Introduction to Differential Geometry

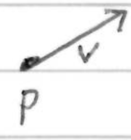
(Material largely adopted from "Elementary Differential Geometry" by B. O'Neill.)

Differential Geometry studies the motions possible in a space.

Some key concepts:

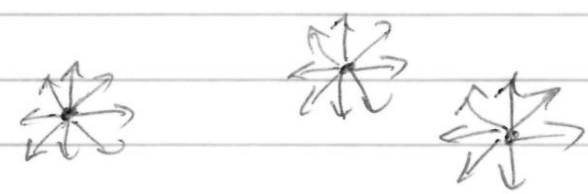
Tangent vector v_p: A vector anchored at a particular point:

Set of all possible v_p for a given p is called the tangent space T_p at p.

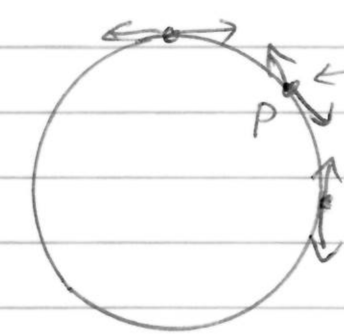


Tangent Bundle: A space along with all its tangent vectors.

Ex: If $\mathbb{R}^n$ is the underlying space then we have another $\mathbb{R}^n$ at each point $p \in \mathbb{R}^n$, consisting of all the tangent vectors anchored at p. So we get $\mathbb{R}^n \times \mathbb{R}^n$ overall, just like a state space. For $n=2$:



Ex: The tangent bundle associated with a circle looks like $S^1 \times \mathbb{R}^1$:



tangent space at p describes motions possible (differentially) that remain on the circle.

<u>Vector field</u> A function $M \to T(M)$

M: underlying space, often called a <u>manifold</u> (e.g., $\mathbb{R}^n$, S^n, even matrix groups, e.g. $so(n)$)

$T(M)$: tangent bundle

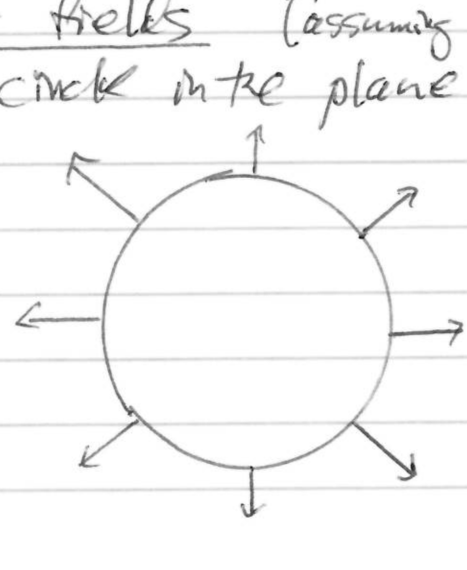
$$p \longmapsto v_p \in T_p$$

often denoted by $V(p)$ or V_p. (We generally require V to be smooth, meaning as many derivatives as we need.)

One classic question is whether a manifold has a continuously varying vector field that is never zero.

We just drew one for $M = S^1$.
Impossible for $M = S^2$
(of course there is a 2D tangent space T_p for each $p \in S^2$, but we can't find a function $V: S^2 \to T(S^2)$ that is nonvanishing and continuous. Proof is beyond these lectures; it entails studying antipodal maps & fixed point theorems.)

When we have one manifold embedded in another we can ask whether it is possible to find ~~nonvanishing~~ smooth unit <u>normal vector fields</u> (assuming we have defined an inner product).
E.g., for the circle in the plane:



Possible for <u>orientable</u> submanifolds.

(This leads to ideas like the Gauss map and Gaussian curvature.)

Geometry of Curves in $\mathbb{R}^3$

We will consider parameterized curves and we will assume that they are sufficiently smooth to give us as many derivatives as we need. (e.g., C^3)

Def's A curve is a smooth function $\alpha: I \to \mathbb{R}^3$, with I some interval in $\mathbb{R}^1$.
We often write $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$.

The velocity vector of α at time t is the tangent vector of $\mathbb{R}^3$ given by $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$ [here $'$ means $\frac{d}{dt}$]. This vector is also tangent to $\alpha(t)$.

The speed of α at time t is $v(t) = \|\alpha'(t)\|$.

The arclength traversed between time t_0 and time t_1 is

$$\int_{t_0}^{t_1} v(t)\, dt$$

$\boxed{\text{Th}^{\underline{m}}}$ Suppose $\alpha: [a,b] \to \mathbb{R}^3$ is a curve for which $\alpha'(t)$ is not ever 0 on $[a,b]$. Then one can reparameterize $\alpha(t)$ as $\beta(s)$ with s measuring arclength. Note that β gives a unit-speed parametrization of the curve.

[So $\beta(s) = \alpha(t(s))$ — the time at which the curve α would have ~~reached~~ reached/traversed arclength s.]

<u>Proof</u> Define $s(t) = \int_a^t \|\alpha'(u)\|\,du$, for $t \in [a,b]$

Then $s'(t) = \|\alpha'(t)\| > 0$.

So $s(t)$ is strictly monotone, meaning the inverse $t(s)$ exists. (not always easy to calculate, of course)

Let $\beta(s) = \alpha\big(t(s)\big)$, $s \in [0, s(b)]$.

Note that

$$\beta'(s) = \frac{d}{ds}\alpha(t(s))$$
$$= \alpha'(t(s))\frac{dt}{ds}(s).$$

So $\|\beta'(s)\| = \|\alpha'(t(s))\| \frac{dt}{ds}(s)$ since $\frac{ds}{dt} > 0$, $\frac{dt}{ds} > 0$.

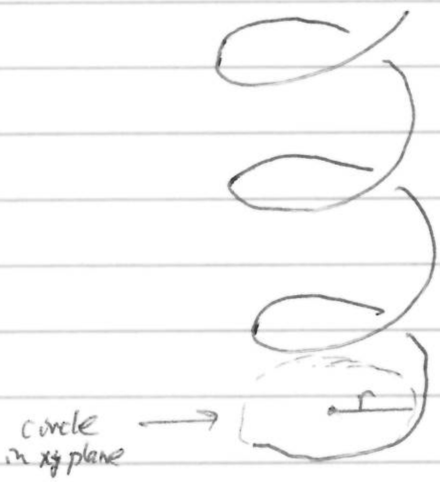
$$= \frac{ds}{dt}(t(s)) \cdot \frac{dt}{ds}(s)$$

$$= 1.$$

Example: A helix in $\mathbb{R}^3$.

$$\alpha(t) = (\underbrace{r\cos t,\ r\sin t}_{\text{circular part}},\ \underset{\text{rise/fall}}{\overset{}{qt}}),\quad r>0,\ q\neq 0.$$

$t \in [0,\infty]$ for simplicity (but could use any interval of course, including $[-\infty,\infty]$, but then might also want arclength to go from $-\infty$ to $+\infty$; just break into two parts joined at $t=0$ / $s=0$)



$$\alpha'(t) = (-r\sin t,\ r\cos t,\ q)$$

$$v(t) = \sqrt{r^2+q^2} = c \quad \text{(constant speed)}$$

$$s(t) = \int_0^t c\,du = ct. \quad \text{Thus } t(s) = \frac{s}{c}.$$

So can reparameterize as

$$\beta(s) = \alpha\left(\frac{s}{c}\right) = \left(r\cos\frac{s}{c},\ r\sin\frac{s}{c},\ \frac{qs}{c}\right).$$

Now have a unit speed curve giving the same shape.

(Not usually so easy to reparameterize this way.)

Observe: Suppose we have a curve $\alpha : I \to \mathbb{R}^3$ and a smooth int function that assigns to each point $\alpha(t)$ a vector $V(t)$ of $\mathbb{R}^3$ ($V(t)$ need not be tangent to $\alpha(t)$, merely a tangent vector of $\mathbb{R}^3$). Differentiating $V(t)$ allows us to obtain information about α.

Let's first look at a unit-speed curve $\beta : I \to \mathbb{R}^3$ (so $\beta(s)$ is a parameterization in terms of arclength s).

Define the following three vector fields on β:

$T = \beta'$ called the unit tangent vector field of β

$N = \dfrac{T'}{\|T'\|}$ called the principal normal vector field of β

$B = T \times N$ called the binormal vector field of β

The quantity $\|T'\|$ also has a name:

$\kappa(s) = \|T'(s)\|$ is the curvature function of β.

Note: Since β is unit-speed, T is a unit vector. It could be that $T' = 0$, in which case N & B are not well-defined. This occurs for instance when β is a straight-line, or when it is instantaneously linear. So, let's assume $\kappa > 0$ over the entire curve segment I that we are considering.

Th$^{\underline{m}}$ Let $\beta: I \to \mathbb{R}^3$ be a unit-speed curve with nonzero curvature (i.e., $\kappa > 0$) for all $s \in I$.
Then $[T, N, B]$ is an orthonormal set for all $s \in I$.

Def $[T, N, B]$ is called the <u>Frenet frame field</u> of β.

Proof T, N, B are all well-defined since the curvature is nonzero. They all have unit length by construction. Moreover, B is orthogonal to T & N by construction.
So we need only show that T & N are orthogonal.
Observe:

$T = T(s)$ $\quad \kappa = \kappa(s)$
$N = N(s)$
$B = B(s)$

$$T \cdot T = 1$$

So $$\frac{d}{ds}(T \cdot T) = 0$$

So $$2T' \cdot T = 0,$$

telling us that T' & T are orthogonal, since $T' \neq 0$ by the curvature assumption.
That means N & T are orthogonal.

(And of course, B is orthogonal to both T & N, and all three vectors, T, N, B, have unit length.)

This type of analysis gives us more information:

- Since $B \cdot B = 1$, we also see that $B' \cdot B = 0$.

  Since $B \cdot T = 0$, we see that $B' \cdot T + B \cdot T' = 0$.

  So $B' \cdot T = -B \cdot T' = -B \cdot (\kappa N)$

  since $T' = \kappa N$ by construction.

  $B \cdot N = 0$ by construction, so in fact

  $B' \cdot T = 0$.

  That means, since $[T, N, B]$ is an orthonormal frame, that B' must be a scalar multiple of N. One writes

  $B' = -\tau N$, with τ a real-valued function on β, called the <u>torsion</u> of β.

- Finally, we know that

  $N \cdot T = 0$ & $N \cdot N = 1$ & $N \cdot B = 0$.

  So $N' \cdot T + N \cdot T' = 0$ $\quad$ $2N \cdot N' = 0$ $\quad$ $N' \cdot B + N \cdot B' = 0$

  $\Downarrow$ $\quad$ $\Downarrow$ $\quad$ $\Downarrow$

  $N' \cdot T = -N \cdot T' = -N \cdot (\kappa N) = -\kappa$ (since $N \cdot N = 1$)

  $N \cdot N' = 0$

  well, I meant to write $N' \cdot N$

  $N' \cdot B = -N \cdot B'$
  $= -N \cdot (-\tau N)$
  $= \tau$.

So $N' \cdot T = -\kappa$, $N' \cdot N = 0$, & $N' \cdot B = \tau$.

We have just established the Frenet Formulas:

Th$^{\underline{m}}$ If $\beta : I \to \mathbb{R}^3$ is a unit-speed curve with curvature $\kappa > 0$ & torsion τ, then:

$$\begin{aligned} T' &= \qquad\qquad \kappa N \\ N' &= -\kappa T \qquad\qquad\quad \tau B \\ B' &= \qquad\quad -\tau N \end{aligned}$$

They generalize to arbitrary parameterizations $\alpha(t)$, assuming $\alpha'(t) \neq 0$ & $\kappa > 0$ as follows:

$$\begin{aligned} T' &= \qquad\qquad \kappa v N \\ N' &= -\kappa v T \qquad\qquad\quad \tau v B \\ B' &= \qquad\quad -\tau v N \end{aligned}$$

($[T, N, B]$ still an orthonormal frame field.)

where now $\quad v = \|\alpha'\|$ (& that gives T via $\downarrow$)

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$$

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$$

$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$$

$$N = B \times T \qquad\qquad T = \frac{\alpha'}{\|\alpha'\|}$$

Also have: $\alpha' = vT$ & $\alpha'' = v'T + \kappa v^2 N$

<u>Example:</u> Unit-speed helix: $\beta(s) = \left(r\cos\frac{s}{c}, r\sin\frac{s}{c}, \frac{qs}{c}\right)$,

$r > 0$, $q \neq 0$,

$c = \sqrt{r^2+q^2}$.

' means $\frac{d}{ds}$ here, i.e., differentiate wrt curve parameter

$$T(s) = \beta'(s) = \left(-\frac{r}{c}\sin\frac{s}{c}, \frac{r}{c}\cos\frac{s}{c}, \frac{q}{c}\right)$$

$$T'(s) = \left(-\frac{r}{c^2}\cos\frac{s}{c}, -\frac{r}{c^2}\sin\frac{s}{c}, 0\right)$$

$$\kappa(s) = \|T'(s)\| = \frac{r}{c^2} = \frac{r}{r^2+q^2} \quad \text{(curvature)}$$

$$N(s) = \frac{T'(s)}{\kappa(s)} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$$

[this vector points at the z-axis, parallel to xy plane]

$$B(s) = T(s) \times N(s) = \left(\frac{q}{c}\sin\frac{s}{c}, -\frac{q}{c}\cos\frac{s}{c}, \frac{r}{c}\right)$$

$$B'(s) = \left(\frac{q}{c^2}\cos\frac{s}{c}, \frac{q}{c^2}\sin\frac{s}{c}, 0\right)$$

Frenet tells us that $B' = -\tau N$,

so the torsion $\tau(s) = \frac{q}{c^2} = \frac{q}{r^2+q^2}$.

Also, just to verify the Frenet formula for N', which says $N' = -\kappa T + \tau B$, note that:

$$N' = \left(\frac{1}{c} \sin \frac{s}{c}, -\frac{1}{c} \cos \frac{s}{c}, 0 \right)$$

Also:

$$-\kappa T = \left(\frac{r^2}{c^3} \sin \frac{s}{c}, -\frac{r^2}{c^3} \cos \frac{s}{c}, -\frac{qr}{c^3} \right)$$

$$\tau B = \left(\frac{q^2}{c^3} \sin \frac{s}{c}, \frac{-q^2}{c^3} \cos \frac{s}{c}, \frac{qr}{c^3} \right)$$

$$-\kappa T + \tau B = \left(\frac{r^2+q^2}{c^3} \sin \frac{s}{c}, -\frac{r^2+q^2}{c^3} \cos \frac{s}{c}, 0 \right)$$

$$\frac{r^2+q^2}{c^3} = \frac{c^2}{c^3} = \frac{1}{c}$$, so all good.

If we started with the parametrization $\alpha(t) = (r\cos t, r\sin t, qt)$, then we would use the general formulas to obtain:

$$\alpha'(t) = (-r\sin t, r\cos t, q)$$
$$\alpha''(t) = (-r\cos t, -r\sin t, 0)$$
$$\alpha'''(t) = (r\sin t, -r\cos t, 0).$$

So $\alpha' \times \alpha'' = (rq\sin t, -rq\cos t, r^2)$

$\|\alpha' \times \alpha''\| = r\sqrt{q^2+r^2} = rc$

$$v = \|\alpha'(t)\| = \sqrt{r^2+q^2} = c,$$ as before, i.e., on p. 5.

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{rc}{c^3} = \frac{r}{c^2} = \frac{r}{r^2+q^2},$$ as before,

$$\tau = \frac{(\alpha' \times \alpha'')\cdot\alpha'''}{\|\alpha' \times \alpha''\|^2} = \frac{r^2 q}{r^2 c^2} = \frac{q}{c^2} = \frac{q}{r^2+q^2},$$ as before.

$$T = \frac{1}{v}\alpha' = \left(-\frac{r}{c}\sin t, \frac{r}{c}\cos t, \frac{q}{c}\right)$$

$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} = \left(\frac{q}{c}\sin t, -\frac{q}{c}\cos t, \frac{r}{c}\right)$$

* $$N = B \times T = (-\cos t, -\sin t, 0),$$

all as before, now with t in place of $\frac{s}{c}$.

(Note that $\|\alpha' \times \alpha''\| = r\sqrt{r^2+q^2} = rc$.)

Let's see what curvature & torsion mean more generally.
Suppose $\beta: I \to \mathbb{R}^3$ is a unit-speed curve
(assume nonzero curvature throughout).

Let's do a Taylor expansion around $s=0$:

$$\beta(s) = \beta(0) + s\beta'(0) + \frac{s^2}{2}\beta''(0) + \frac{s^3}{6}\beta'''(0) + \cdots$$

Abbreviate: $T_0 = T(0)$, $N_0 = N(0)$, $B_0 = B(0)$
$\kappa_0 = \kappa(0)$, $\tau_0 = \tau(0)$.

Assume $\kappa_0 > 0$ & $\tau_0 \neq 0$.

Then $\beta'(0) = T_0$
$\beta''(0) = \kappa_0 N_0$

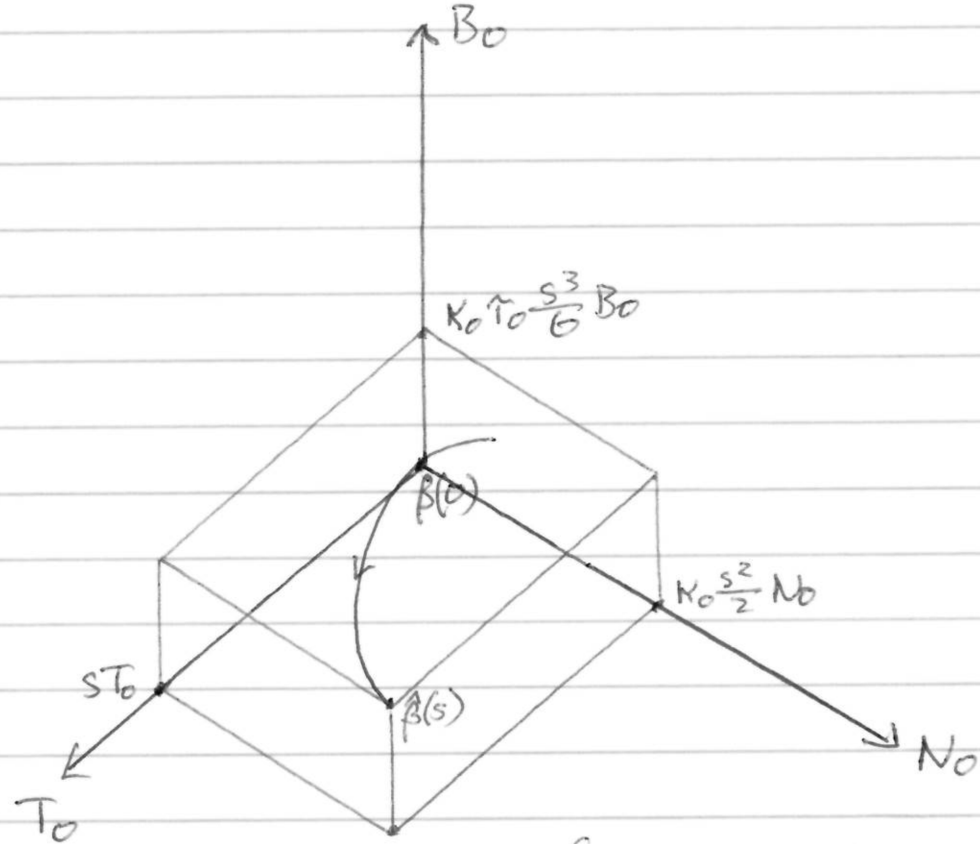
Also $\beta'''(s) = (\kappa N)' = \frac{d\kappa}{ds} N + \kappa N'$
By the Frenet Formulas, $N' = -\kappa T + \tau B$,
so $\beta'''(0) = -\kappa_0^2 T_0 + \frac{d\kappa}{ds}(0) N_0 + \kappa_0 \tau_0 B_0$.

$$\text{So } \beta(s) \approx \underbrace{\beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0}$$

This curve is called the
<u>Frenet approximation</u> of β near $s=0$.

Along each of the directions T_0, N_0, B_0
it retains only the most significant
power of s.

Visualization (picture taken from O'Neill's Elementary Differential Geometry book)



(meaning in a Taylor series sense)

$\beta(0) + sT_0$ is the best linear approximation to β near $\beta(0)$.

$\beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0$ is the best quadratic approximation.

The best cubic approximation is actually

$$\beta(0) + \left(s - \kappa_0^2 \frac{s^3}{6}\right) T_0 + \left(\kappa_0 \frac{s^2}{2} + \frac{d\kappa}{ds}(0)\frac{s^3}{6}\right) N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0,$$

but that includes terms in the directions T_0 & N_0 that are not dominant in those directions, so the Frenet approximation is merely

$$\hat{\beta}(s) = \beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0.$$

Again, coming back to the example helix:

$$\beta(s) = \left(r\cos\tfrac{s}{c},\ r\sin\tfrac{s}{c},\ \tfrac{qs}{c}\right)$$

Then κ is constant, so $\frac{d\kappa}{ds} = 0$.

So the best cubic approximation (in a Taylor series sense) is (at $s=0$)

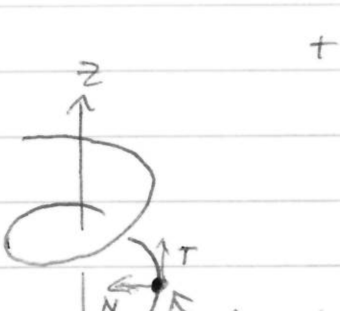
$$(r,0,0) + \left(s - \kappa\tfrac{s^3}{6}\right)\left(0, \tfrac{r}{c}, \tfrac{q}{c}\right)$$
$$+ \quad \kappa\tfrac{s^2}{2}\left(-1, 0, 0\right)$$
$$+ \quad \kappa\tau\tfrac{s^3}{6}\left(0, \tfrac{-q}{c}, \tfrac{r}{c}\right)$$

while the Frenet approximation is

$$(r,0,0) + s\underbrace{\left(0, \tfrac{r}{c}, \tfrac{q}{c}\right)}_{T}$$ (tangent to curve)

$$+ \kappa\tfrac{s^2}{2}\underbrace{\left(-1, 0, 0\right)}_{N}$$ (points at z-axis) (parallel to x-axis)

$$+ \kappa\tau\tfrac{s^3}{6}\underbrace{\left(0, -\tfrac{q}{c}, \tfrac{r}{c}\right)}_{B}$$ (perpendicular to x-axis & to tangent) $\left(c = \sqrt{r^2+q^2}\right)$.



note orthonormality of these two directions.

$$\kappa = \frac{r}{r^2+q^2} = \frac{r}{c^2}$$

$$\tau = \frac{q}{r^2+q^2} = \frac{q}{c^2}$$

$\boxed{\text{Lemma}}$ Let β be a unit-speed curve.

① β is a straight line iff $\kappa = 0$ (meaning: identically zero).

② Suppose $\kappa > 0$. Then β is a planar curve iff $\tau = 0$ (").

> So, κ measures the extent to which a curve is not a line, and τ measures the extent to which a curve is not planar.

③ If $\kappa > 0$ & $\tau = 0$ & κ constant, then β is part of a circle of radius $\frac{1}{\kappa}$.

Note: when $\kappa = 0$, T is constant, but N & B are undefined, so technically τ is undefined. It makes sense to let $\tau = 0$, consistent with the formula on page ~~12~~ 9, since have now $\beta'' = 0$. well, I'm not exactly sure what that means since p.9 would give $\frac{0}{0}$, but ok.

$\boxed{\text{Theorem}}$ Let $\alpha : I \to \mathbb{R}^3$ & $\beta : I \to \mathbb{R}^3$ be arbitrary-speed curves. If $v_\alpha = v_\beta > 0$, $\kappa_\alpha = \kappa_\beta > 0$, and $\tau_\alpha = \pm \tau_\beta$, then α & β are congruent.

Notation:
- v_α is the velocity of α, κ_α its curvature, τ_α its torsion. These are functions of $t \in I$. Similarly for $v_\beta, \kappa_\beta, \tau_\beta$.
- So $v_\alpha = v_\beta$ means $v_\alpha(t) = v_\beta(t)$ for all t in I.
- "congruent" means α & β are related by a translation, rotation, and possibly a reflection.

> So, v, κ, τ characterize the curves.
> These functions are complete invariants (for space curves).
> Indeed, for unit-speed curves, κ & τ are complete invariants.

(The proofs involve some calculations, with the Frenet Formulas being the key insights.)

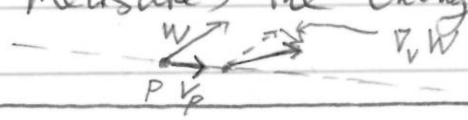
Covariant Derivatives & Lie Brackets

Def Suppose V & W are two vector fields in $\mathbb{R}^n$.
So, for each point $p \in \mathbb{R}^n$, $V(p)$ & $W(p)$ are vectors in $\mathbb{R}^n$.

Then the <u>covariant derivative of W with respect to V</u> is

$$(\nabla_V W)(p) = \frac{d}{dt} W(p + t v_p)\Big|_{t=0}, \text{ with } v_p = V(p).$$

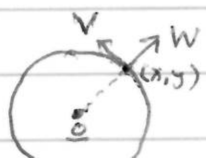
$\nabla_V W$ measures the change in W as one moves along V.



Ex In $\mathbb{R}^2$: $W(p) = (1,0)$ for all p in $\mathbb{R}^2$.
$V(p) = (0,1)$

Then $\nabla_V W = \underset{\text{vector}}{\underline{0}} = \nabla_W V$.

Ex



For each $p = (x,y) \in \mathbb{R}^2$,
$$W = \frac{(x,y)}{\sqrt{x^2+y^2}}, \quad V = \frac{(-y,x)}{\sqrt{x^2+y^2}}$$
(unit normal & unit tangent vector fields to circles of radius $\sqrt{x^2+y^2}$)

Remember: For a unit-speed curve: $N = \frac{T'}{\kappa}$. $T = V$ here & $N = -W$ & $T' = \nabla_V V = \frac{-W}{\sqrt{x^2+y^2}}$, consistent with $\kappa = \frac{1}{\sqrt{x^2+y^2}}$.

Then $\nabla_V W = \frac{V}{\sqrt{x^2+y^2}}$. In particular, on the unit circle, the change in the normal vector field as one moves around the circle is simply the tangent vector field.
(kind of like $-\alpha'''$ on p.12)

Q: What is $\nabla_W V$? A: $\underline{0}$ (as one can see intuitively)
(see p.19 for calculations) (similarly, $\nabla_W W = \underline{0}$.)

Some facts that help with calculations

- $\nabla_V W$ is an n-dimensional vector, whose i^{th} component is

$$\underbrace{\left(\nabla W_i\right)}_{\substack{\text{gradient of the } i^{th} \\ \text{component of } W\text{, viewed as} \\ \text{a function on } \mathbb{R}^n}} \cdot \, V_p$$

tangent vector assigned to point p by vector field V

- $\nabla_V (aW + bU) = a\nabla_V W + b\nabla_V U$,

  for all (real) numbers a & b.

- $\nabla_{fV+gU} W = f\nabla_V W + g\nabla_U W$,

  for all (smooth/differentiable) functions $\mathbb{R}^n \to \mathbb{R}$.

Filling in details for the second example of p. 17:

$$\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}} = \frac{y^2}{(x^2+y^2)^{3/2}} \qquad \frac{\partial}{\partial x}\frac{y}{\sqrt{x^2+y^2}} = -\frac{xy}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial}{\partial y}\frac{x}{\sqrt{x^2+y^2}} = \frac{-xy}{(x^2+y^2)^{3/2}} \qquad \frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}} = \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$\nabla_V W = \left((\nabla W_x)\cdot v_p \ , \ (\nabla W_y)\cdot v_p \right)$$

$$(\nabla W_x)\cdot v_p = \frac{y^2(-y)+(-xy)x}{(x^2+y^2)^2} = \frac{-y}{x^2+y^2}$$

$$(\nabla W_y)\cdot v_p = \frac{(-xy)(-y)+x^2\cdot x}{(x^2+y^2)^2} = \frac{x}{x^2+y^2}$$

So $\nabla_V W = \dfrac{V}{\sqrt{x^2+y^2}}$.

Similarly, $\nabla_W V = \left((\nabla V_x)\cdot w_p , (\nabla V_y)\cdot w_p \right)$

$$= \left(\frac{(xy)x-(x^2)y}{(x^2+y^2)^2} , \frac{y^2x-(xy)y}{(x^2+y^2)^2} \right)$$

$$= (0, 0).$$

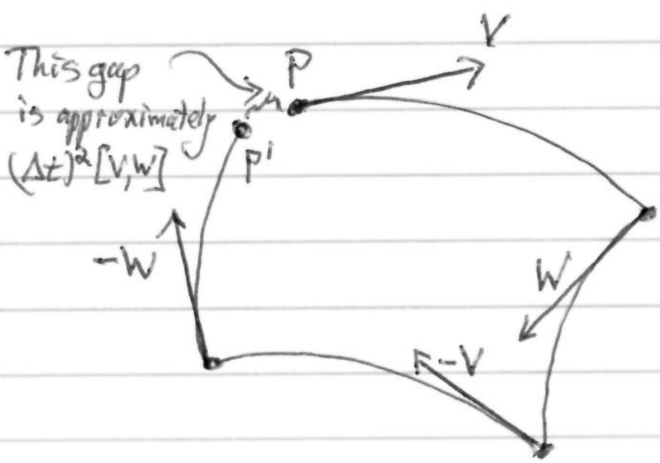
And $\nabla_V V = \left((\nabla V_x)\cdot v_p , (\nabla V_y)\cdot v_p \right)$

$$= \left(\frac{(xy)(-y)-(x^2)x}{(x^2+y^2)^2} , \frac{y^2(-y)-(xy)x}{(x^2+y^2)^2} \right) = \frac{1}{x^2+y^2}(-x,-y) = \frac{-W}{\sqrt{x^2+y^2}}$$

<u>Def</u> The <u>Lie Bracket</u> $[V, W]$ of two vector fields is defined to be

$$[V, W] = \nabla_V W - \nabla_W V.$$

Intuitively, it measures the following:



- Flow along V for duration Δt
- Flow along W for duration Δt
- Flow along $-V$ for duration Δt
- Flow along $-W$ for duration Δt

The net motion is approximately $(\Delta t)^2 \, [V, W]$

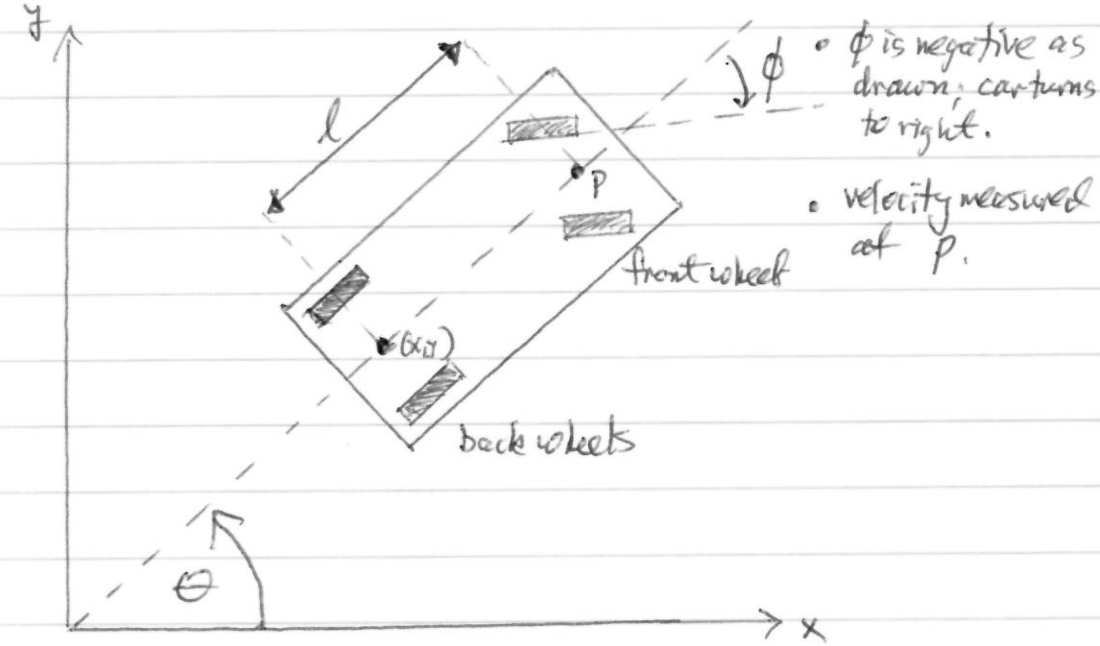
(See Murray, Li, & Sastry for a proof, based on Taylor expansions.)

In non-holonomic control, $[V, W]$ (& higher order variants) can provide additional motions not directly possible via individual controls.

Examples:
- Parallel parking
- Satellite control when some thrusters fail.

[of course, for many "simple" vector fields, $[V, W] = 0$, e.g., if V & W are axis-parallel. Or $[V, W]$ is in span$\{V, W\}$, as in Ex on p. 17.]

Parallel Parking Example



- Configuration of car: (x, y, θ) $\boxed{3D}$
- Controls: (v, ϕ), with ϕ the steering angle and v velocity of point p midway at front.

Only $\boxed{2D}$ of controls.

Yet, can place the car in any configuration, assuming no obstacles. In fact, can do so with two "independent" vector fields and their negations. E.g. consider two controls $(1, \phi_1)$ & $(1, \phi_2)$ with ϕ_1 & ϕ_2 two different steering angles. And, for the negative vector fields, we allow the car to move backwards, corresponding to controls $(-1, \phi_1)$ & $(-1, \phi_2)$.

If we assume the wheels make point contacts & do not slip $\otimes$, then control (v, ϕ) changes the car's configuration differentially as follows:

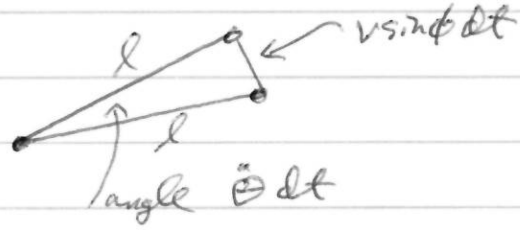
$$\dot{x} = v\cos\phi\cos\theta$$
$$\dot{y} = v\cos\phi\sin\theta$$
$$\dot{\theta} = \frac{v}{\ell}\sin\phi.$$

$\otimes$ well, what I'm really assuming is that p & (x, y) move according to

(why? In time dt, the front wheels and hence the point p move distance $v\,dt$ along the direction $(\cos\phi, \sin\phi)$ relative to the center line of the car, which is aligned with $(\cos\theta, \sin\theta)$.

In other words $v\cos\phi\,dt$ is parallel to the car, giving $\dot{x}$ & $\dot{y}$ as above.

And $v\sin\phi\,dt$ is perpendicular to the car's centerline. That turns the car. The new orientation satisfies:



So $\dot{\theta} = \frac{v}{\ell}\sin\phi$.)

So our two controls $(1, \phi_1)$ & $(1, \phi_2)$ give us the two vector fields

$$V_i = V_i(x, y, \theta) = \left(\cos\phi_i \cos\theta, \cos\phi_i \sin\theta, \frac{\sin\phi_i}{\ell}\right), \ i = 1, 2.$$

We are interested in $[V_1, V_2] = \nabla_{V_1} V_2 - \nabla_{V_2} V_1$.

$$\nabla_{V_1} V_2 = \left(\nabla(\cos\phi_2 \cos\theta) \cdot V_1,\ \nabla(\cos\phi_2 \sin\theta) \cdot V_1,\ \nabla\left(\frac{\sin\phi_2}{\ell}\right) \cdot V_1\right).$$

gradients are wrt x, y, θ (not ϕ_i) :

$$\nabla(\cos\phi_2 \cos\theta) = \begin{pmatrix} 0 \\ 0 \\ -\cos\phi_2 \sin\theta \end{pmatrix}$$

$$\nabla(\cos\phi_2 \sin\theta) = \begin{pmatrix} 0 \\ 0 \\ \cos\phi_2 \cos\theta \end{pmatrix}$$

$$\nabla\left(\frac{\sin\phi_2}{\ell}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $\nabla_{V_1} V_2 = \dfrac{\sin\phi_1 \cos\phi_2}{\ell}(-\sin\theta, \cos\theta, 0)$

Similarly, $\nabla_{V_2} V_1 = \dfrac{\sin\phi_2 \cos\phi_1}{\ell}(-\sin\theta, \cos\theta, 0)$.

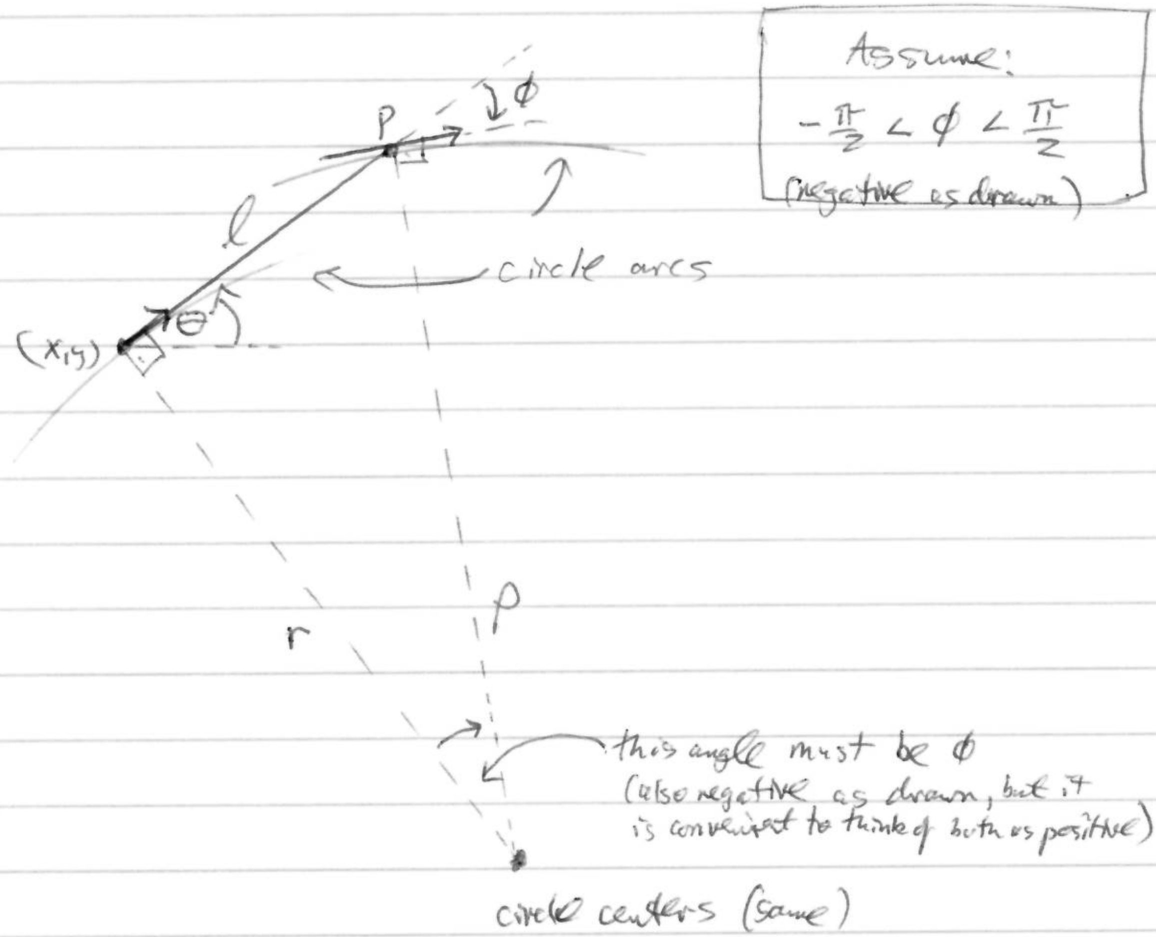
$$\text{So } [V_1, V_2] = \frac{\sin\phi_1 \cos\phi_2 - \sin\phi_2 \cos\phi_1}{\ell}(-\sin\theta, \cos\theta, 0)$$

$$= \frac{\sin(\phi_1 - \phi_2)}{\ell}(-\sin\theta, \cos\theta, 0).$$

In other words, so long as $\sin(\phi_1 - \phi_2)$ (so $\phi_1 - \phi_2$ is not a multiple of π.) is not zero, the Lie Bracket says that one can move <u>perpendicular</u> to the car's centerline. That is a new direction, not directly attainable by any single control (v, ϕ).

Coming back to p.22, we should be able to obtain these differential motions as well by assuming that (x,y) & p each move differentially on a circle, one circle per point, with the circle centers identical (that's the "no slip" assumption, differentially).

So:



In other words, the car rigidly rotates around the circle center(s). So that changes its orientation by $\dot{\theta} dt$. And the point (x,y) moves distance $|r\dot{\theta}dt|$ while the point p moves distance $|\rho\dot{\theta}dt|$, with r & ρ the corresponding circle radii.

We also know that $l^2 + r^2 = \rho^2$.

And from the drawing we see that $|\rho \sin\phi| = l$.

$|\rho\cos\phi| = r$, so $\rho\cos\phi = r$, given $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

If ϕ is negative as drawn, then $-\sin\phi = \frac{l}{\rho}$.

we know that $-v\,dt = \rho\,\dot{\theta}\,dt$ (the minus sign accounts for positive velocity along the arrow through ρ giving a negative change in orientation)

so $\dot{\theta} = \frac{-v}{\rho} = \frac{v}{l}\sin\phi$

↑ using last equality on p. 24.

The signed distance that (x, y) moves forward along the car's center line is similarly

$$
\begin{aligned}
d &= -r\dot{\theta}\,dt \\
&= v\frac{r}{\rho}\,dt \\
&= v\cos\phi\,dt \qquad (\text{since } \rho\cos\phi = r).
\end{aligned}
$$

That motion occurs along direction vector $(\cos\theta, \sin\theta)$. So we see that

$$\dot{x} = v\cos\phi\cos\theta,$$

$$\dot{y} = v\cos\phi\sin\theta,$$

$$\dot{\theta} = \frac{v}{l}\sin\phi.$$

Pf of the intuitive picture on p. 45 — (p. 20 in these notes)
(This is adapted from pp. 323 & 4 of "Robotic Manipulation" by Murray, Li, & Sastry.)

First, let's look at what happens when we follow a vector field V, starting from some point p.
Let's denote the resulting curve by $\alpha(t)$.
Taylor's theorem tells us that for small Δt,

$$\alpha(\Delta t) = \alpha(0) + \alpha'(0)\Delta t + \alpha''(0)\frac{(\Delta t)^2}{2} + O((\Delta t)^3)$$

where $\alpha(0) = p$ i.e., the starting point
$\alpha'(0) = V(p)$ i.e., the direction of the vector field at p.
& $\alpha''(0) = \frac{d}{dt}V(\alpha(t)) = \nabla_V V$, i.e., the change of V as one moves along V.

That's the general form. Now we want to look at the following concatenation of motions:

(i) from time 0 to Δt, flow along V,
(ii) from time Δt to $2\Delta t$, flow along W,
(iii) from time $2\Delta t$ to $3\Delta t$, flow along $-V$,
(iv) from time $3\Delta t$ to $4\Delta t$, flow along $-W$.

Let's write $q(t)$ as the configuration at time t resulting from this composite motion. What we want to show is that

$$q(4\Delta t) - q(0) \approx \underset{\uparrow \text{constant}}{C}\,(\Delta t)^2[V, W]$$

(in fact, $C = 1$)

(i) By our reasoning above:

$$q(\Delta t) = q(0) + \Delta t\, V\Big|_{q(0)} + \tfrac{1}{2}(\Delta t)^2\, \nabla_V V\Big|_{q(0)} + \mathcal{O}((\Delta t)^3)$$

(ii) Similarly:

$$q(2\Delta t) = q(\Delta t) + \Delta t\, W\Big|_{q(\Delta t)} + \tfrac{1}{2}(\Delta t)^2\, \nabla_W W\Big|_{q(\Delta t)} + \mathcal{O}((\Delta t)^3)$$

$$= q(0) + \Delta t\, V\Big|_{q(0)} + \tfrac{1}{2}(\Delta t)^2\, \nabla_V V\Big|_{q(0)}$$

$$+ \Delta t \Big(W\Big|_{q(0)} + \Delta t\, \underbrace{\frac{d}{dt} W(q(t))\Big|_{t=0}}_{\text{This is } \nabla_V W|_{q(0)}} + \cdots \Big)$$

$$+ \tfrac{1}{2}(\Delta t)^2 \Big(\nabla_W W\Big|_{q(0)} + \cdots \Big) + \mathcal{O}((\Delta t)^3)$$

$$= q(0) + \Delta t \Big(V\Big|_{q(0)} + W\Big|_{q(0)} \Big)$$

$$+ \tfrac{1}{2}(\Delta t)^2 \Big(\nabla_V V + 2\, \nabla_V W + \nabla_W W \Big)\Big|_{q(0)}$$

$$+ \mathcal{O}((\Delta t)^3)$$

(iii) Continuing:

$$q(3\Delta t) = \underline{q(2\Delta t)} + \Delta t\,(-V)\Big|_{q(2\Delta t)} + \frac{1}{2}(\Delta t)^2\,\nabla_{-V}(-V)\Big|_{q(2\Delta t)} + O((\Delta t)^3)$$

$$= q(0) + \Delta t\,(V+W) + \frac{1}{2}(\Delta t)^2\left(\nabla_V V + 2\nabla_V W + \nabla_W W\right) + \cdots$$

using $q(2\Delta t) \approx q(0) + \Delta t\,(V+w) + \cdots$

$$+ \Delta t\left(-V\Big|_{q(0)} + \Delta t\,\nabla_{V+W}(-V)\Big|_{q(0)} + \cdots\right)$$

$$+ \frac{1}{2}(\Delta t)^2\left(\nabla_V V + \cdots\right)$$

$$+ O((\Delta t)^3)$$

$$= q(0) + \Delta t\,W + \frac{1}{2}(\Delta t)^2\left(\cancel{\nabla_V V} + 2\nabla_V W + \nabla_W W - 2\nabla_{V+W} V + \cancel{\nabla_V V}\right) + O((\Delta t)^3)$$

This is $\cancel{-2\nabla_V V} - 2\nabla_W V$

so $$= q(0) + \Delta t\,W + \frac{1}{2}(\Delta t)^2\left(\nabla_W W + 2\nabla_V W - 2\nabla_W V\right) + O((\Delta t)^3)$$

(where I leave off $\Big|_{q(0)}$ it means evaluate V or W or whatever at $q(0)$.

Otherwise, I'll write e.g. $\Big|_{q(2\Delta t)}$ or whatever)

(iv) And finally:

$$q(4\Delta t) = q(3\Delta t) + \Delta t(-W)\Big|_{q(3\Delta t)} + \tfrac{1}{2}(\Delta t)^2 \nabla_{-W} -W\Big|_{q(3\Delta t)} + \mathcal{O}((\Delta t)^3)$$

$$= q(0) + \cancel{\Delta t W} + \tfrac{1}{2}(\Delta t)^2\left(\nabla_W W + 2\nabla_V W - 2\nabla_W V\right) + \cdots$$

using $q(3\Delta t) \approx q(0) + \Delta t W + \cdots$

$$+ \Delta t\left(\cancel{-W\Big|_{q(0)}} + \Delta t \nabla_W(-W)\Big|_{q(0)} + \cdots\right)$$

$$+ \tfrac{1}{2}(\Delta t)^2\left(\nabla_W W\Big|_{q(0)} + \cdots\right)$$

$$+ \mathcal{O}((\Delta t)^3)$$

$$= q(0) + \tfrac{1}{2}(\Delta t)^2\left(\cancel{\nabla_W W} + 2\nabla_V W - 2\nabla_W V - \cancel{2\nabla_W W} + \cancel{\nabla_W W}\right) + \mathcal{O}((\Delta t)^3)$$

---

so, indeed

$$q(4\Delta t) - q(0) \approx (\Delta t)^2\left(\nabla_V W - \nabla_W V\right)$$

$$= (\Delta t)^2\,[V, W]$$

Shape Operators

We would like to generalize our ideas/methods for measuring the bending of curves to surfaces.

Let M be a surface[*] in $\mathbb{R}^3$.

Suppose Z is a vector field defined on M (perhaps tangent or perhaps normal or a mix).

Z might not be defined on $\mathbb{R}^3 \setminus M$, so taking arbitrary derivatives doesn't make sense. But we can take derivatives wrt to any motions that remain in the surface.

Let's adapt our ∇ notation from pp. 17 & 18:

Suppose v is tangent to M at point p. Imagine a curve $\alpha(t)$ such that $\alpha(0)=p$ and $\alpha'(0) = v$.

Now define $\nabla_v Z = \frac{d}{dt} Z(\alpha(t))\Big|_{t=0}$.

So $\nabla_v Z$ measures the change in Z as one moves along M in the direction v, differentially.

[*] There are some technical details, but intuitively this is a 2D manifold, meaning locally we can think of M as given by a smooth function $f(u,v)$ in $\mathbb{R}^3$.

For calculation purposes the following is useful:

Z is technically only defined on M, so we can't necessarily differentiate wrt motions that move off M.

However, sometimes Z is definable more generally in $\mathbb{R}^3$ (assuming $M \subseteq \mathbb{R}^3$) and differentiable. In that case,

For $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$,

(Maybe better to use U or W instead of Z.)

$$\nabla_v Z = \begin{pmatrix} (\nabla z_1)\cdot v \\ (\nabla z_2)\cdot v \\ (\nabla z_3)\cdot v \end{pmatrix}$$

($z_i = z_i(x, y, z)$, $i = 1, 2, 3$)

where ∇ is the usual gradient, ie, $\nabla z_i = \begin{pmatrix} \frac{\partial z_i}{\partial x} \\ \frac{\partial z_i}{\partial y} \\ \frac{\partial z_i}{\partial z} \end{pmatrix}$ z coord not big Z

and v is the tangent vector at p.

(Note also: $\nabla_v Z$ is linear in v and linear in Z.)

Assumption (for simplicity): M is connected.

Side comments:

- M may or may not be orientable (formal definition is in terms of the existence of a smooth nonvanishing 2-form on M, but is equivalent to the existence of a smooth unit normal vector field on M).

  Since M is connected, if there is such a unit normal vector field $\mathcal{U}$, then there are exactly two unit normal vector fields: $\pm\mathcal{U}$.

- If M is defined by an implicit equation (& if M is a surface, meaning the gradient is nonvanishing) then it is orientable. (so this is immediate, of course)

- Even if M is not orientable, for each $p \in M$, there exists a neighborhood of p that is locally orientable.

Def Let M be a surface in $\mathbb{R}^3$.
Let $p \in M$ and suppose $\mathcal{U}$ is a smooth unit normal vector field on M, defined in a neighborhood of p.

Define S_p by $S_p(v) = -\nabla_v \mathcal{U}$

for each tangent vector $v \in T_p(M)$.

The shape operator S of M is the collection of all these S_p.
(And, yes, there is an ambiguity in sign, since we could use $-\mathcal{U}$ in place of $\mathcal{U}$, but locally in a neighborhood we can make the sign be consistent.)
Measures how the tangent plane to M changes as one moves in M.

Lemma For each $p \in M$, S_p is a linear operator $T_p(M) \to T_p(M)$

Proof Linearity in v follows from linearity of ∇_v in v (basically because dot product with ∇ is linear) (see p. 18).
But how do we know S_p maps tangent vectors to tangent vectors?
By an argument similar to one we used when computing the Frenet formulas:

$$\mathcal{U} \cdot \mathcal{U} = 1$$

$$\text{so } \frac{d}{dt}\left(\mathcal{U}(\alpha(t)) \cdot \mathcal{U}(\alpha(t))\right)\Big|_{t=0} = 0,$$

with $\alpha(t)$ a curve in M such that $\alpha(0) = p$ & $\alpha'(0) = v$.

$$\text{That means } \quad 2\,\mathcal{U}(\alpha(t)) \cdot \underbrace{\frac{d}{dt}\left(\mathcal{U}(\alpha(t))\right)}_{\nabla_v \mathcal{U}} = 0.$$

So $S_p(v)$ is perpendicular to $\mathcal{U}$ at p, meaning $S_p(v) \in T_p(M)$.

<u>Lemma</u> For each $p \in M$, $S_p : T_p(M) \to T_p(M)$ is a symmetric linear operator, meaning

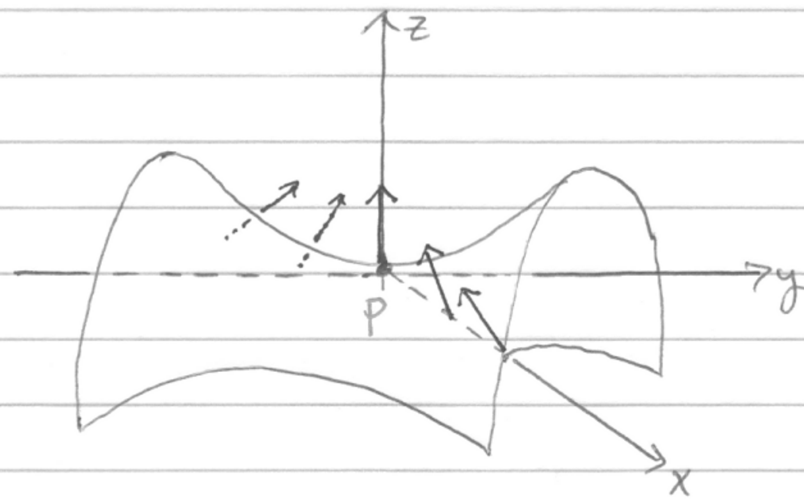
$$S_p(v) \cdot w = S_p(w) \cdot v$$

for any two tangent vectors v & w in $T_p(M)$.

(We omit the proof. See O'Neill, § 5.4, Corollary 4.1. & Lemma 4.2)

So the shape operator has a description at each p in terms of a symmetric 2×2 matrix, meaning we should be able to describe the shape operator in terms of two eigenvectors and two eigenvalues at each point of M (varying smoothly over M).

<u>Ex</u> Consider the saddle surface $M: z = xy$



Let $p = (0,0,0)$.

M includes the x and y axis, so, let $u_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ & $u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$; they are tangent to M at p.

The upward normal at p is $U(p) = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

In fact, the implicit equation for M is $g(x,y,z) = 0$ with $g(x,y,z) = z - xy$. $\nabla g = \begin{pmatrix}-y\\-x\\1\end{pmatrix}$, so

$$U(x,y,z) = \frac{1}{\sqrt{1+x^2+y^2}} \begin{pmatrix}-y\\-x\\1\end{pmatrix}.$$

Then $\nabla_{u_1} U = \frac{\partial}{\partial x} U = \frac{1}{(1+x^2+y^2)^{3/2}} \begin{pmatrix}xy\\-1-y^2\\-x\end{pmatrix}$

So at $p = (0,0,0)$, $\nabla_{u_1} U = \begin{pmatrix}0\\-1\\0\end{pmatrix}$. ← (Intuition: As one moves in x direction, normal bends toward $-y$ direction.)

Similarly, at $p = (0,0,0)$, $\nabla_{u_2} U = \begin{pmatrix}-1\\0\\0\end{pmatrix}$.

So S_p can be written as the matrix $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ in u_1, u_2 coordinates (describing the tangent plane $T_{(0,0,0)}(M)$).

BTW, this is also Hessian of $f(x,y) = xy$. More generally, one needs to look at 1st & 2nd fundamental forms.

In other words, at $p = (0,0,0)$, $S_p(au_1 + bu_2) = bu_1 + au_2$.

Ex Consider the sphere $x^2+y^2+z^2=r^2$ for M.

For each $p \in M$, let $\mathcal{U}(p) = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the unit outward normal, with $p=(x,y,z)$.

Then $\mathcal{U}(p) = \frac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, with $p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

If v is tangent to the sphere at p, then

$$\nabla_v \mathcal{U} = \frac{1}{r}\begin{pmatrix} (\nabla x)\cdot v \\ (\nabla y)\cdot v \\ (\nabla z)\cdot v \end{pmatrix}.$$

$$\nabla x = \begin{pmatrix} \frac{\partial x}{\partial x} \\ \frac{\partial x}{\partial y} \\ \frac{\partial x}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly, $\nabla y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ & $\nabla z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

So $\nabla_v \mathcal{U} = \frac{v}{r}$.

So $S_p(v) = -\frac{v}{r}$.

In other words, as a matrix in local tangent space coordinates, $S_p = -\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$, multiplication by $-\frac{1}{r}$.

__Lemma__ If α is a curve in $M \subset \mathbb{R}^3$, then

$$\alpha'' \cdot \mathcal{U} = S(\alpha') \cdot \alpha'$$

Here $\mathcal{U}$ means the unit normal vector field on M at the point $\alpha(t)$ & S means $S_{\alpha(t)}$.

(As usual α means $\alpha(t)$ and differentiation is wrt t.)

__Interpretation:__

$\alpha'' \cdot \mathcal{U}$ is the acceleration of the curve normal to the surface.

$S(\alpha') \cdot \alpha'$ consists of Coriolis & centripetal terms.

So the shape operator can be used to compute the generalized forces required to maintain contact with M (perhaps these are simply internal constraint forces).

__Pf__ α is a curve in M, so $\alpha' \cdot \mathcal{U} = 0$.
So $\alpha'' \cdot \mathcal{U} + \alpha' \cdot \mathcal{U}' = 0$
$\mathcal{U}'$ means $\frac{d}{dt}\mathcal{U}(\alpha(t))$ which is $\nabla_{\alpha'(t)} \mathcal{U}$

So $\alpha'' \cdot \mathcal{U} = -\alpha' \cdot \nabla_{\alpha'} \mathcal{U} = \alpha' \cdot S(\alpha')$.

A curve in a surface is called a __geodesic__ if its acceleration α'' is always normal to M.

With that fact in mind, let's consider some curvatures:

Def Let $u \in T_p(M)$ be a unit vector, for some $p \in M$.
(Caution: u is a unit tangent vector, not to be confused with $\mathcal{U}$.)

Define $k(u) = S(u) \cdot u$.

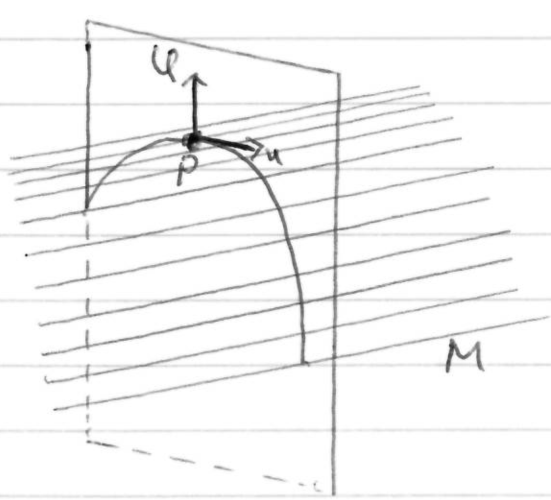
(Again, really we should write k_p & S_p. That is implicit.)

This is a number. It is called the normal curvature of M at p in the direction u.

Intuition:

If we intersect M with a plane at p spanned by $\mathcal{U}$ & u, then the resulting curve has curvature (in the curve sense) that is either $k(u)$ or $-k(u)$, depending on which unit normal vector field we chose for $\mathcal{U}$.

If $k(u) > 0$, then in the u direction M is bending toward $\mathcal{U}$.
If $k(u) < 0$, then in the u direction M is bending away from $\mathcal{U}$.



(as drawn, $k(u) < 0$)

<u>Def</u> Let $p \in M$.

The minimum & maximum values of $k(u)$ as u varies over $T_p(M)$ (Note: u is a unit vector.) are called the <u>principal curvatures of M at p</u>, denoted by k_1 & k_2.

The directions u at which these extreme values occur are called <u>principal directions of curvature</u>

<u>Th^m</u> The principal curvatures of M at p are eigenvalues of S_p whose eigenvectors are the principal directions of curvature.

<u>Corollary</u> When $k_1 \neq k_2$, then the principal directions are orthogonal (bear in mind that S_p is symmetric). (If $k_1 = k_2$, we can of course find two orthogonal principal directions as well. See comment below.)

<u>Comment:</u> When $k_1 = k_2$, then locally (at p), M looks like a sphere of radius $\frac{1}{k_1}$. All tangent directions are principal directions.

Proof of theorem

In tangent space coordinates, we can think of $k(u)$ as a function $k(\theta)$, with θ giving the tangent vector $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$, $\theta \in [0, 2\pi]$.

Since $[0,2\pi]$ is compact, $k(\theta)$ will have a maximum & minimum value.

Sp in local coordinates

Write $k(\theta) = (\cos\theta \ \ \sin\theta)\begin{pmatrix} a & b \\ b & d\end{pmatrix}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$

$$= a\cos^2\theta + 2b\cos\theta\sin\theta + d\sin^2\theta$$
$$= (a-d)\cos^2\theta + b\sin 2\theta + d$$

So $k'(\theta) = -2(a-d)\cos\theta\sin\theta + 2b\cos 2\theta$

Suppose we choose our angular coordinate system so that $k(\theta)$ is a max at $\theta=0$. Then $k'(0)=0$, meaning $b=0$.

So then $S_p = \begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}$ & $k(\theta) = a\cos^2\theta + d\sin^2\theta$.

We see that $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$ are eigenvectors of S_p. These occur at $\theta=0$ & $\theta=\frac{\pi}{2}$, so are orthogonal. Moreover, $k_1 = a$ & $k_2 = d$, as one sees from the formula for $k(\theta)$. Perhaps easiest to see as $k(\theta) = (a-d)\cos^2\theta + d$, noting that $a-d \geq 0$ since max at $\theta=0$.

//

Def Let M be a surface in $\mathbb{R}^3$.

The <u>Gaussian curvature</u> of M is the real valued function $K(p) = \det S_p$.

The <u>mean curvature</u> is the function

$$H(p) = \tfrac{1}{2}\,\mathrm{trace}(S_p)$$

sum of diagonal elements.

Lemma $K = k_1 k_2$

$$H = \frac{k_1 + k_2}{2}$$

A surface is called <u>flat</u> if $K=0$ everywhere. It is called <u>minimal</u> if $H=0$ everywhere.

this excludes any surface for which $K>0$.

Proof In a basis given by principal directions,

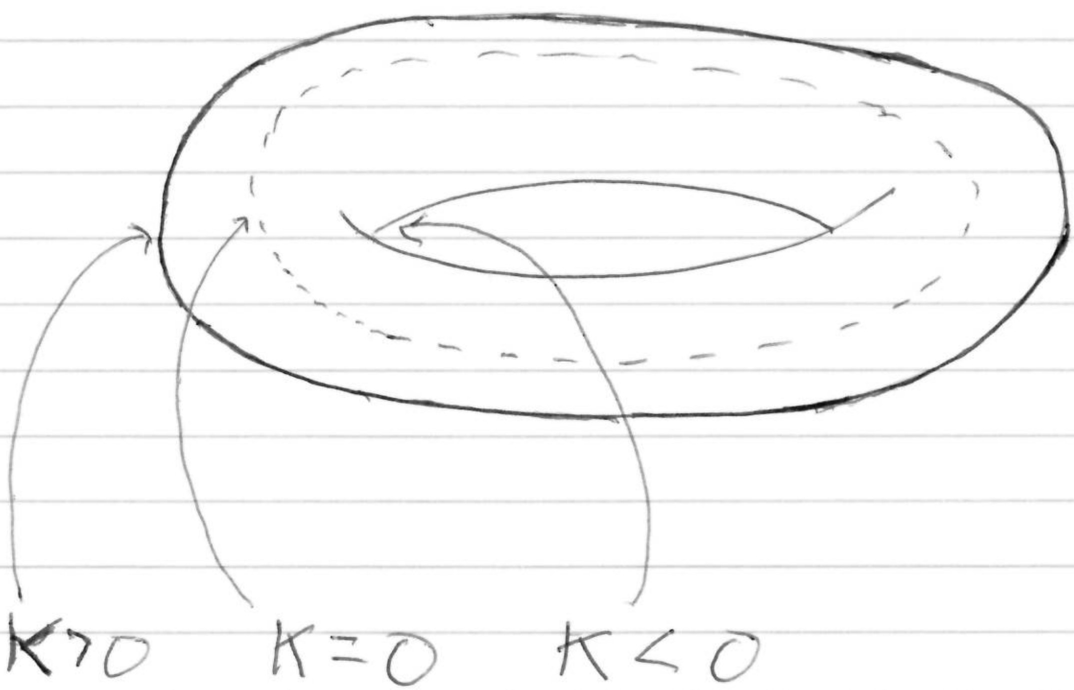
$$S_p = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Note: K is independent of the direction of U chosen to define S_p.

The sign of K tells us a lot about the surface at p:

$K(p) > 0$	$K(p) < 0$	$K(p) = 0$ ("flat") locally: only one of k_1 & k_2 is 0 (say $k_1 \neq 0$)	$K(p) = 0$ ("flat") locally: $k_1 = k_2 = 0$
locally parabolic surface bends in same way for both principal directions	locally saddle-like "opposite bendings"	locally cylindrical (but need higher order data in k_2 direction)	need higher order information (but locally like a plane to 2nd order

Ex The torus exhibits all three signs for the Gaussian curvature:



Taylor Expansion

Suppose we describe a surface around the origin (p) by $z = f(x,y)$, with $T_p(M)$ the xy plane.

Taylor says:

$$f(x,y) \approx f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right)$$

$$= \frac{1}{2}\left((x,y) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

If we further arrange the coordinates so that the x & y axes are the principal directions of curvature, then $f_{xy} = 0$
(very similar to proof on p. 33.1)

$\therefore$

$$f(x,y) = \frac{1}{2} (x,y) \begin{pmatrix} f_{xx} & 0 \\ 0 & f_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

i.e., $f(x,y) = \frac{1}{2}\left(k_1 x^2 + k_2 y^2 \right)$

with $k_1 = f_{xx}$ & $k_2 = f_{yy}$
(either of these could be min or max).

The surface $z = \frac{1}{2}\left(k_1 x^2 + k_2 y^2\right)$ is called the <u>quadratic approximation of M at p</u>.

<u>Def</u> Let M be an orientable surface in $\mathbb{R}^3$ with a unit normal vector field $\mathcal{U}$.

The <u>Gauss map</u> is the function

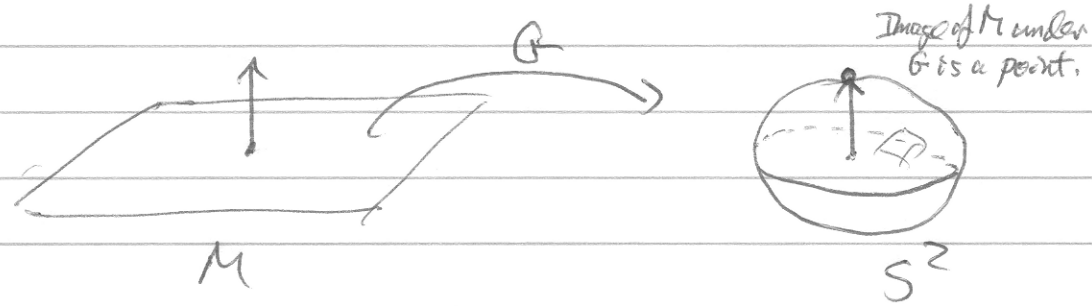
$$\begin{aligned} G: M &\longrightarrow S^2 \\ p &\longmapsto \mathcal{U}(p) \end{aligned}$$

<u>Examples:</u>

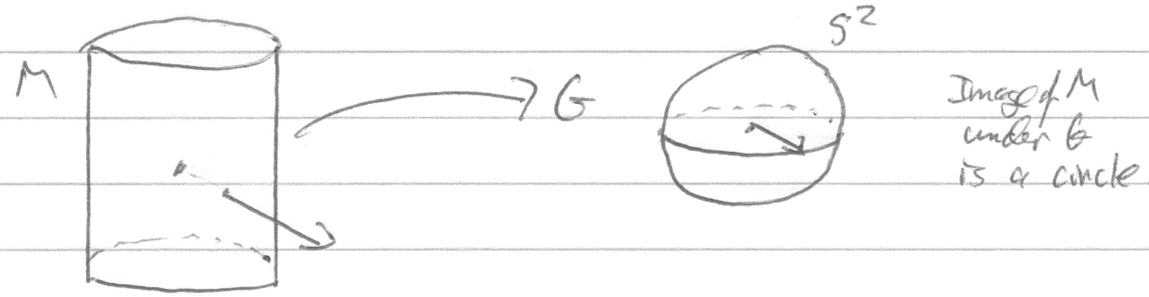
① M is xy plane, with unit normal $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then G is the constant function $G(x,y) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

I.e.,



② Suppose M is the cylinder $x^2+y^2=r^2$ (some $r>0$).



$$(r\cos\theta, r\sin\theta, z) \longmapsto (\cos\theta, \sin\theta, 0)$$

(3) Suppose M is S^2 itself, with outward normal unit vector field U.
Then G is the identity function.

(3)' Suppose M is S^2, now with inward normal unit vector field U.

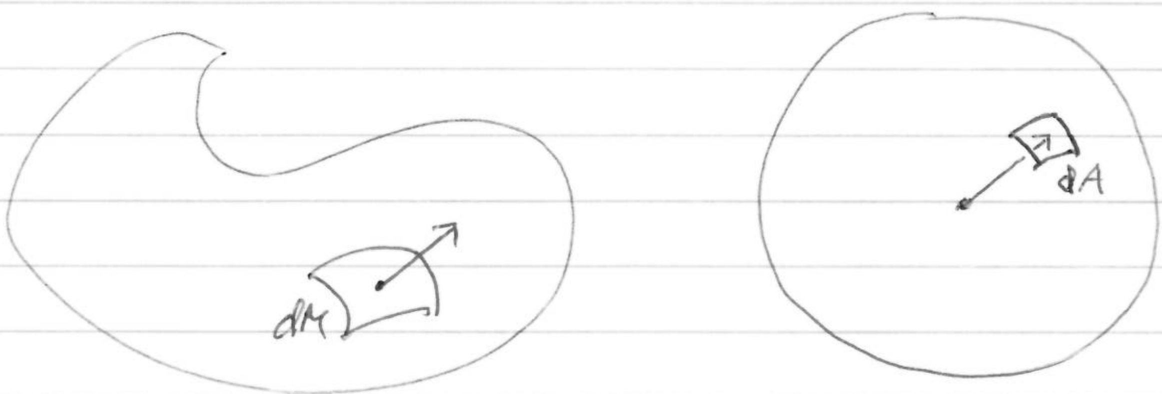
Then $G(p) = -p$.

Theorem

$$K = \det\left(J_G\right)$$

Jacobian of the Gauss map.

(Assumes an orientable surface with a unit normal vector field, smoothly varying.)



$$K = \lim_{dM \to 0} \frac{dA}{dM}$$

(Proof involves some explicit computations of how area is transformed; standard stuff for higher-dim calculus.)

<u>Def</u> $\iint_M K\,dM$ is known as the <u>total Gaussian curvature</u> of M.

<u>Theorem (Corollary to the previous theorem)</u>

The total Gaussian curvature of an orientable surface in $\mathbb{R}^3$ is the <u>algebraic area</u> of the image $G(M)$.

This means that one may count some areas in S^2 more than once and possibly with opposite sign, e.g., if the Gauss map is not 1-1 (think again about xoring areas).
[Stokes again.]

There is an even more general theorem:

<u>Gauss-Bonnet</u> Let M be a compact orientable surface in $\mathbb{R}^3$.
(well, a corollary to an even more general theorem)

$$\iint_M K\,dM = 2\pi\,\chi(M)$$

Euler characteristic of M.

For many years these invariants were part of vision research. (See Horn & Ikeuchi, for instance.)

<u>Example</u> The total Gaussian curvature of any sphere in $\mathbb{R}^3$ is 4π.
So that tells us the area of a sphere of radius r must be $4\pi r^2$. ☺

since $\iint_M K\,dM = \frac{1}{r^2}\iint_M dM = \frac{\text{area}}{r^2} = 2\pi\,\chi(M)$, $\chi(M) = 2$.