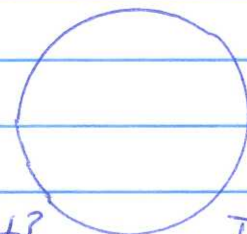
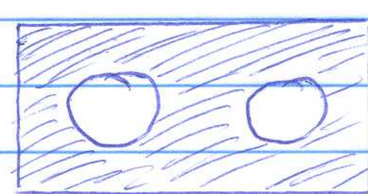
Homology

How do we see that  has a hole, but  does not? Intuitively, the first one is a one-dimensional cycle. And, even though we see such a cycle in , that cycle is the boundary of a two-dimensional region. <u>That is the core idea.</u>

How do we make that work in more general settings, such as

(one hole)

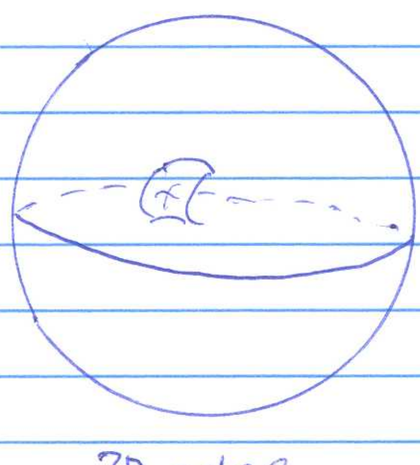
and



(two holes)

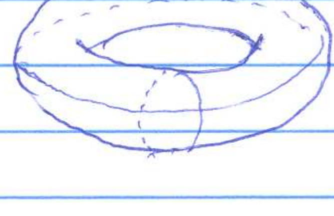
?

And what about in higher dimensions, such as



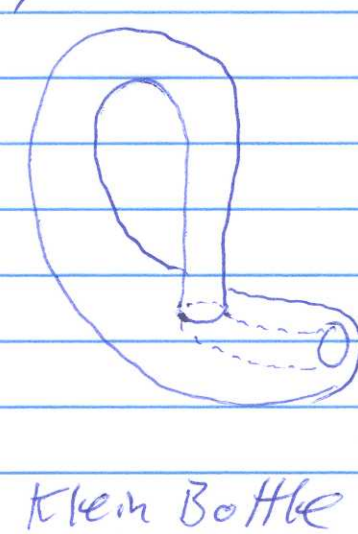
2D sphere

One 2D hole



2D Torus

One 2D hole & two 1D holes



Klein Bottle

?

We need to model geometric boundaries computationally. We will do so using linear maps.

A hint appears in the familiar integration formula

$$\int_a^b f'(x)\,dx = f(b) - f(a),$$

whereas

$$\int_b^a f'(x)\,dx = f(a) - f(b) = -\int_a^b f'(x)\,dx.$$

We think of the interval $[a,b]$ as <u>oriented</u> from a to b:

$$\underset{a}{\bullet}\longrightarrow\underset{b}{\bullet}\,.$$

The geometric boundary consists of the two endpoints $\underset{a}{\bullet}$ & $\underset{b}{\bullet}$, but with signs attached to tell us how those points relate to the orientation of the edge.

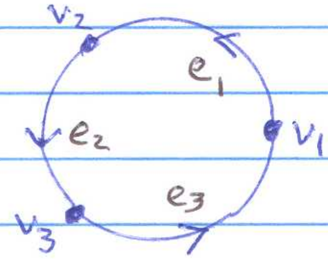
We will define a boundary operator as follows:

$$\partial\left(\underset{a}{\bullet}\longrightarrow\underset{b}{\bullet}\right) = \left(-\underset{a}{\bullet}\right) + \left(+\underset{b}{\bullet}\right)$$

$$\&\quad \partial\left(\underset{a}{\bullet}\longleftarrow\underset{b}{\bullet}\right) = \left(+\underset{a}{\bullet}\right) + \left(-\underset{b}{\bullet}\right).$$

We will extend ∂ to general figures (geometric shapes) by triangulating the shape, then viewing ∂ as a linear map from oriented edges to vertices. Each edge is a basis element in the domain and each vertex is a basis element in the range. We just defined how ∂ behaves on basis elements, Let's try this for the circle.

Suppose we have the following triangulation:



Then
$$\partial(e_1) = v_2 - v_1$$
$$\partial(e_2) = v_3 - v_2$$
$$\partial(e_3) = v_1 - v_3$$

∂ is a linear map from Edges (E) to Vertices (V)

$\partial : E \to V$

Let's use $\{e_1, e_2, e_3\}$ as a basis for E

& $\{v_1, v_2, v_3\}$ as a basis for V.

Then we can represent ∂ via a matrix relative to those bases:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$$

This column for instance tells us that $\partial(e_2) = v_3 - v_2$.

<u>What is the null space of this matrix?</u>

It is spanned by the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$! That represents the circle as a whole, i.e., the 3 oriented edges together!

$(e_1 + e_2 + e_3)$

Comments

- The dimensionality of the matrix for ∂ depended on the triangulation, i.e., the number of edges & vertices. In fact, it is not hard to see that no matter what the dimension, the null space will always be one-dimensional. This is key to realizing there is a single 1D hole.

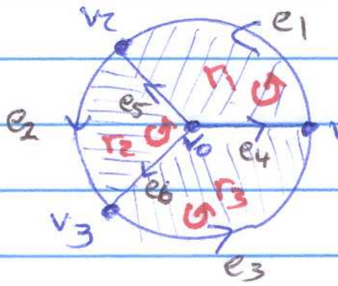
- I haven't really said what E & V are. We've tended to use $\mathbb{R}$ & $\mathbb{C}$ as our vector space scalars, so maybe $\mathbb{R}^3$ is a natural space for each of E & V.

  In fact, we can use a different underlying field. If all we care about is parity, i.e., whether an edge or vertex appears in a particular geometric shape or its boundary, then it makes sense to use $\mathbb{Z}/2$ as the underlying field. So then E & V are each $(\mathbb{Z}/2)^3$.

  Often, one uses an abelian group, such as the integers, in place of a field. So then E & V would each be $\mathbb{Z}^3$. Using a group is useful for detecting something called "torsion" in a shape, as we will see with the Klein Bottle.

Let's now look at the filled-in disk to understand how our computational tool reveals the absence of a hole.

Again, we need to triangulate, and now we have 2D regions as well:



We now have
3 regions
6 edges
4 vertices

Let's orient the regions by the right-hand rule as shown (though any orientation will work, as long as we define boundary operators correctly).

We now have two boundary operators:

$$\partial_2 : \text{Regions} \longrightarrow \text{Edges}$$
$$\partial_1 : \text{Edges} \longrightarrow \text{Vertices}$$

$$\partial_2(r_1) = e_1 - e_5 + e_4$$
$$\partial_2(r_2) = e_5 + e_2 - e_6$$
$$\partial_2(r_3) = e_6 + e_3 - e_4$$

As a matrix

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

$$\partial_1(e_1) = v_2 - v_1$$
$$\partial_1(e_2) = v_3 - v_2$$
$$\partial_1(e_3) = v_1 - v_3$$
$$\partial_1(e_4) = v_1 - v_0$$
$$\partial_1(e_5) = v_2 - v_0$$
$$\partial_1(e_6) = v_3 - v_0$$

$$\begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Some Observations

- $\partial_1 \circ \partial_2 = 0$

(The matrix multiplication is zero:

$$\begin{pmatrix} 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \underline{0}.)$$

- $\dim(\text{null space }(\partial_1)) = 3$, spanned by

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}}.$$

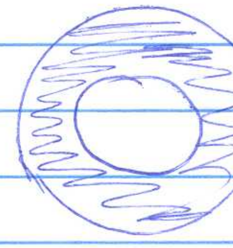
Note that this includes our previous $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

- These are just the columns of the matrix for ∂_2. In other words $\text{col space}(\partial_2) = \text{null space}(\partial_1)$.

Comment: Since $\partial_1 \circ \partial_2 = 0$, we know $\text{colspace}(\partial_2) \subseteq \text{nullspace}(\partial_1)$. The fact that we have equality means no part of the null space of ∂_1 lies outside the colspace of ∂_2.

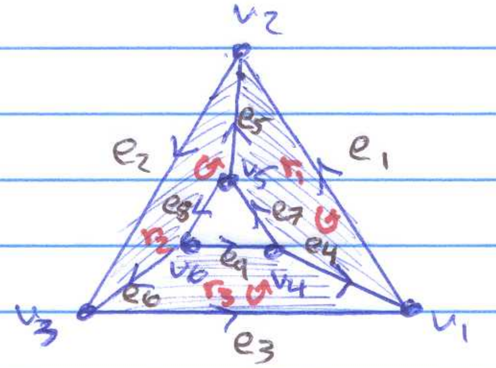
→ When there is a part of the null space of ∂_1 outside the col space of ∂_2, <u>that</u> is when we have holes.

Another example



One of the theorems one proves is that these computations are unchanged by continuous invertible transformations (called homeomorphisms).

So let me work with this shape:



(And I won't completely triangulate but use some other regions. Results are unaffected.)

$$\partial_1(e_1) = v_2 - v_1$$
$$\partial_1(e_2) = v_3 - v_2$$
$$\partial_1(e_3) = v_1 - v_3$$
$$\partial_1(e_4) = v_1 - v_4$$
$$\partial_1(e_5) = v_2 - v_5$$
$$\partial_1(e_6) = v_3 - v_6$$
$$\partial_1(e_7) = v_5 - v_4$$
$$\partial_1(e_8) = v_6 - v_5$$
$$\partial_1(e_9) = v_4 - v_6$$

$$\begin{array}{c|c} & \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{pmatrix} -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

$$\partial_2(r_1) = e_1 - e_5 - e_7 + e_4$$
$$\partial_2(r_2) = e_5 + e_2 - e_6 - e_8$$
$$\partial_2(r_3) = e_3 + e_6 - e_4 - e_9$$

$$\begin{array}{c|c} & \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$$

Observations

- Again, $\partial_1 \circ \partial_2 = 0$
- $\dim(\text{null space}(\partial_1)) = 4$, spanned by

$$\underbrace{\begin{pmatrix}1\\0\\0\\1\\-1\\0\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\\-1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\1\\0\\0\\-1\end{pmatrix}}_{\substack{\text{These are in the column space of } \partial_2 \\ \text{(in fact, their span is that column space).}}}, \underbrace{\begin{pmatrix}1\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}}_{\substack{\text{This is not in} \\ \text{the column space} \\ \text{of } \partial_2\text{. Therefore} \\ \text{there is 1 1D hole.}}}$$

Btw, one might be tempted to also regard

$$\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\1\\1\end{pmatrix}$$

as a hole-spanning null space vector. Indeed, it is. However, it is in the span of the four nullspace vectors shown above, namely "4th vector minus sum of the other three".

Some Surfaces

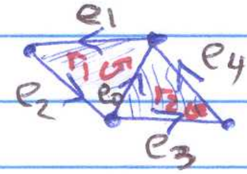
- Sphere

We triangulate the sphere (cover it with spherical triangles), two of which are shown.
We orient the triangles via their outward normals.
Relative to that basis $\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ represents the sum of all the triangles.

We observe that $\partial_2\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = 0$, so the sphere has a 2D hole. This is much like the argument that a circle had a 1D hole.

Why is $\partial_2\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = 0$?

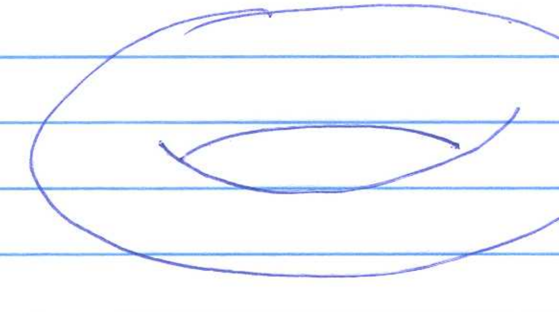
Locally we have this picture



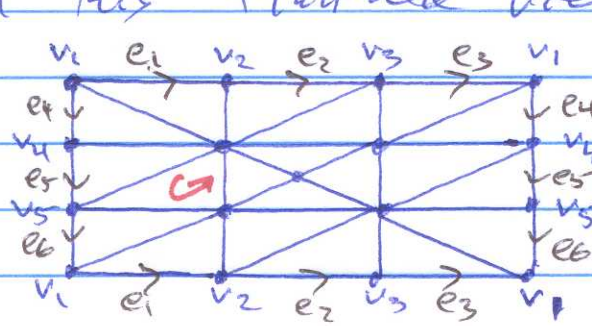
$$\partial_2(r_1) = e_1 + e_2 + e_0$$
$$\partial_2(r_2) = e_3 + e_4 - e_0,$$

so $\partial_2(r_1 + r_2) = e_1 + e_2 + e_3 + e_4$.
Notice how e_0 appeared twice, but with opposite sign, so cancelled.
When we compute ∂_2(sum of all oriented triangles), every edge will appear twice with opposite sign, so the result is 0.

Torus



It's useful to flatten out the torus so we see the triangulation. Note that the top & bottom edge are the same and the left & right edge are the same, in this flattened view:



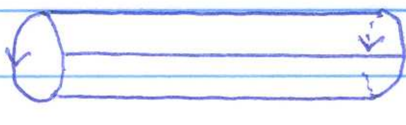
I've only labelled some of the vertices & edges, oriented as shown. All the triangles are oriented CCW, representing outward normal in the torus.

As with the sphere, ∂_2(sum of triangles) $= 0$.
So there is one 2D hole.
← one and not more because we could check that $\dim(\text{null space}(\partial_2)) = 1$.

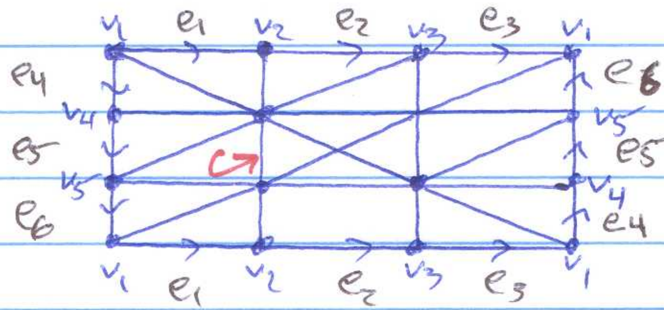
The null space of ∂_1 includes $e_1 + e_2 + e_3$ & $e_4 + e_5 + e_6$. There are numerous other combinations in the null space, e.g., all three edges around any one triangle. However, all those combinations lie in the combined span of the column space of ∂_2 (e.g., the boundary of a triangle = cycle of its edges) and the span of the cycles $e_1+e_2+e_3$ & $e_4+e_5+e_6$.
There are in fact two dimensions of null space(∂_1) that are not in the colspace(∂_2). Therefore there are two 1D holes.
(A basis for these holes is given precisely by $e_1+e_2+e_3$ & $e_4+e_5+e_6$.)

Klein Bottle

We can think of creating the torus by taking a cylinder and bending it to glue together the left & right sides, preserving orientation of the circles:



To create a Klein Bottle one reverses the orientation of the circles, (during gluing) much like when creating a Möbius band.
In the flattened out view, this shows up as follows:



Note the reversal of vertex & edge labels here, compared to the left side.

This has significant consequences for hole presence/detection.
Let σ denote the sum of all the oriented triangles (oriented ccw).
Let $w_1 = e_1 + e_2 + e_3$ & $z_1 = e_4 + e_5 + e_6$.

Then $\partial_2(\sigma) = 2z_1$. $\qquad \partial_1(w_1) = 0$, $\partial_1(z_1) = 0$ as before.
↑ the number 2

This is new. Unless our field or group of scalars has $2=0$ (in $\mathbb{Z}/2$, $2=0$), that means $\partial_2(\sigma) \neq 0$, which means ∂_2 has no null space, meaning no 2D hole is present/apparent. It also means that now $2z_1$ is in the column space of ∂_2. If we are working with a field of scalars that means there is a single 1D hole visible, with basis w_1.
Torsion But if we are working with $\mathbb{Z}$ as our scalars there is a "mod 2" hole visible: z_1 is a 1D hole, but $2z_1$ is not.

since then $\partial_2(\frac{1}{2}\sigma) = z_1$. Note the importance of the "$\frac{1}{2}$" to hit z_1 exactly.

<u>Def</u> Suppose X is a topological space and G an abelian group. Then the k^{th} homology group

$$H_k(X;G) = Z_k(X;G) / B_k(X;G).$$

Here "Z" stands for "cycle" (or "zero"). It is the null space of ∂_k, viewed as a group.

"B" stands for "boundary". It is the column space of ∂_{k+1}, viewed as a group.

"/" refers to quotient groups.

There are lots of technical details having to do with triangulations and in particular how these definitions do not depend on them in detail.

We view H_k as measuring holes in X at dimension k, as detectable by group G.

actually "isomorphism"

$$H_k(X;G) = H_k(Y;G)$$

whenever X & Y are homeomorphic topological spaces. (That is <u>not</u> an iff.)

We have seen

	Sphere	Torus	Klein Bottle
$H_1(X;\mathbb{R})$	0	$\mathbb{R}\times\mathbb{R}$	$\mathbb{R}$
$H_2(X;\mathbb{R})$	$\mathbb{R}$	$\mathbb{R}$	0

$H_1(X;\mathbb{Z})$	0	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}/2$
$H_2(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$	0

In fact, also

$H_1(X;\mathbb{Z}/2)$	0	$\mathbb{Z}/2\times\mathbb{Z}/2$	$\mathbb{Z}/2\times\mathbb{Z}/2$
$H_2(X;\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Euler Characteristic

Suppose we have triangulated a space X.
Assume the triangulation is finite.

Def The Euler Characteristic $\chi(X)$ is the alternating sum of the number of "faces" of different dimensions in the triangulation of X, i.e.,

$$\begin{aligned}\chi(X) = &\ \#\text{ of vertices}\\ &- \#\text{ of edges}\\ &+ \#\text{ of triangles}\\ &- \#\text{ of tetrahedra}\\ &\ \vdots\end{aligned}$$

(It turns out that this number does not depend on the specific triangulation.)

Theorem $\chi(X) = \sum_{k=0}^{\infty} (-1)^k \beta_k(X)$

where β_k = dimension of the non-torsion part of $H_k(X; \mathbb{Z})$

called the "k^{th} betti number".

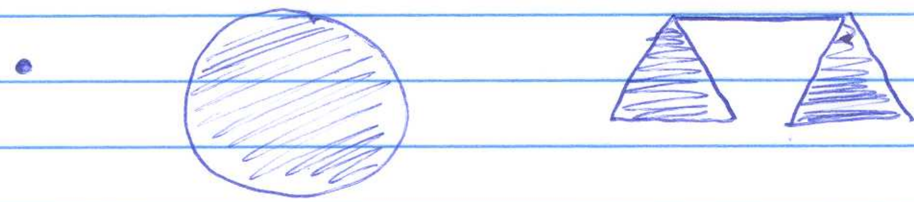
Here β_0 = # of components of X.

(One can extend the definition to more general spaces than triangulated spaces.)

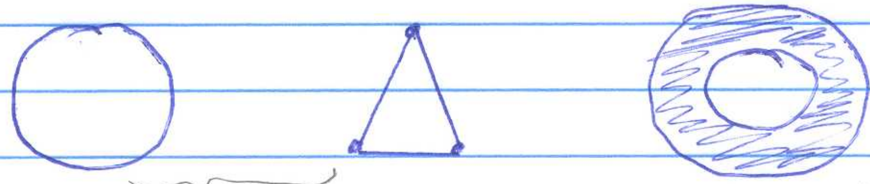
Facts

- The Euler characteristic of a shape is unchanged by a certain kind of continuous transformation called a <u>deformation retraction</u>. (not here required to be invertible) (but in that case occurring "within" the shape itself, roughly speaking)

- So the Euler characteristic of these shapes is all the same, namely 1:

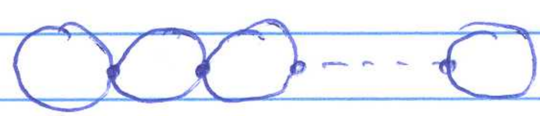


- Similarly, these shapes all have Euler characteristic 0:

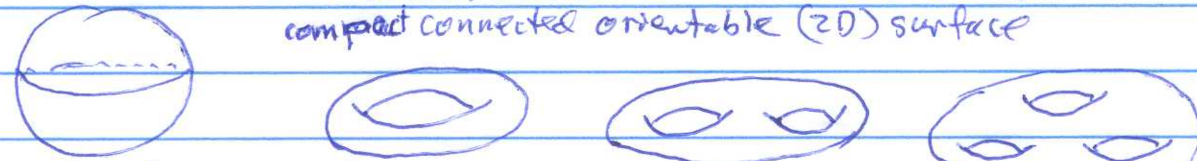


these by homeomorphism   by deformation retraction

- A collection of n circles touching as indicated has Euler characteristic $1-n$:



- A toroidal-like shape with g punctures (g for "genus") has Euler characteristic $2-2g$:
  ↳ counts the number of "handles" on a compact connected orientable (2D) surface



sphere $g=0,\ \chi=2$ torus $g=1,\ \chi=0$ "teapot" $g=2,\ \chi=-2$ "pretzel" $g=3,\ \chi=-4$.

Combinatorial Topology

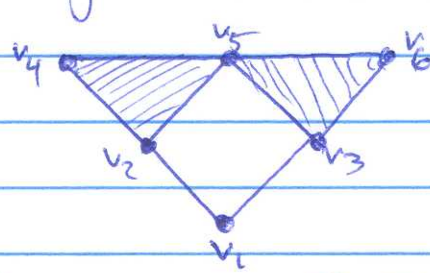
We have seen how triangulations of spaces allow us to measure holes in those spaces.

Triangulations are combinatorial objects.

<u>One can go the other way:</u>
Start with a combinatorial property that resembles a triangulation and thus view the combinatorial property as a shape (ie, something with geometry or topology).

<u>Def</u> An <u>abstract simplicial complex</u> Σ <u>with underlying vertex set</u> V is a collection of finite subsets of V such that if σ is in Σ, then every subset of σ is also in Σ.

<u>Example</u> Let Σ consist of all vertices, edges, and triangles in this diagram:



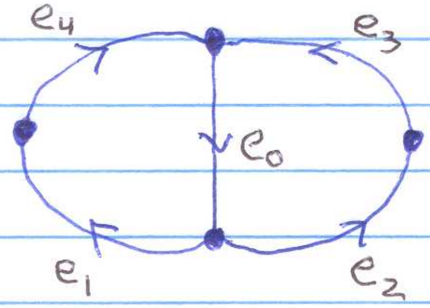
Maximal simplices:
Two triangles & two edges.

We might write
$$\Sigma = \{ \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_2, v_4, v_5\}, \{v_3, v_5, v_6\} \}.$$

(Some people also include the empty set ϕ explicitly in Σ.)

Another Example

Here is a directed graph:



There are 4 vertices and 5 edges.
I have intentionally labelled only the edges.

The edges will form the underlying vertex set of a simplicial complex.

This perspective is a first look at modeling motions via simplicial complexes.
(We will look at another example in which motions are uncertain and in which an agent can make decisions.)

What is the simplicial complex?

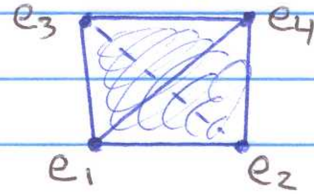
Its simplices are sets of edges:

A set of edges is a simplex iff no subset of these edges forms a cycle. [$\sigma \in \Sigma$ iff no set of edges in σ forms a cycle.]

Notice how the property "no cycle" is preserved by subsets and thus is suitable for defining a simplicial complex.

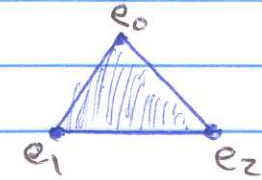
Formally, the simplicial complex consists of these maximal sets of edges, along with their subsets:

$\{e_1, e_2, e_3, e_4\}$



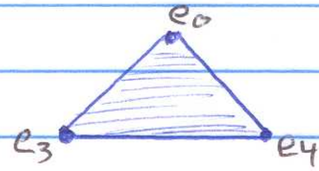
(tetrahedron)

$\{e_0, e_1, e_2\}$



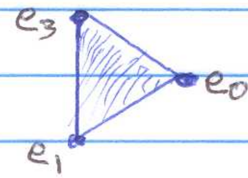
(triangle)

$\{e_0, e_3, e_4\}$



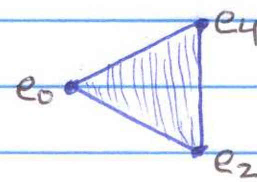
(triangle)

$\{e_0, e_1, e_3\}$



(triangle)

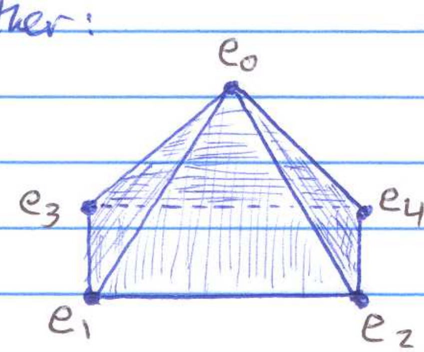
$\{e_0, e_2, e_4\}$



(triangle)

We can envision those simplices together:

The four triangles form a "pointed hat" above the base. (The base is a tetrahedron which I've drawn as a rectangle for visual simplicity.)



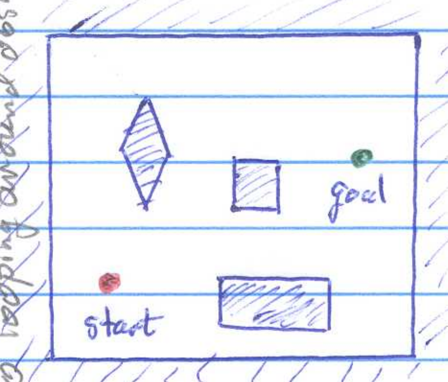
The interior is hollow, so the simplicial complex has one 2D hole.

It turns out that that fact reflects the strong connectivity of the original directed graph. (A directed graph is <u>strongly connected</u> iff there exists for each pair of vertices in the graph a path of directed edges leading from the first vertex to the second (and another path the other way).)

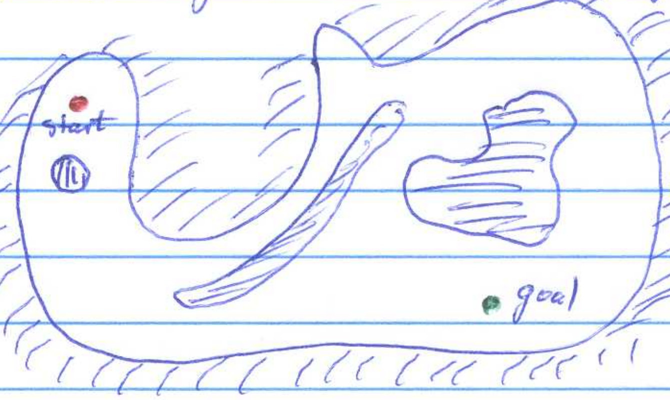
We would like a notion of similarity between spaces that preserves holes.

For instance, we would like to see that the following two 2D environments each offer 4 classes(*) of paths for a robot to move from start to goal: (doing so might allow for some high-level generic motion planning, followed by local fine-motion adjustments)

(*) The idea is that any two paths in one class can be continuously morphed into each other, but two paths in different classes cannot be. And "4 classes" assumes no looping around obstacles. (way more otherwise.)



X



Y

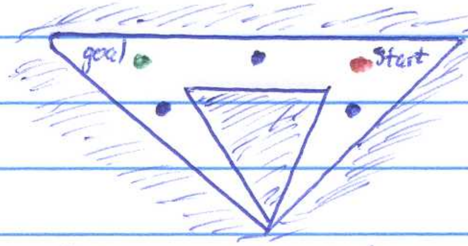
Here we could ask for (at least) two continuous transformations

$$f: X \to Y \quad \& \quad g: Y \to X$$

such that $g \circ f = id_X$ & $f \circ g = id_Y$,

but that is stronger than one wants. (It requires invertibility of continuous functions.)

For instance, we would also want to see that the two spaces above are similar to this space



(the blue dots are point obstacles)

(We could for instance imagine collapsing objects (obstacles in the previous pictures) to points, or expanding points into objects. While the collapse is continuous, the expansion is not.)

One addresses that issue by defining similarity of functions first and then similarity of spaces. Intuitively one thinks of morphing one transformation into another. For instance, we could imagine shrinking this ring (wheel) into a circle over time:

That process (in a continuous way, i.e., with a continuous function) is not invertible, but it captures the 1D homology in both the ring (wheel) and the circle.

<u>Def</u> Suppose $f : X \to Y$ & $g : X \to Y$ are two continuous maps.

We say f & g are <u>homotopic</u>, and write $f \simeq g$, if there is a continuous map

$$F : X \times [0,1] \longrightarrow Y$$

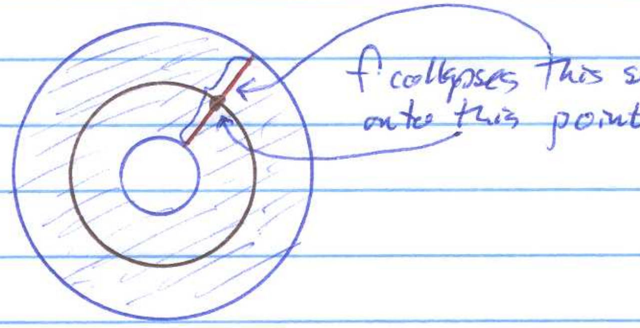
such that

$$\left.\begin{aligned} F(x,0) &= f(x) \\ \& \quad F(x,1) &= g(x), \end{aligned}\right\} \text{for all } x \in X.$$

<u>Def</u> We say two spaces X & Y are <u>homotopy equivalent</u> (or of the <u>same homotopy type</u>, or even <u>homotopic</u>) and write $X \simeq Y$, if there are continuous functions $f : X \to Y$ & $g : Y \to X$, such that $g \circ f \simeq id_X$ & $f \circ g \simeq id_Y$.

<u>Example</u> Suppose the wheel & ring from before are co-centric around the origin. So:

X is the wheel
Y is the circle



Define $f: X \to Y$ by collapsing ray segments onto the circle, as shown. Define $g: Y \to X$ to simply be the identity function.

Then $f \circ g = id_Y$, so certainly $f \circ g \simeq id_Y$.

To see that $g \circ f \simeq id_X$ formally, define

$$F : X \times [0,1] \longrightarrow X$$

$$F(x,t) = \frac{x}{\|x\|}\left(tr + (1-t)\|x\|\right),$$

for all $x \in X$ & $t \in [0,1]$,

Here r is the radius of the circle that is Y.

Observe that F is continuous (since $x \neq 0$ anywhere in X ← uppercase), $F(x,0) = x$ ← lowercase and $F(x,1) = \frac{x}{\|x\|}r$, which lies on the circle of radius r, namely at the point along the direction $\frac{x}{\|x\|}$.

So $F(x,0)$ = identity function on X
& $F(x,1) = g \circ f$

telling us $g \circ f \simeq id_X$, as desired.

(well, per the definition on p.5, we've shown $id_X \simeq g \circ f$, but $\simeq$ is an equivalence relation, so $\simeq$ is symmetric.)

We mentioned the phrase "deformation retraction" earlier. Formally,

<u>Def</u> Suppose $A \subseteq X$, X a topological space. A <u>deformation retraction</u> of X onto A is a continuous map

$$F: X \times [0,1] \to X$$

such that

$$F(x,0) = x \quad \text{for all } x \in X,$$
$$F(x,1) \in A \quad \text{for all } x \in X,$$
$$\& \quad F(a,t) = a \quad \text{for all } a \in A \;\&\; t \in [0,1].$$

One says A is a <u>deformation retract</u> of X.

<u>Example:</u> Y is a deformation retract of X on p. 6.

<u>Theorem</u> If A is a deformation retract of X, then $A \simeq X$.

<u>Theorem</u> $X \simeq Y$ (X & Y topological spaces)

iff

there exists a topological space Z along with imbeddings $h: X \to Z$ & $k: Y \to Z$ such that $h(X)$ & $k(Y)$ are both deformation retracts of Z.

(An "imbedding" is a continuous 1-1 function which has a continuous inverse defined on its image, ie, it is a homeomorphism onto its image.)

Theorem Homotopic spaces have identical homology groups.
(isomorphic)

The proof is complicated, but see for instance
Elements of Algebraic Topology by James R. Munkres

Corollary If $X \simeq Y$, then $\chi(X) = \chi(Y)$.
(and if both have finite triangulations)
(maybe enough to assume only finitely many homology groups are nonzero)

Def A space which is homotopy equivalent to a point is called contractible.

Corollary A contractible space has Euler characteristic 1.

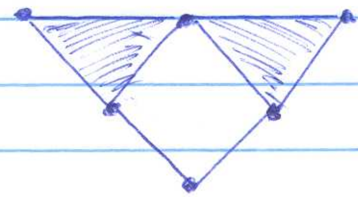
Def Suppose Σ is a simplicial complex and suppose σ & τ are simplices in Σ such that σ is a proper subset of τ but of no other simplex in Σ.

(*) Then the simplicial complex $\Sigma' = \Sigma \setminus \{\sigma, \tau\}$ is homotopy equivalent to Σ.

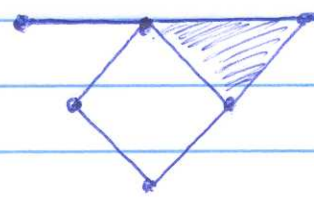
Removing σ & τ is called an elementary collapse.

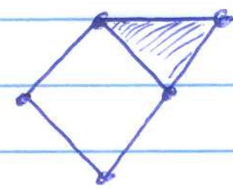
(*) Easy to check that Σ' is indeed a simplicial complex (since τ must be maximal) and then one can visualize a deformation to create Σ' from Σ, thereby establishing homotopy equivalence.

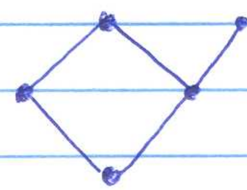
<u>Example</u> Earlier we saw

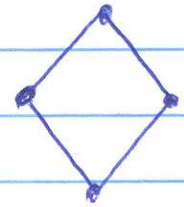


Here is a sequence of successive elementary collapses:
(the first complex is obtained by removing one edge and one triangle, etc)







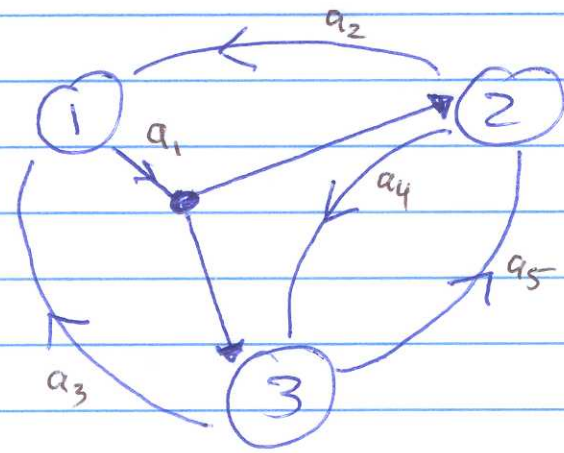


No further elementary collapses are possible.

One sees clearly that the original simplicial complex has the homotopy type of a circle.

Earlier we considered directed graphs.
Let's look at graphs in which arcs/transitions might have uncertain outcomes.

For instance, in this graph there are



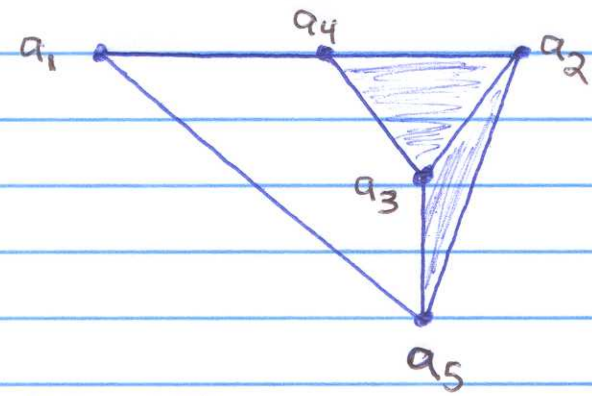
4 directed edges along with one uncertain "action". Specifically the one outgoing edge from state 1 could lead to either state 2 or state 3.

This is a bit like what we saw with Markov chains, except here we will view the uncertainty as potentially adversarial: some adversary might choose the outcome 2 or 3 in a worst-case way for us (depending on our underlying goals) rather than stochastically.

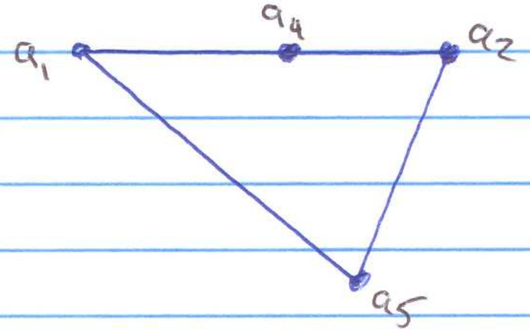
We call such a graph a <u>nondeterministic graph</u>.

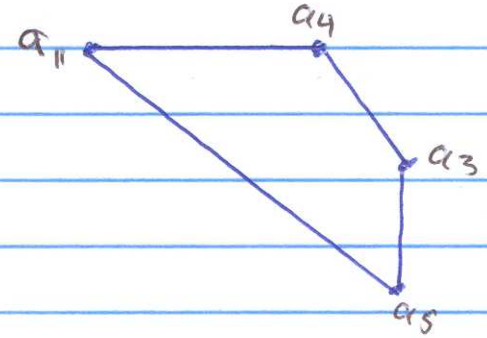
We can again define a simplicial complex whose underlying vertex set consists of the possible directed edges and uncertain transitions (think of all as "actions").

Again, we will consider a set of actions to be a simplex iff the underlying arrows do not create a cycle. So for the previous graph we obtain this simplicial complex (called a <u>strategy complex</u>)



Notice that we have different sequences of possible elementary collapses, eventually giving us one of these two simplicial complexes: (actually, we have even more; see p. 10.5)

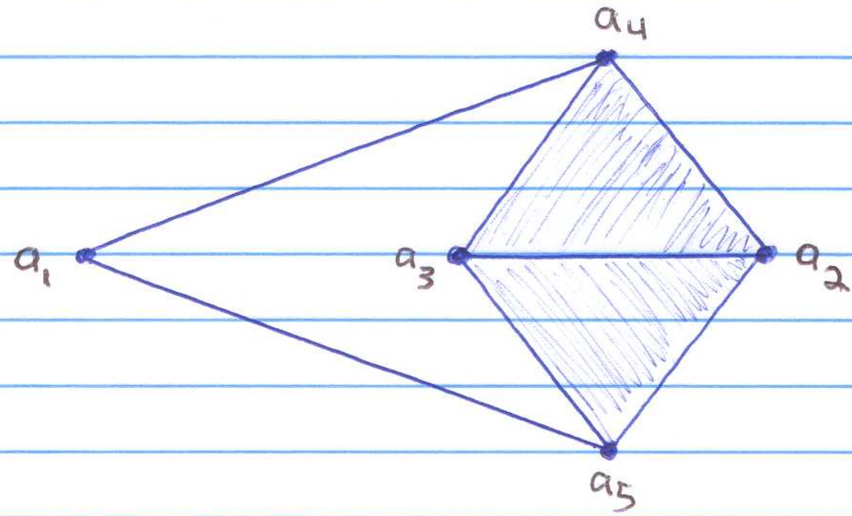




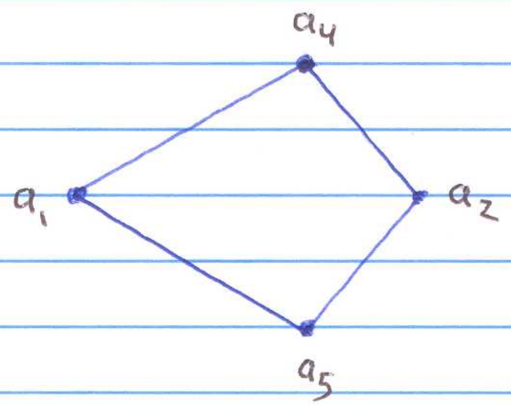
Regardless, we see that all three complexes have the homotopy type of a circle.

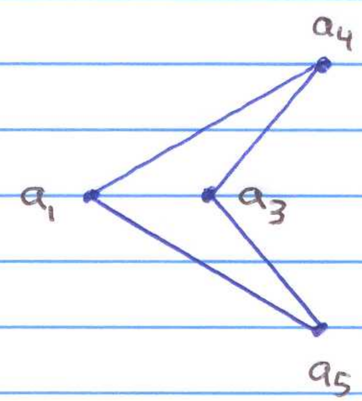
The following drawings of the simplicial complexes from page 10 are more symmetric looking:

Original strategy complex:

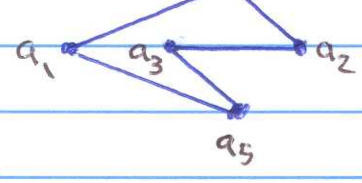


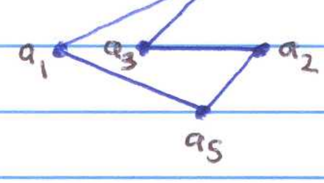
The two possible complexes obtained via elementary collapses:





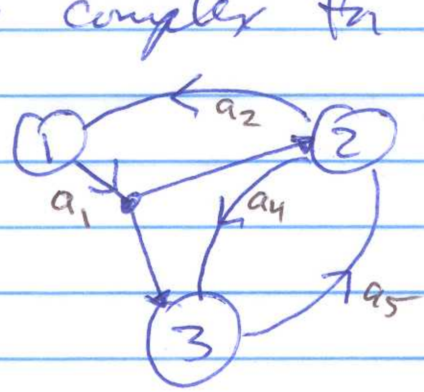
Here are two other possible complexes one might obtain via elementary collapses:



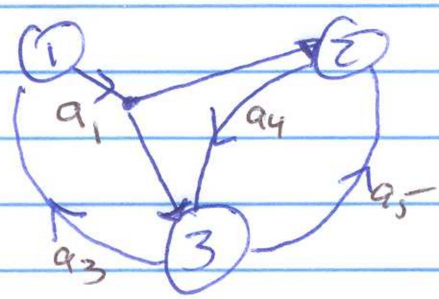


Again, these are homeomorphic to circles. They still contain all the original actions so are themselves not full strategy complexes.

The first of the reduced complexes from p. 10 would be the strategy complex for this graph:



The second would be the strategy complex for this graph:



Q: What do the three graphs have in common?

A: Despite uncertainty in actions, for any start state and any goal state there exists a strategy for moving from start to goal (assuming the system (agent) can sense its current state as it moves).

("moving from start to goal" means we are certain that the system will eventually be at the goal.)

By a "strategy" we mean a simplex of actions in the strategy complex, with the system (agent) choosing any relevant action from that simplex at any state it encounters.

We say such a graph is <u>"fully controllable"</u>.

<u>Theorem</u> Let G be a nondeterministic graph with n states.

Then G is fully controllable iff G's strategy complex is homotopy equivalent to S^{n-2}, the sphere of dimension $n-2$.

<u>Corollary</u>

Suppose G is a nondeterministic graph with n states.

Let g be a desired goal state in G. Define G_g to consist of G with directed edges added from g to each of the other $n-1$ states.

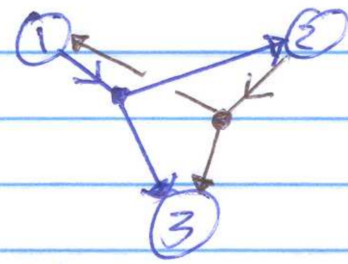
"moving to g" means moving there certainly; no adversary can prevent it forever.

Then there exists a strategy for moving to g from anywhere in the graph G iff G_g's strategy complex is homotopic to S^{n-2}. Moreover, if this is not the case, then G_g's strategy complex is contractible.

<u>Corollary</u> Let Δ be G_g's strategy complex. If $\chi(\Delta)$ is even, then there exists a strategy for attaining g from anywhere in G. Otherwise, no such strategy exists.

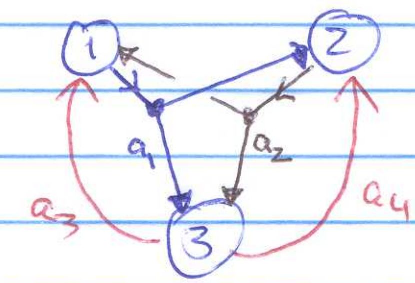
$\boxed{\text{Pf.}}$ $\chi(\Delta)$ is one of 0, 2 for spheres & 1 for contractible.

Example G:

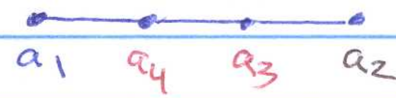


Let $g = 3$. Observe that an adversary can prevent attainment of g.

G_g:



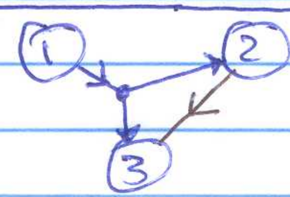
Δ:



$\chi(\Delta) = 4 - 3 = 1$.

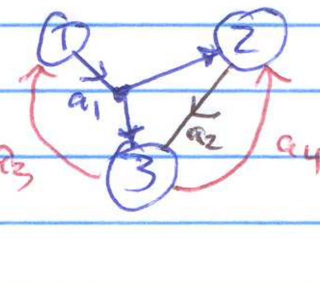
contractible

In contrast, for this G:

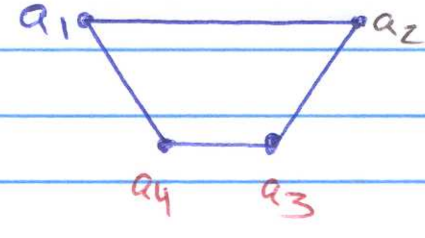


one can be certain of reaching state 3.

Now, for $g = 3$, G_g:



Δ:



$\chi(\Delta) = 4 - 4 = 0$.

$\Delta \simeq S^1$