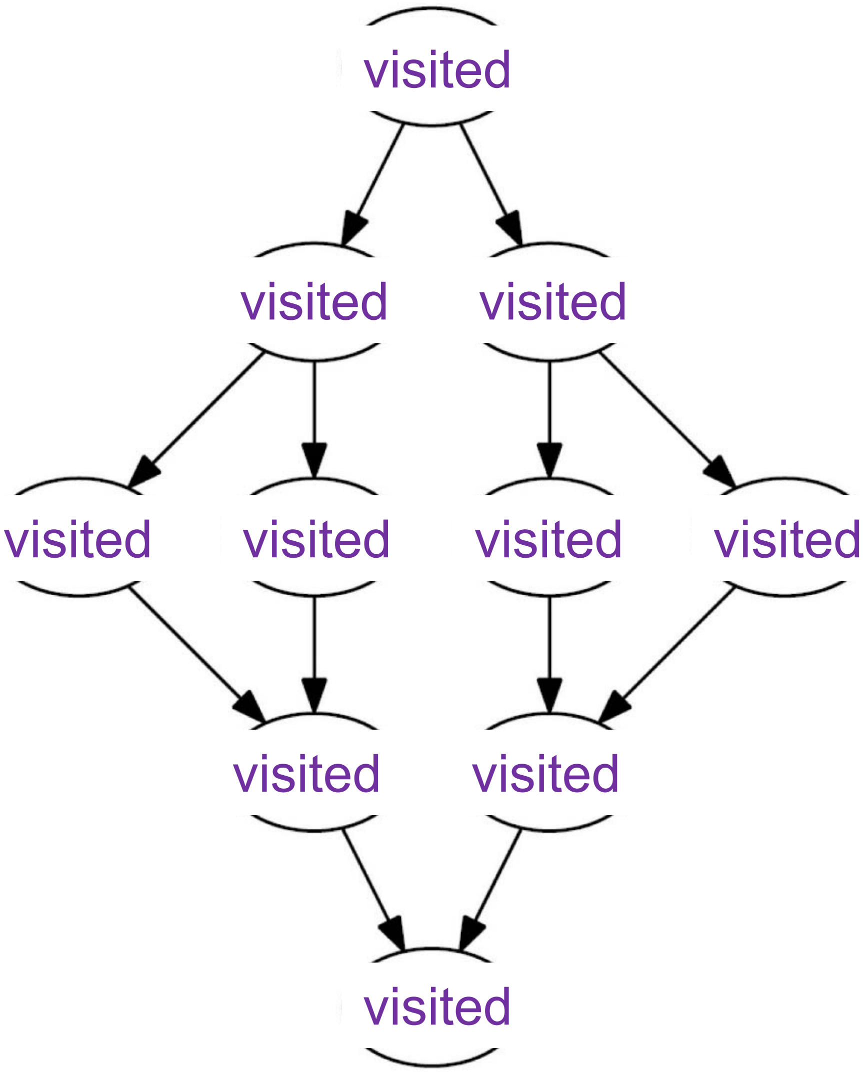
Breadth-First Pebbling Algorithm



processors

time	1	2
1	a	
2	b	g
3	c	d
4	h	i
5	e	j
6		f

Work = 10, Span = 5, so Brent's Theorem suggests
$5 = \max(10/2, 5)$ time steps might be possible.
But we have some idle times, so get 6.
(Also: Brent's Theorem is only an asymptotic assertion.)



time	processors 1	processors 2
1	a	
2	b	g
3	c	d
4	h	i
5	e	j
6		f

Sequences

- We will present (part of the) **`SEQUENCE`** signature.
- We will describe the **work** and **span** of some sequence functions via **cost graphs**.
- Sequences are abstract. Hidden implementation.
- For reasoning purposes, we write a sequence of length **n** containing elements $\mathtt{x_0, \ldots, x_{n-1}}$ as

  $$\mathtt{<x_0, \ldots, x_{n-1}>}\ .$$

- Two sequence values are extensionally equivalent iff they have the same length and contain extensionally equivalent values at corresponding positions.

```
signature SEQUENCE =
sig
  type 'a seq      (* abstract *)

  exception Range of string

  val empty : unit -> 'a seq

  val tabulate : (int -> 'a) -> int -> 'a seq
  val length : 'a seq -> int
  val nth    : 'a seq -> int -> 'a

  val map : ('a -> 'b) -> 'a seq -> 'b seq
  val reduce : ('a * 'a -> 'a) -> 'a -> 'a seq -> 'a

  val mapreduce :
          ('a -> 'b) -> 'b -> ('b * 'b -> 'b) -> 'a seq -> 'b

  val filter : ('a -> bool) -> 'a seq -> 'a seq

  ...
end
```

Most of those functions should seem familiar from lists.

One difference is that instead of **`foldr`** and **`foldl`** we now have **`reduce`**.   We will talk more about that.

You probably never used **`List.tabulate`**.
We will discuss **`tabulate`** for sequences.

Unlike lists, sequences support parellization, giving good span costs for many functions.

sequence type

$$\texttt{<x}_0\texttt{, …, x}_{n-1}\texttt{> : t seq}$$

if $\mathtt{x_i : t}$,

for $\mathtt{i = 0, ..., n-1}$.

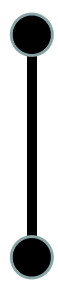
`empty`

`empty ()`

returns a sequence of length **`0`**,
containing no elements.

The type can be **`t seq`**, for any type **`t`**.

Cost Graph:

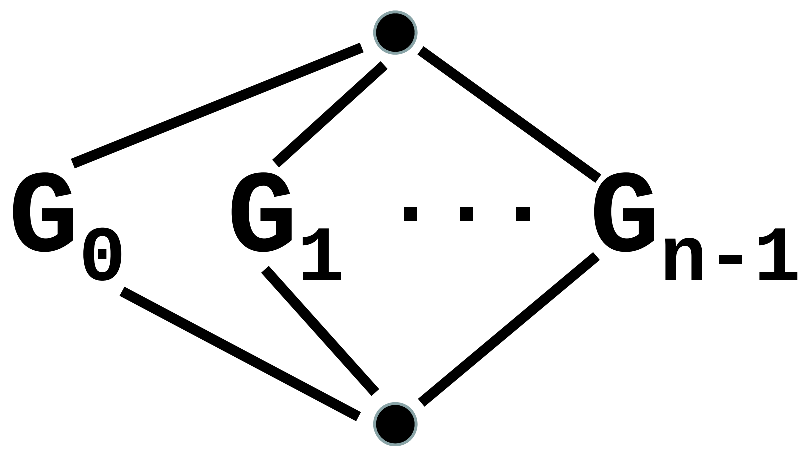


So O(1) work and span.

tabulate

tabulate f n $\cong$ **<f(0), …, f(n-1)>**

Cost Graph:



Here G_i is the cost graph for evaluating **f(i)**.

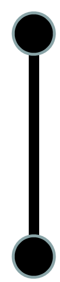
If **f(i)** has O(1) work and span for all **i**, then **tabulate f n** has O(n) work and O(1) span.

nth

$\texttt{nth}\ \langle \mathtt{x}_0,\ \ldots,\ \mathtt{x}_{n-1} \rangle\ \mathtt{i} \cong \mathtt{x}_\mathtt{i}$, if $\mathbf{0} \le \mathtt{i} < \mathtt{n}$,

raises **Range** otherwise.

Cost Graph:

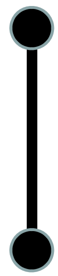


So O(1) work and span.

In other words, constant time access
to elements (unlike lists).

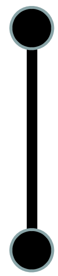
`length`

`length <x`$_0$`, …, x`$_{n-1}$`>` $\cong$ `n` .

Cost Graph:  Again, O(1) work and span.

length

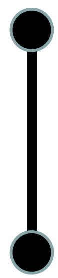
$\texttt{length}\ \langle x_0, \ldots, x_{n-1} \rangle \cong n.$

Cost Graph:  Again, O(1) work and span.

Question: How could one achieve this?

`length`

$\texttt{length}\ \langle x_0,\ \ldots,\ x_{n-1}\rangle \cong n.$

Cost Graph:  Again, O(1) work and span.

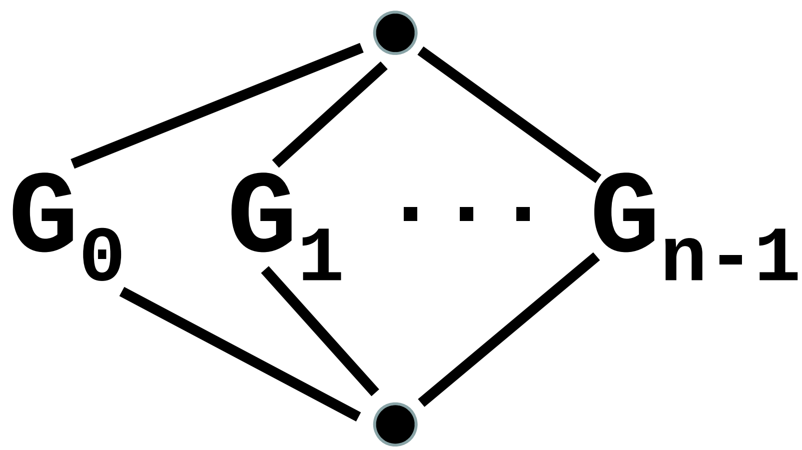
Question: How could one achieve this?

Answer: Keep track of length explicitly
in the underlying representation of sequences.

map

$\mathtt{map\ f\ <x_0,\ldots,x_{n-1}>} \cong \mathtt{<f\ x_0\ ,\ldots,\ f\ x_{n-1}>}$

Cost Graph:



Here $\mathtt{G_i}$ is the cost graph for evaluating $\mathtt{f(x_i)}$.

If $\mathtt{f(x)}$ has O(1) work and span for all $\mathtt{x}$, then $\mathtt{map\ f\ <x_0,\ldots,x_{n-1}>}$ has O(n) work & O(1) span.

reduce

Recall the type:

```
reduce : ('a * 'a -> 'a) -> 'a -> 'a seq -> 'a
```

That is more restrictive than the type of **`foldr`** was:

```
foldr : ('a * 'b -> 'b) -> 'b -> 'a list -> 'b
```

Let’s explore that.

`reduce : ('a * 'a -> 'a) -> 'a -> 'a seq -> 'a`

reduce

$$\texttt{reduce g z <}x_0,\ldots,x_{n-1}\texttt{>} \cong x_0 \odot \cdots \odot x_{n-1} \odot z$$

We assume that **g** is *associative*, meaning **g(g(x,y),w) ≅ g(x,g(y,w))**, for all values **x**,**y**,**w** of the correct type. So no parentheses are needed on the right, where we represent **g** by the infix operator $\odot$.

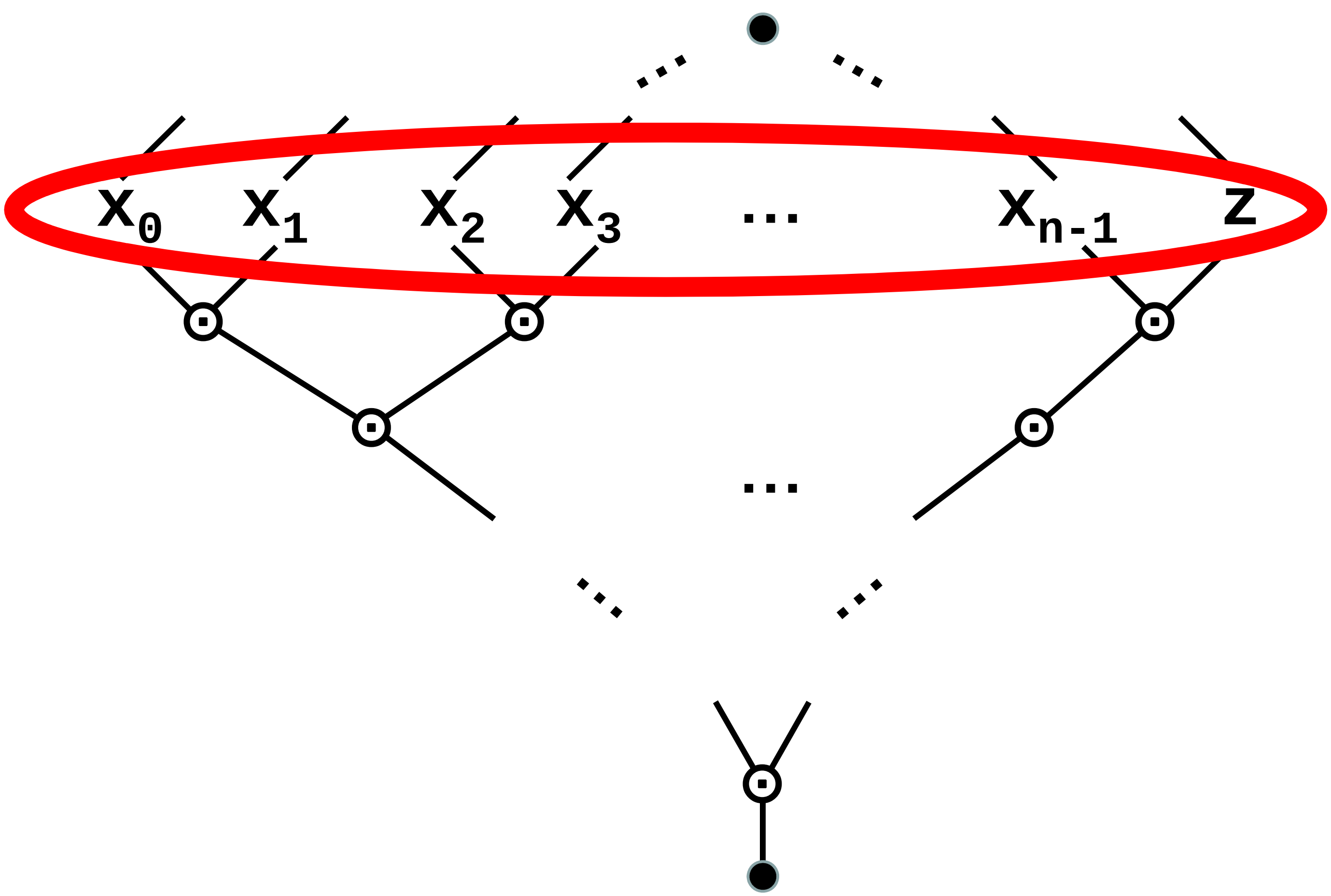
[In 15-210 you will generally assume as well that **z** is an *identity* (also called a *zero*) for **g**, meaning **g(x,z) ≅ x ≅ g(z,x)**, for all values **x** of the correct type. We do that sometimes in 15-150 but it can be useful to allow more general **z** (thus mimicking a list **foldr**).]

`reduce`

`reduce g z <`x_0`,…,`x_{n-1}`>` $\cong$ $x_0 \odot \cdots \odot x_{n-1} \odot z$

Cost Graph:

(forking abbreviated)



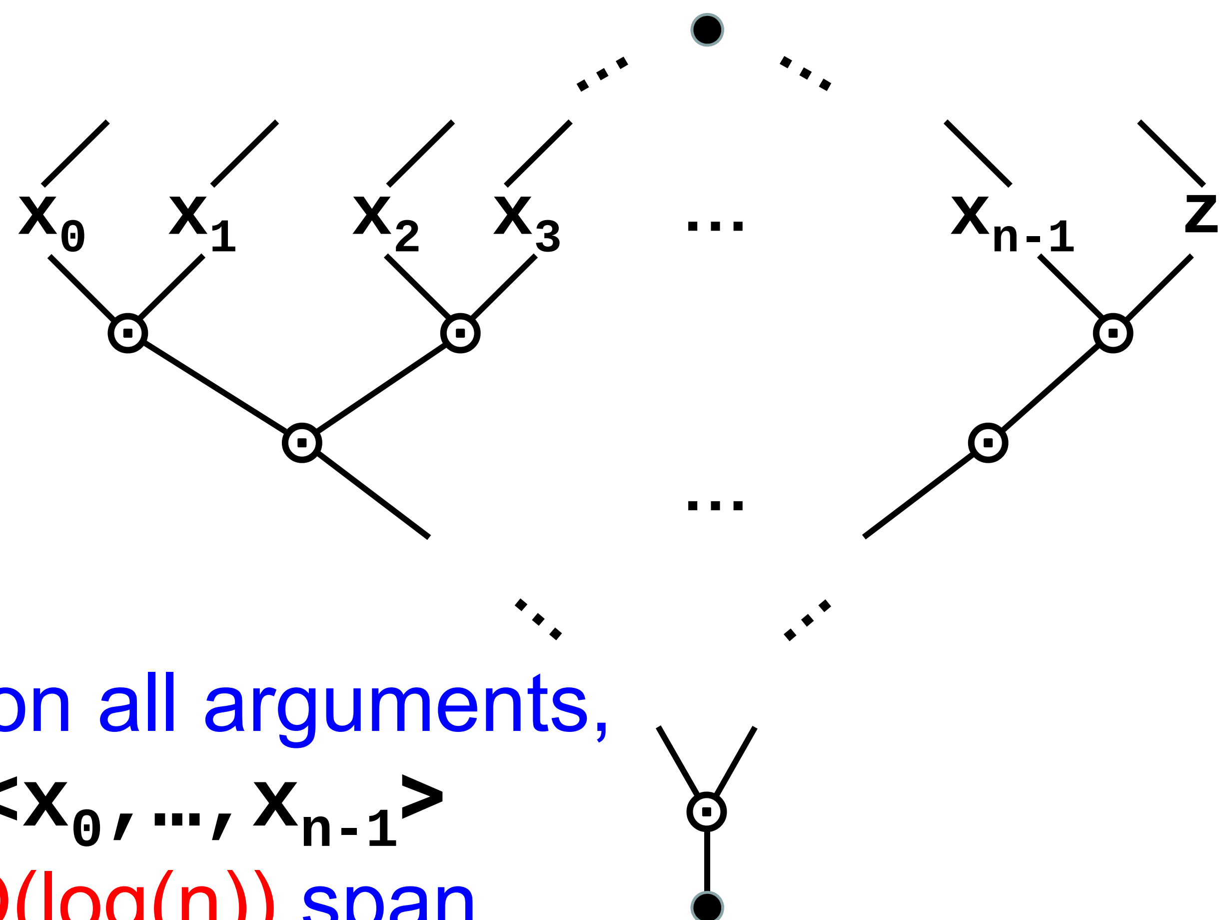
reduce

$$\texttt{reduce g z} \langle x_0, \ldots, x_{n-1} \rangle \cong x_0 \odot \cdots \odot x_{n-1} \odot z$$

Cost Graph:

(forking abbreviated)

O(log(n)) levels)



If **g** is constant time on all arguments, then $\texttt{reduce g z} \langle x_0, \ldots, x_{n-1} \rangle$ has O(n) work and O(log(n)) span.

mapreduce

mapreduce combines **map** and **reduce**:

$$\texttt{mapreduce f z g } \langle x_0, \ldots, x_{n-1} \rangle$$

$$\cong$$

$$(\texttt{f } x_0) \odot \cdots \odot (\texttt{f } x_{n-1}) \odot \texttt{z}$$

(here we again represent **g** by the infix operator $\odot$)

So, if **f** and **g** have O(1) work and span on all arguments, then $\texttt{mapreduce f z g } \langle x_0, \ldots, x_{n-1} \rangle$ has O(n) work and O(log(n)) span.

filter

filter p s $\cong$ **s'**,

with **s'** a sequence consisting of all $\mathbf{x_i}$ in **s** such that $\mathbf{p(x_i)} \cong$ **true** . The order of retained elements in **s'** is the same as in **s**.

If **p** has O(1) work and span on all arguments, then **filter p s** has O(n) work and O(log(n)) span (this is not obvious; you will learn more in 15-210).

Example (recall also Lecture 1):

```
fun sum (s : int Seq.seq) : int =
      Seq.reduce (op +) 0 s

type row = int Seq.seq
type room = row Seq.seq

fun count (class : room) : int =
      sum (Seq.map sum class)
```

(Here we are assuming a structure **`Seq`** ascribing to signature **`SEQUENCE`**.)

Example (recall also Lecture 1):

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Let value **`c:room`** contain **`n`** **`row`**s of length **`n`** each.
What is the work and span to evaluate **`count(c)`**?

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```

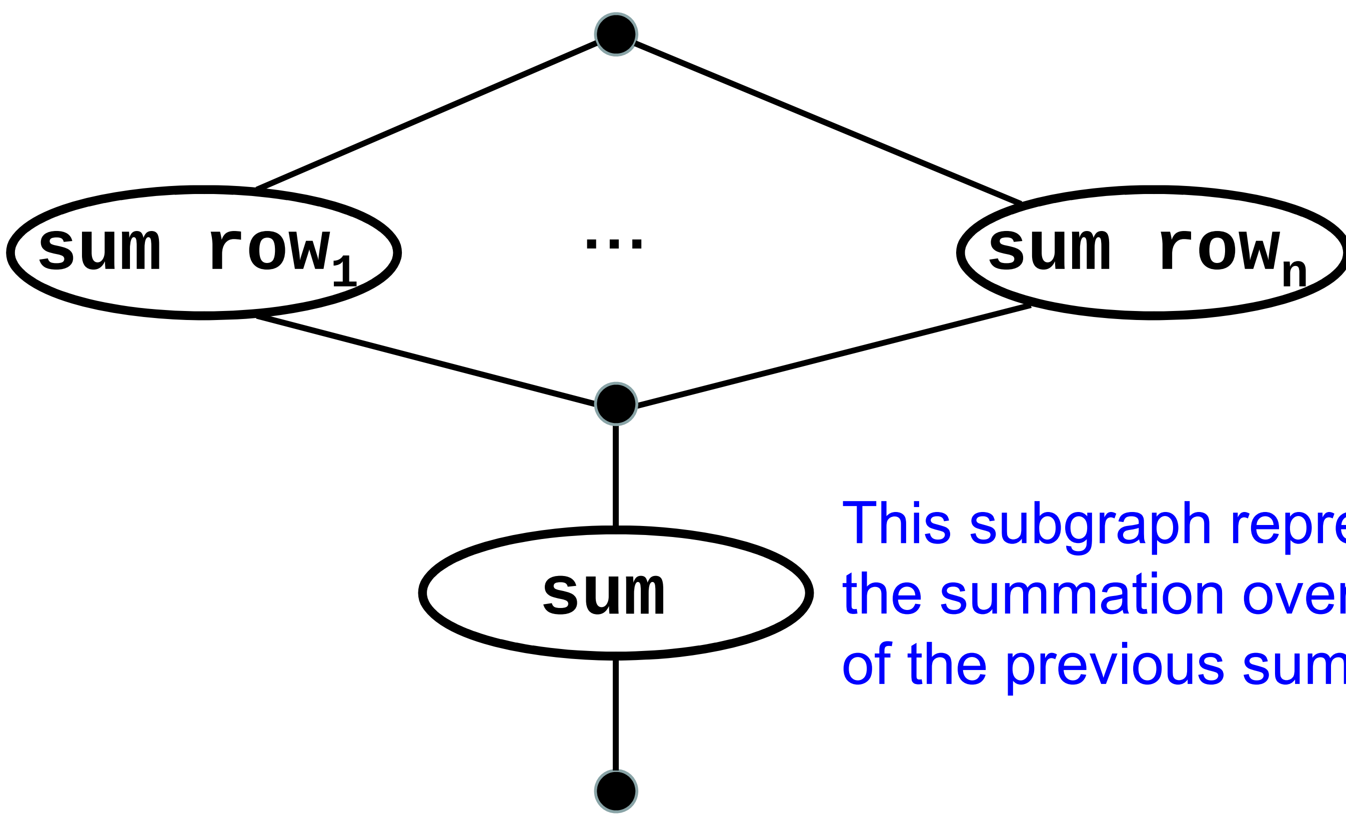
Let value **`c:room`** contain **`n`** **`row`**s of length **`n`** each.
What is the work and span to evaluate **`count(c)`**?

Answer: $O(n^2)$ work and $O(\log(n))$ span.

Answer: $O(n^2)$ work and $O(\log(n))$ span.

To see that, construct a cost graph.

Suppose `c = <row`$_1$`, …, row`$_n$`>`:



This subgraph represents the summation over the results of the previous summations.

Example (recall also Lecture 1):

```
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type row = int Seq.seq
type room = row Seq.seq

fun count (class : room) : int =
      sum (Seq.map sum class)
```

We could also have implemented **count** as:

```
val count : room -> int =
      Seq.mapreduce sum 0 (op +)
```

That is all.

Once, again, please have a good Wednesday.

See you Thursday, when we will start talking about games.