Computing LDU decomposition of

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

Loop invariant: $LA' = A$.

(Step)	L	A'	
(0)	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{matrix}$	(Start)
(1)	$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & -2 & 1 \\ 0 & 4 & 1 \\ 0 & 7 & 2 \end{matrix}$	(Clear first column of A below diagonal)
(2)	$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{7}{4} & 1 \end{matrix}$	$\begin{matrix} 1 & -2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{1}{4} \end{matrix}$	(Clear second col below diag)

So $$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{7}{4} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

To solve $Ax = b$ we first solve $Ly = b$, then solve $Ux = D^{-1}y$, both of which are easy (substitution).

Example: $b = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}$

① $Ly = b$ is $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{7}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$.

Can "read off" the solution y:

$$y_1 = 5$$
$$y_2 = 9 - y_1 = 4$$
$$y_3 = 2 - 2y_1 - \tfrac{7}{4}y_2 = -15$$

So $y = \begin{bmatrix} 5 \\ 4 \\ -15 \end{bmatrix}$

(2) $\mathcal{U}x = D^{-1}y$ becomes

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = D^{-1} \begin{bmatrix} 5 \\ 4 \\ -15 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -60 \end{bmatrix}$$

Now read off solution x from bottom to top:

$$x_3 = -60$$
$$x_2 = 1 - \tfrac{1}{4}x_3 = 16$$
$$x_1 = 5 - x_3 + 2x_2 = 97$$

So $x = \begin{pmatrix} 97 \\ 16 \\ -60 \end{pmatrix}$.

Comment

Ask yourself how you might compute A^{-1} using this idea.

(If you do that, you should find

$$A^{-1} = \begin{pmatrix} 2 & 11 & -6 \\ 0 & 2 & -1 \\ -1 & -7 & 4 \end{pmatrix}.)$$

Computing $PA = LDU$ for $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

Loop invariant: $LA' = PA$.

(step)	L	A'	P	
(0)	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	(start)
(1)	$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 3 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	(Clear first column of A below diagonal)
(2)	$\begin{matrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 2 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$	(swap)

2nd diagonal element is 0 so can't use it to clear below. Instead swap rows 2&3.

If the matrix was larger we might have to clear more entries below diagonal in columns 2, 3, …, but here we are done.

$$\text{So} \quad \underset{P}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} A = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \underset{U}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}}$$

Computing $PA = LDU$ for $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

Loop invariant: $LA' = PA$.

(step)	L	A'	P	
(0)	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	(Start)
(1)	$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 3 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	(Clear first column of A below diagonal)
(2)	$\begin{matrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 2 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$	(swap)

2nd diagonal element is 0 so can't use it to clear below. Instead swap rows 2 & 3.

Notice that we only swap subpart of L that we've already worked on, to preserve loop invariant.

If the matrix was larger we might have to clear more entries below diagonal in columns 2, 3, ..., but here we are done.

So

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P} A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$