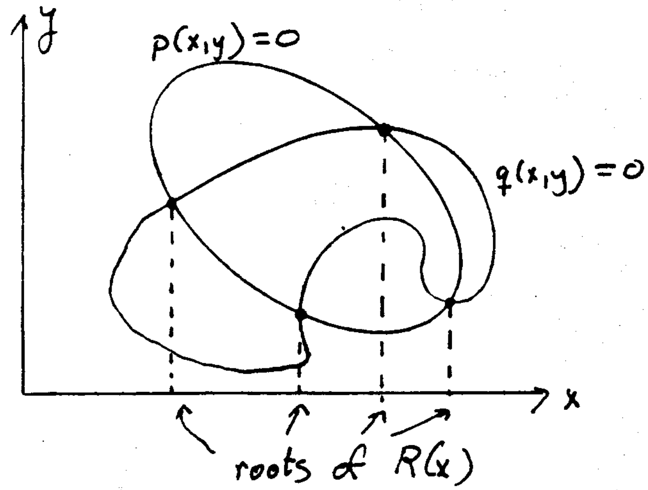
Resultants and Elimination Theory

One technique with several variations.

Isolating simultaneous zeros



$p(x,y)$ & $q(x,y)$ are bivariate polynomials in x & y. We seek their simultaneous zeros.

The method of resultants produces a single univariate polynomial $R(x)$ such that

$$R(x) = 0 \quad \text{iff} \quad p(x,y) = 0 \ \& \ q(x,y) = 0 \text{ for some } y$$

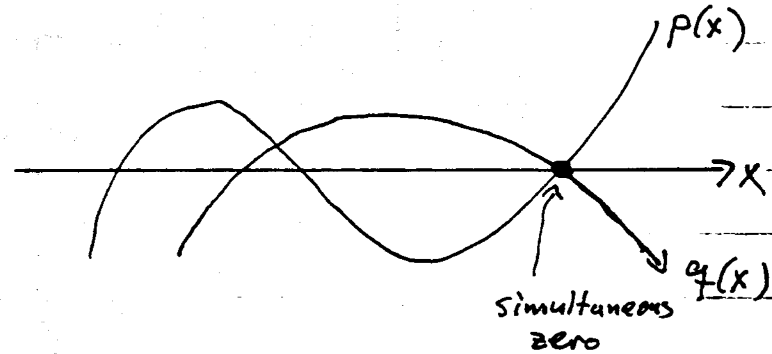
So, find the roots of $R(x)$. For each of these, solve a poly in y to get the simultaneous roots of p & q.

Application: One can use this technique to split a robot's configuration space into critical sections within which the topology doesn't change — Algebraic Cylindrical Decomposition.

Singular Simultaneous Zeros

The method tells us when an overconstrained algebraic system has a simultaneous zero.

E.g. $p(x)=0$ $\qquad$ (2 equations, 1 unknown)
$q(x)=0$



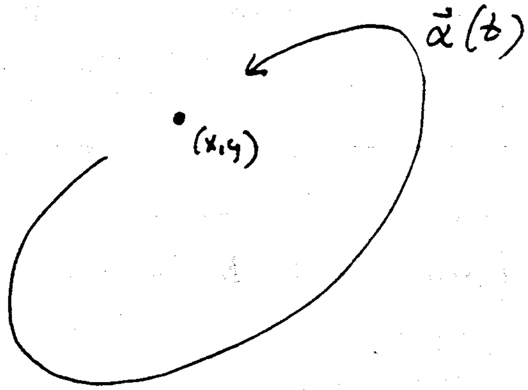
This is parameter elimination, i.e., the parameter x is "removed."

Really, what we get is a resultant R(coefficients of p & q) that tells us whether or not p & q have a simultaneous root. If R(coefficients of p & q)$=0$ then yes, otherwise no.

If the coefficients are known numbers, then we just plug in and test.

If the coefficients are symbols, then the equation $\quad R$(coefficients of p & q) $=0$ provides constraints on the coefficients that tell us the conditions under which a simultaneous zero exists.

Implicitizing Parametric Equations



Suppose we have a parameterized curve in 2D
$\vec{\alpha}(t) = (p(t), q(t))$ with p & q polys in t.

We might want an implicit equation $F(x,y)=0$ for the same curve, with F a poly in x & y.

Elimination theory will give this to us, basically by constructing the system of equations
$$p(t) - x = 0$$
$$q(t) - y = 0.$$

If we think of this as 2 equations in 1 unknown, t, then we are back in the "Singular Simultaneous Zeros" scenario. In other words, we have two polynomial equations in t. We think of x & y as part of the coefficient set of these equations.

The result is a polynomial equation $F(x,y)=0$ that provides constraints on x & y for a simultaneous zero in t to exist. In other words, it gives us our desired implicit equation.

Example Application (very quick overview of the paper by Ponce + Kriegman)

Goal: P&K would like to recognize objects by matching image observables directly to 3D models (rather than build intermediate representations).

- Motivated by the use of planes, quadrics, superquadrics, etc. in CAD systems, P&K assume that their models have rational parametric descriptions.

Thus their models consist of surface patches described as

$$\vec{x}(s,t) = \frac{\sum s^i t^j \, \vec{x_{ij}}}{\sum s^i t^j \, w_{ij}}$$

where the $\vec{x_{ij}}$ & w_{ij} are vectors & numbers derived from their model data,

$\vec{x}$ is of the form (x, y, z), ie, 3D.

s, t are the patch parameters.

- Next, P&K define

  $\vec{\vartheta}$ — vector of observables (e.g., intensity, intensity gradients, 3D data, ...)

  $\vec{P}$ — vector of viewing parameters (e.g., pose of an object relative to the camera, direction of the light source, etc.)

- The relationship between the observables $\vec{\vartheta}$ & the viewing parameters $\vec{P}$ depends on the surface $\vec{x}(s,t)$ observed. For instance, intensity data depends both on the light source and the surface normal.

  The trick is to have enough observables & viewing parameters so that one can construct three such relationships, described implicitly by three equations that relate $s, t, \vec{\vartheta}, \& \vec{P}$:

$$f_1(s,t,\vec{\vartheta},\vec{P}) = 0$$
$$f_2(s,t,\vec{\vartheta},\vec{P}) = 0$$
$$f_3(s,t,\vec{\vartheta},\vec{P}) = 0$$

(There are 3 such functions for each surface patch in the model database.)

For instance, here is an example from the P&K paper:

Observables: $\vec{O} = (x, y, I)$

where I is the intensity observed in the image at image coords (x, y).

Viewing Parameters: $\vec{P} = (x_0, y_0, \vec{l}, \vec{w}, \vec{u})$

where (x_0, y_0) — is the world origin in the camera's coord system (so won't matter)

$\vec{l}$ — is a unit vector describing the direction of the light source (situated at infinity)

$\vec{w}$ — unit vector describing the world orientation of the image x-axis.

$\vec{u}$ — unit vector describing the world orientation of the image y-axis.

(Note: together $\vec{w}$ & $\vec{u}$ have 3 dofs, and thus are often replaced by three angles (α, β, γ) that describe the orientation of the camera. For simplicity of presentation, we won't worry about that here.)

The three equations relating $\vec{\theta}$ & $\vec{P}$ in terms of a given surface patch $\vec{x}(s,t)$ are then

$$\boxed{\begin{aligned} x &= \vec{x}(s,t) \cdot \vec{w} + x_0 \\ y &= \vec{x}(s,t) \cdot \vec{u} + y_0 \\ I &= \vec{N} \cdot \vec{l} \end{aligned}}$$

where $\vec{N}$ is the surface normal at $\vec{x}(s,t)$, obtained in the usual way.

(This derivation assumes a matte surface and orthographic projection, for simplicity)

- So, P & K have three equations in $\vec{\theta}, \vec{P}$ & (s,t). Since the parameters s & t aren't intrinsic they are a nuisance — after all, the same surface can be parameterized in different ways.

  P & K get rid of s & t using Elimination Theory. Again, this is much like case of "Singular Simultaneous Zeros." (see next page)

We think of the equations

$$\left.\begin{array}{l} f_1(s,t,\vec{\theta},\vec{p}) \\ f_2(s,t,\vec{\theta},\vec{p}) \\ f_3(s,t,\vec{\theta},\vec{p}) \end{array}\right\} \begin{array}{l} =0 \\ =0 \\ =0 \end{array}$$

as 3 equations in 2 unknowns, ie, an overconstrained system.

Elimination theory allows us to construct a "resultant" equation in the coefficients of s & t in f_1, f_2, f_3. $\vec{\theta}$ & $\vec{p}$ are some of these coefficients.

We thus get a single implicit equation

$$F(\vec{\theta},\vec{p}) = 0$$

that relates $\vec{\theta}$ & $\vec{p}$.

The data $\{\vec{x}_{ij}\}$ & $\{w_{ij}\}$ appear inside this equation, but the parameters s & t have been eliminated.

In the image intensity example, we thus get a standard implicit equation $F(x, y, I, x_0, y_0, \vec{l}, \vec{w}, \vec{u}) = 0$ relating observed intensity to camera and light source, parametrized by the observed surface point.
In other words F is an implicit equation for the intensity surface $I(x,y)$, parameterized by all the other quantities.

- Beyond eliminating s & t, how is this useful to P & K?

They would like to do two things:

Given some observed data,

(1) Determine which object they are looking at
(2) Determine unknown viewing parameters (e.g. the orientation of the camera to the object, i.e., the pose of the object)

Here is the approach P & K take:

(i) First, they have a collection of models $M_1, \ldots, M_m$. For simplicity, let's think of each as being a single surface patch $\vec{X}_j(s,t)$.

Thus, using elimination theory, P & K construct a collection of implicit functions

$$F_1(\vec{O}, \vec{P}) = 0$$
$$\vdots$$
$$F_m(\vec{O}, \vec{P}) = 0,$$

one for each model. F_j relates observables & viewing parameters whenever the camera is looking at an object corresponding to model M_j.

(ii) Next, P & K collect data. Specifically they look at an unknown object with unknown viewing parameters. They look at several points on the object, and thus acquire a collection of data points:

$$\vec{\mathcal{O}}_1, \ldots, \vec{\mathcal{O}}_n.$$

(For instance, in the image intensity example, each triple (x, y, I) as (x, y) varies over the image is a data point.)

(iii) Finally, P & K run the following recognition algorithm:

1. For each model M_j, determine the viewing parameters $\vec{P}_j$ that minimize the squared error

$$E_j = \sum_{i=1}^{n} F_j^2(\vec{\mathcal{O}}_i, \vec{P})$$

(so $\vec{P}$ is the minimization variable)

2. For each model M_j, and each data point $\vec{\mathcal{O}}_i$, let d_{ij} be the distance between $\vec{\mathcal{O}}_i$ and the surface in $\vec{\mathcal{O}}$-space defined implicitly by the equation $F_j(\mathcal{O}, \vec{P}_j) = 0$.
Then compute the cumulative distance
$$D_j = \sum_{i=1}^{n} d_{ij}$$

3. Let j_0 be the index that minimizes D_j. Model M_{j_0} is reported back.

Brief Intro to Resultants

Good reference: Sederberg, Anderson, & Goldman "Implicit Representation of Parametric Curves and Surfaces" in <u>Computer Vision, Graphics, and Image Processing</u> 28, 72-84 (1984).

<u>Motivation:</u>
- Determine when a system of equations has a simultaneous zero
- Use this, e.g., to implicitize parametric equations

Recall how this works for linear systems:

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \vdots \qquad \cdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{matrix} \qquad \text{ie } Ax = 0$$

This system has a <u>non-trivial</u> solution iff A is singular, ie., iff $\det A = 0$.

Furthermore, if we have a reduced system ($n-1$ equations, n unknowns)

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \qquad\qquad \vdots \\ a_{n-1,1}x_1 + \cdots + a_{n-1,n}x_n = 0 \end{matrix}$$

then the ratio $\frac{x_i}{x_j}$ of any solution is given by

$$\frac{x_i}{x_j} = (-1)^{i+j} \frac{\det A_i}{\det A_j}$$

where $A_i = \begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & & & & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,i-1} & a_{n-1,i+1} & \cdots & a_{n-1,n} \end{pmatrix}$

ie, A_i is A with its i^{th} column removed (A_i is $(n-1) \times (n-1)$).

<u>Example</u>

$$\begin{aligned} x + y + z &= 0 \\ 2x + y &= 0 \end{aligned}$$

$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A_2 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ $\quad A_3 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

$\det A_1 = -1$ $\quad \det A_2 = -2$ $\quad \det A_3 = -1$

So this system has a line of solutions of the form $c(1,-2,1)$. [Set $z=1$ & note $\frac{x}{z} = (-1)^4 \frac{-1}{-1}$, $\frac{y}{z} = (-1)^5 \frac{-2}{-1}$

Indeed, if we did some linear algebra by hand on the original two equations that's the result we would get as well.

<u>The big question:</u> How does one generalize this approach to non-linear systems?

For polynomial systems, Resultants and Elimination Theory are the answer.

We will illustrate the approach with low degree polynomials.

<u>Sylvester's Method</u>

Suppose we have two uni-variate quadratics:

$$\begin{aligned} ax^2 + bx + c &= 0, \quad & a \neq 0 \\ a'x^2 + b'x + c' &= 0, \quad & a' \neq 0 \end{aligned}$$

Does this system have a simultaneous zero? Well, here's a trick:

The system above has a simultaneous zero iff there is some x such that

$$\begin{bmatrix} a & b & c & 0 \\ 0 & a & b & c \\ a' & b' & c' & 0 \\ 0 & a' & b' & c' \end{bmatrix} \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \underline{0}.$$

Write this as $Q\vec{x}^3 = \underline{0}$ where $\vec{x}^3 = \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix}$.

By analogy to the linear case <u>we claim</u> that this is possible iff $\det Q = 0$.

[If the x^3, x^2, x, & 1 really were independent variables, then this would be clear, but they are not, so let us prove our claim.

<u>Claim:</u> $Q\vec{x}^3 = 0$ iff $\det Q = 0$

<u>Proof:</u>

(i) Necessity is clear. After all if $Q\begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = 0$ for some x, then $\det Q = 0$ by linear alg. since $\begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix}$ is a non-trivial vector.

(ii) What about sufficiency?

If $\det Q = 0$ can we show that there is an x for which $Q\vec{x}^3 = 0$?

Yes, here's how:

Since $\det Q = 0$ we know there is some vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ such that $Q\vec{v} = 0$.

Assuming that we started with two different equations (which is easy enough to check) we know that rows 1 & 3 of Q are independent, as are rows 2 & 4.

In other words, the linear system

$$a v_1 + b v_2 + c v_3 = 0$$
$$a' v_1 + b' v_2 + c' v_3 = 0$$

has a line of solutions in (v_1, v_2, v_3) space.

In other words $(v_1, v_2, v_3) = \alpha(r, s, t)$ where (r, s, t) is the direction vector of the line and α is a scalar.

Similarly, the system
$$\begin{aligned} a v_2 + b v_3 + c v_4 &= 0 \\ a' v_2 + b' v_3 + c' v_4 &= 0 \end{aligned}$$
has a line of solutions in (v_2, v_3, v_4) space.

Of course, since the coefficients are the same, this line is the same as in the previous case, just shifted by one coordinate.

In other words $(v_2, v_3, v_4) = \beta(r, s, t)$ for some β.

So, if $Qv = 0$ then

$$\begin{aligned} (v_1, v_2, v_3) &= \alpha(r, s, t) \\ \text{\&} \quad (v_2, v_3, v_4) &= \beta(r, s, t) \end{aligned}$$

Therefore $v_1 = \frac{\alpha}{\beta} v_2 = \left(\frac{\alpha}{\beta}\right)^2 v_3 = \left(\frac{\alpha}{\beta}\right)^3 v_4$

β can't be zero, for if it were then we'd be saying $Q\left[\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right] = 0$, which is impossible since a & a' are non-zero.

Also, v_4 can't be zero, for then $v = 0$, and we've assuming $Qv = 0$ has a non-trivial solution.

So, let's take v_4 to be 1 (we can do that since any multiple of v will also satisfy $Qv = 0$).

Then our solution looks like $(x^3, x^2, x, 1)$, with $x = \frac{\alpha}{\beta}$.

$$\boxed{\text{In short, if } \det Q = 0 \text{, then there is a non-trivial solution to } Qv = 0 \text{, and furthermore } v \text{ is of the form } \begin{pmatrix} x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}.}$$ //

Terminology

$$\det Q = 0$$

is called <u>the resultant</u> of the original two equations

$$ax^2+bx+c = 0$$
$$a'x^2+b'x+c' = 0$$

<u>Def</u> A <u>resultant</u> of a set of polynomials is an expression involving the coefficients of the polynomials such that the vanishing of the resultant is a necessary and sufficient condition for the set of polynomials to have a common zero.

(It is a generalization of the idea of a determinant, which is an expression involving the coefficients of a set of linear equations.)

Sylvester's method outlined above for two quadratic polynomials can be generalized.
More generally, Sylvester's method expresses the resultant of two univariate polynomials of degree m and n as a determinant of an $(m+n)\times(m+n)$ matrix.

Numerical Examples

1) $$p(x) = x^2 - 6x + 2$$
$$q(x) = x^2 + x + 5$$

$$\det Q = \det \begin{pmatrix} 1 & -6 & 2 & 0 \\ 0 & 1 & -6 & 2 \\ 1 & 1 & 5 & 0 \\ 0 & 1 & 1 & 5 \end{pmatrix} = 233,$$ which is non-zero

Therefore we know that p & q do not have a common zero.
(Indeed, we could check this directly. $p(x)$ has roots $3 \pm \sqrt{7}$
$q(x)$ has roots $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-19}$)

2) $$p(x) = x^2 - 4x - 5$$
$$q(x) = x^2 - 7x + 10$$

(It's easy to solve this by hand. $x = 5$ is the common root.)

$$\det Q = \det \begin{pmatrix} 1 & -4 & -5 & 0 \\ 0 & 1 & -4 & -5 \\ 1 & -7 & 10 & 0 \\ 0 & 1 & -7 & 10 \end{pmatrix} = 0,$$ so p & q do have a common root.

To find it, consider the partial system $\begin{pmatrix} 1 & -4 & -5 & 0 \\ 0 & 1 & -4 & -5 \\ 1 & -7 & 10 & 0 \end{pmatrix} \begin{pmatrix} x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} = 0$

Using our previous linear algebra result, we know that

$$x = \frac{x^3}{x^2} = (-1)^{1+2} \frac{\det \begin{pmatrix} -4 & -5 & 0 \\ 1 & -4 & -5 \\ -7 & 10 & 0 \end{pmatrix}}{\det \begin{pmatrix} 1 & -5 & 0 \\ 0 & -4 & -5 \\ 1 & 10 & 0 \end{pmatrix}} =$$

$$= (-1) \frac{-375}{75} = 5 \checkmark$$

3) So far the coefficients have all been numbers, but suppose we had an unknown parameter. Then we could use the method of resultants to supply a constraint on the unknown parameter in order for a common zero to exist.

For example, if $p(x) = x^2 - 4x - 5$
& $q(x) = x^2 - 7x + c$

Then we can ask: For what value of c does this system have a common root. In effect, we are <u>eliminating</u> x from the system and determining a constraint on c.

$$\det Q = \det \begin{pmatrix} 1 & -4 & -5 & 0 \\ 0 & 1 & -4 & -5 \\ 1 & -7 & c & 0 \\ 0 & 1 & -7 & c \end{pmatrix} = c^2 - 2c - 80$$

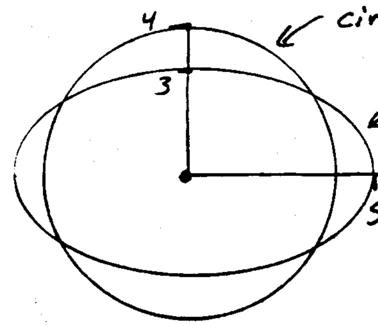
So p & q have a common root iff $\det Q = 0$
iff $c^2 - 2c - 80 = 0$
iff $c = 10$ or $c = -8$

The case $c = 10$ we just saw on the previous page, yielding $x = 5$.
The case $c = -8$ corresponds to the system $x^2 - 4x - 5 = 0$
& $x^2 - 7x - 8 = 0$

which yields the simultaneous root $x = -1$.

4) So we see that what seemed like a simple test merely to decide whether an overconstrained system had a solution is actually quite powerful if some of the coefficients are left symbolic.

In this next example, we actually use the method to solve two simultaneous equations in two unknowns.



Suppose we want to intersect the circle and the ellipse

$$p(x,y) = x^2+y^2-16 = 0$$
$$q(x,y) = 9x^2+25y^2-225=0$$

In order to apply the method of resultants we will think of p & q as two <u>univariate polynomials in y</u>, with coefficients that happen to include the symbol x.

We will construct the resultant à la Sylvester's method. This will produce a single polynomial in x, which is zero iff the original system has a common root in y. We solve for x, then for y.

Here goes. Think of p & q as polys in y only:

$$p(y) = y^2 + \quad + (x^2 - 16)$$
$$q(y) = 25y^2 + \quad + (9x^2 - 225)$$

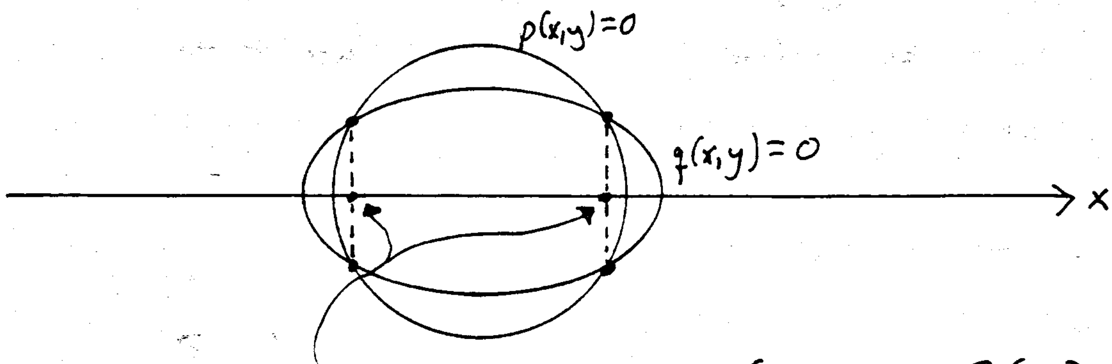
$\uparrow$ quadratic terms $\qquad \uparrow$ linear terms $\qquad \uparrow$ constant terms

$$\det Q = \det \begin{pmatrix} 1 & 0 & x^2-16 & 0 \\ 0 & 1 & 0 & x^2-16 \\ 25 & 0 & 9x^2-225 & 0 \\ 0 & 25 & 0 & 9x^2-225 \end{pmatrix}$$

$$= (175 - 16x^2)^2$$

So $\det Q = 0$ iff $16x^2 = 175$
iff $x = \pm \frac{5}{4}\sqrt{7}$

What does this mean? It means that p & q have a simultaneous zero iff $x = \pm\frac{5}{4}\sqrt{7}$. In effect the resultant $\det Q$ has projected the common roots of $p(x,y)$ & $q(x,y)$ onto the x-axis:



roots of $\det Q = 0$ $\quad \left(\det Q = R(x) = (175-16x^2)^2\right)$

Now plug these values of x back into our system:

$$p(y) = y^2 + (x^2-16) = y^2 - \frac{81}{16}$$

$$q(y) = 25y^2 + (9x^2-225) = 25\left(y^2 - \frac{81}{16}\right)$$

So, indeed p & q have common roots, namely at $y = \pm\frac{9}{4}$! Thus we have the four intersection points

$$\left(-\frac{5}{4}\sqrt{7}, -\frac{9}{4}\right)$$

$$\left(+\frac{5}{4}\sqrt{7}, -\frac{9}{4}\right)$$

$$\left(\frac{5}{4}\sqrt{7}, \frac{9}{4}\right)$$

$$\left(-\frac{5}{4}\sqrt{7}, \frac{9}{4}\right).$$

5) Finally, let's look at an implicitization example. Again the basic technique is to construct resultants, just as before. This time we use it to eliminate a curve parameter.

Consider the parameterized curve

$$x(t) = 5t^2 + t + 3$$
$$y(t) = 5t^2 - t - 1$$

<u>What is the implicit equation $F(x,y)=0$ that describes this curve?</u>

The trick is to look at the system

$$(*) \quad \begin{array}{ccccccc} 5t^2 & + & t & + & (3-x) & = & 0 \\ 5t^2 & - & t & + & (-1-y) & = & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{quadratic terms} & & \text{linear terms} & & \text{constant terms} & & \end{array}$$

as 2 equations in one unknown, namely t. x & y simply play the role of symbolic coefficients, much like the symbol c in example (3) on p. 18.

We know that the system $(*)$ has a solution in t precisely when (x,y) is a point on the curve. In other words $\underbrace{\det Q}_{\text{function of } x \,\&\, y} = 0$ iff (x,y) is a point on the curve.

$$\det Q = \det \begin{pmatrix} 5 & 1 & 3-x & 0 \\ 0 & 5 & 1 & 3-x \\ 5 & -1 & -1-y & 0 \\ 0 & 5 & -1 & -1-y \end{pmatrix}$$

$$= 5(84 + 38y - 42x + 5x^2 + 5y^2 - 10xy)$$

So $\det Q = 0$ iff $\underline{5x^2 + 5y^2 - 42x + 38y - 10xy + 84 = 0}$

↑ this is our desired implicit equation

The discriminant $c_{x^2} \cdot c_{y^2} - \left(\frac{c_{xy}}{2}\right)^2 = 5 \cdot 5 - \left(\frac{10}{2}\right)^2 = 0$, so this is a parabola. — Indeed it turns out that all 2D quadratic parameterizations yield parabolas — can't get hyperbolas or ellipses unless one looks at rational functions. For instance, the parameterization

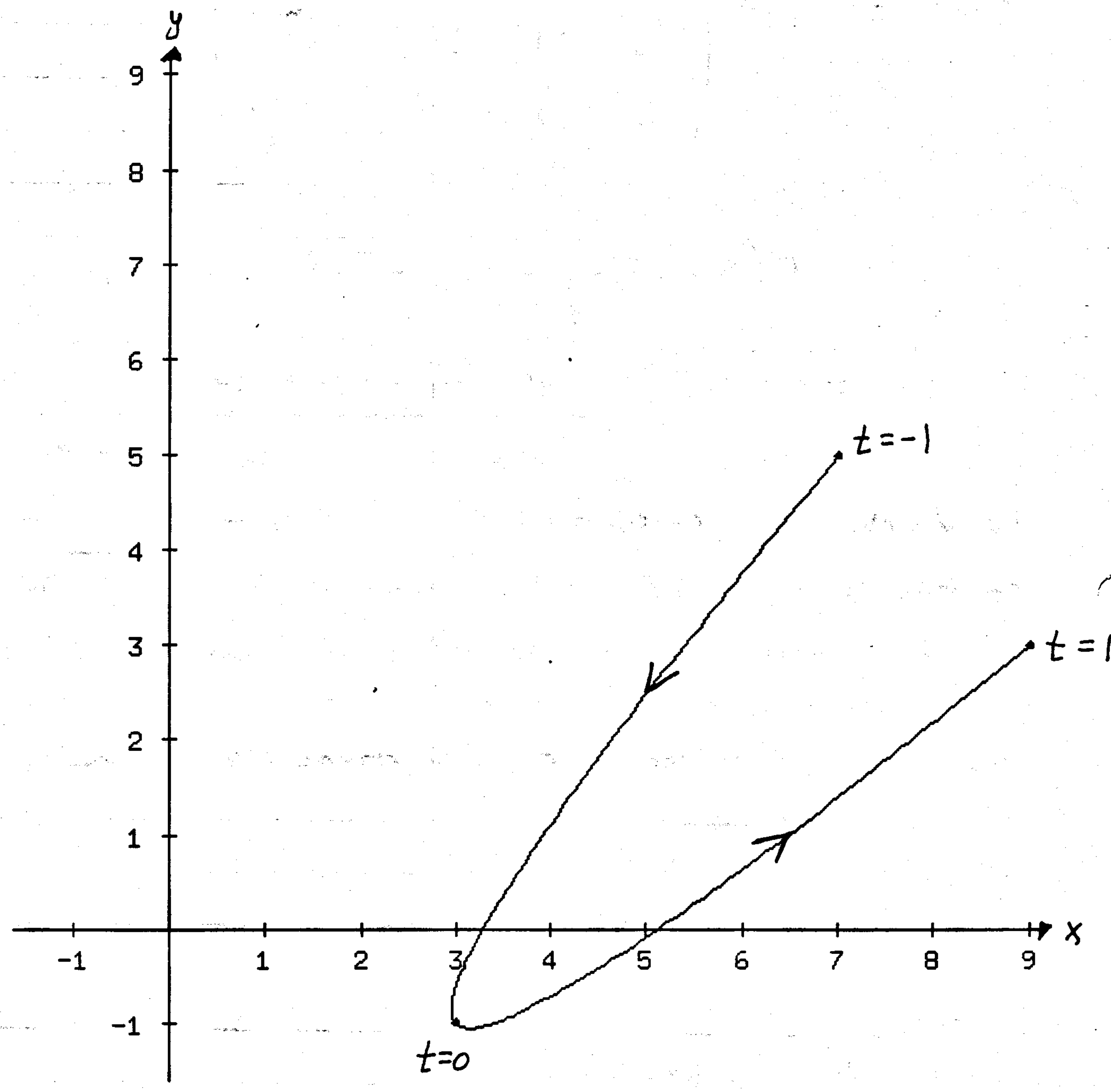
$$x = \frac{2Rt}{1+t^2}$$

$$y = \frac{R(1-t^2)}{1+t^2}$$

yields the circle of radius R.

Btw, some simple additional computations make our techniques work with rational functions (just multiply through by the denominators).

Quick sketch of the parabola



Terminology & Facts

- The zero set of a system of polynomials with rational coefficients is called an algebraic set.

- Resultants give us a way of converting rational parameterized surfaces into implicit equations.

- Interestingly, it is not always possible to go the other way, that is, take an algebraic surface $\{(x,y,z) \mid F(x,y,z)=0\}$ and construct a parameterization of that surface by rational functions.

Example 5 (cont)

Suppose someone gives us a point on the curve. Can we figure out the parameter t corresponding to that point? Yes, using the reduced system.

The point $(x,y) = (7,5)$ is on the curve from p.22. To find t, plug (x,y) into the partial system

$$\begin{pmatrix} 5 & 1 & 3-x & 0 \\ 0 & 5 & 1 & 3-x \\ 5 & -1 & -1-y & 0 \end{pmatrix} \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix} = 0$$

Then
$$t = \frac{t^3}{t^2} = (-1)^{1+2} \frac{\det \begin{vmatrix} 1 & 3-x & 0 \\ 5 & 1 & 3-x \\ -1 & -1-y & 0 \end{vmatrix}}{\det \begin{vmatrix} 5 & 3-x & 0 \\ 0 & 1 & 3-x \\ 5 & -1-y & 0 \end{vmatrix}}$$ with $x=7$ & $y=5$

$$= - \frac{\det \begin{vmatrix} 1 & -4 & 0 \\ 5 & 1 & -4 \\ -1 & -6 & 0 \end{vmatrix}}{\det \begin{vmatrix} 5 & -4 & 0 \\ 0 & 1 & -4 \\ 5 & -6 & 0 \end{vmatrix}}$$

$$= - \frac{-40}{-40}$$

So. $t = -1$

To double-check:
$$x(-1) = 5 - 1 + 3 = 7$$
$$y(-1) = 5 + 1 - 1 = 5$$

Comments

- These ideas generalize to higher dimensions, and higher order polynomials.

- Sylvester's Method, while easy to describe, is not the most compact way to generate the resultant of two univariate polynomials.

  A more compact representation is given by Caley's method. We won't derive the general form here, but see p. 76 of S, A, & G. For two quadratics
  $$ax^2+bx+c=0$$
  $$a'x^2+b'x+c'=0$$
  the method says that one can replace our 4×4 matrix Q with a 2×2 matrix whose entries basically precompute some of the subdeterminants. Thus
  $$\det Q = \det \begin{pmatrix} ab'-a'b & ac'-a'c \\ ac'-a'c & bc'-b'c \end{pmatrix}.$$
  Also known as Bezout's method

- The extension of Cayley's method to three bivariate polys in 2D is known as Dixon's Method. Dixon's method underlies the surface implicitization of Ponce & Kriegman's paper.

For completeness, let's revisit our first three examples, now using Cayley's method:

1) $\left.\begin{array}{l} p(x) = x^2 - 6x + 2 \\ q(x) = x^2 + x + 5 \end{array}\right\}$ $Q = \begin{pmatrix} 7 & 3 \\ 3 & -22 \end{pmatrix}$, $\det Q = -233$,

so there are no common zeros.

2) $\left.\begin{array}{l} p(x) = x^2 - 4x - 5 \\ q(x) = x^2 - 7x + 10 \end{array}\right\}$ $Q = \begin{pmatrix} -3 & 15 \\ 15 & -75 \end{pmatrix}$, $\det Q = 0$,

so there is a common zero.

Furthermore, its value is:

$$x = \frac{x}{1} = (-1)^{1+2} \frac{\det(15)}{\det(-3)} = 5$$

3) $\left.\begin{array}{l} p(x) = x^2 - 4x - 5 \\ q(x) = x^2 - 7x + c \end{array}\right\}$ $Q = \begin{pmatrix} -3 & c+5 \\ c+5 & -4c-35 \end{pmatrix}$

$$\det Q = -(c^2 - 2c - 80)$$

which is what we got before

(i.e., p & q have simultaneous zeros iff $c = 10$ or $c = 8$)