16-350
Planning Techniques for Robotics

Search Algorithms:
A Search*

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Uninformed A* Search

- Computes g*-values for **relevant** (not all) states

Main function

$g(s_{start}) = 0;$ all other g-values are infinite; $OPEN = \{s_{start}\};$

ComputePath();

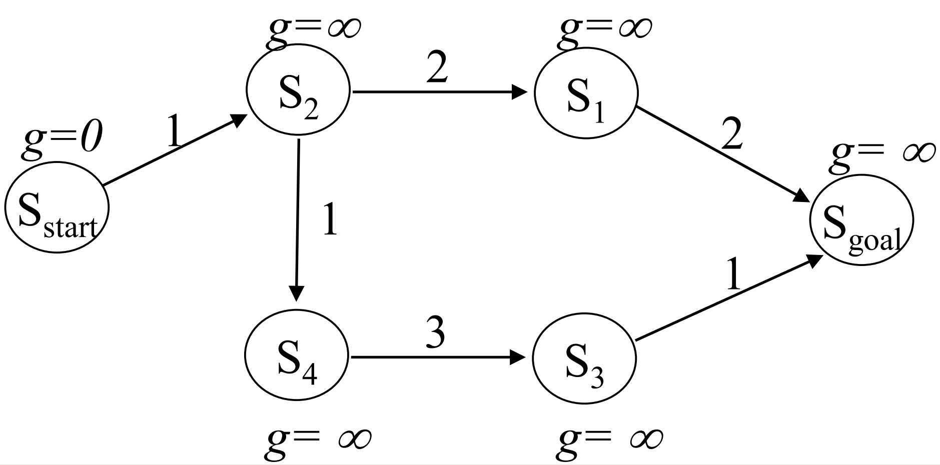
publish solution; //compute least-cost path using g-values

ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

  remove s with the smallest $g(s)$ from $OPEN$;

  expand s;



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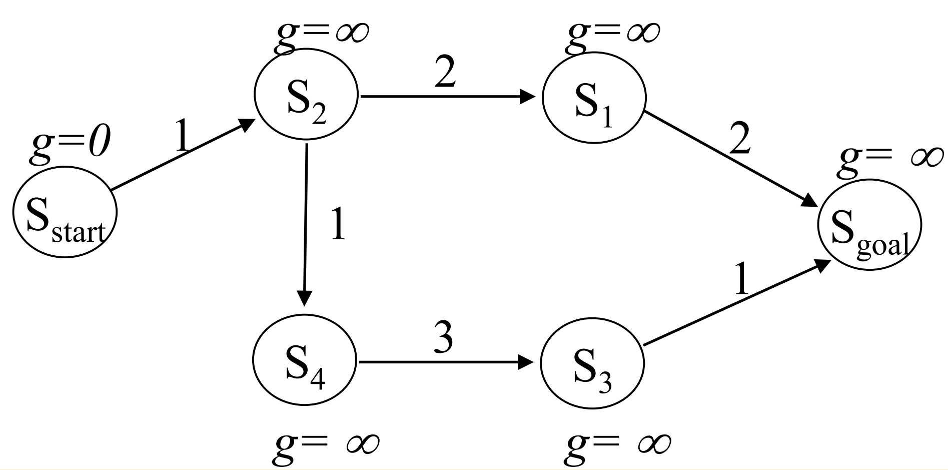

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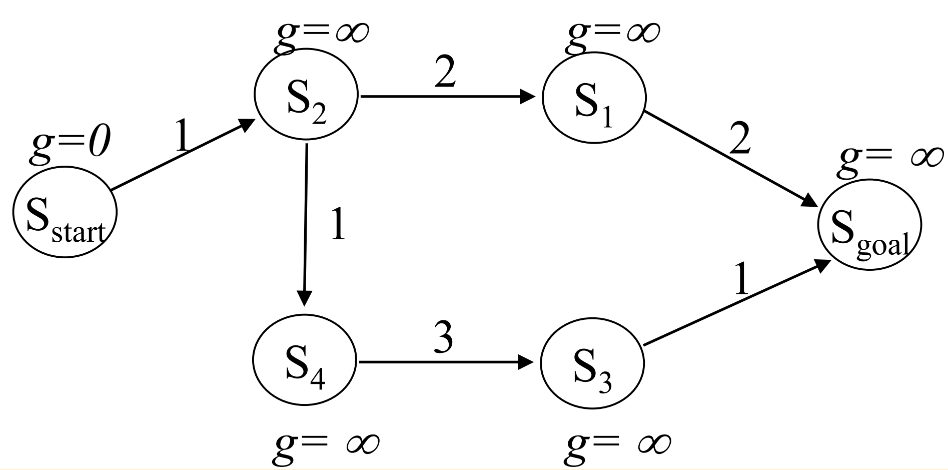
clarification: *updates g(s') if s' is already in OPEN*



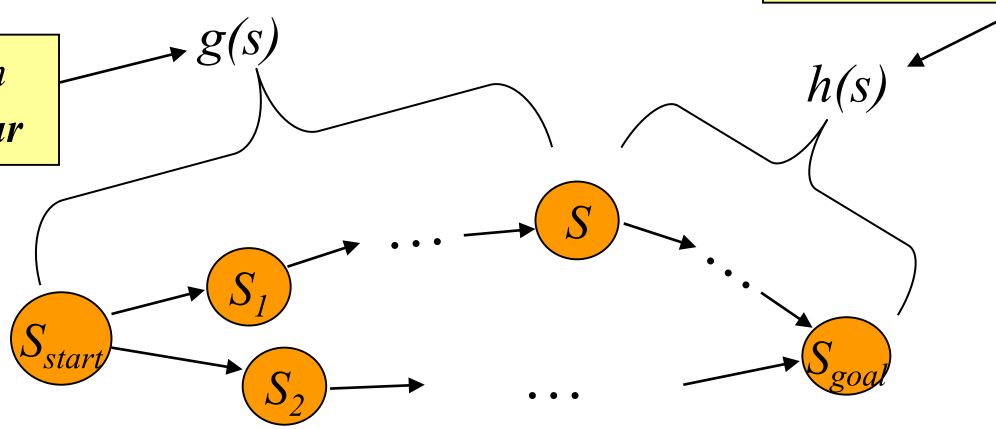
A* Search [Hart, Nillson, Raphael, ‘68]

- Computes optimal g-values for relevant states

at any point of time:

the cost of a shortest path from s_{start} to s ***found so far***

an (under) estimate of the cost of a shortest path from s to s_{goal}



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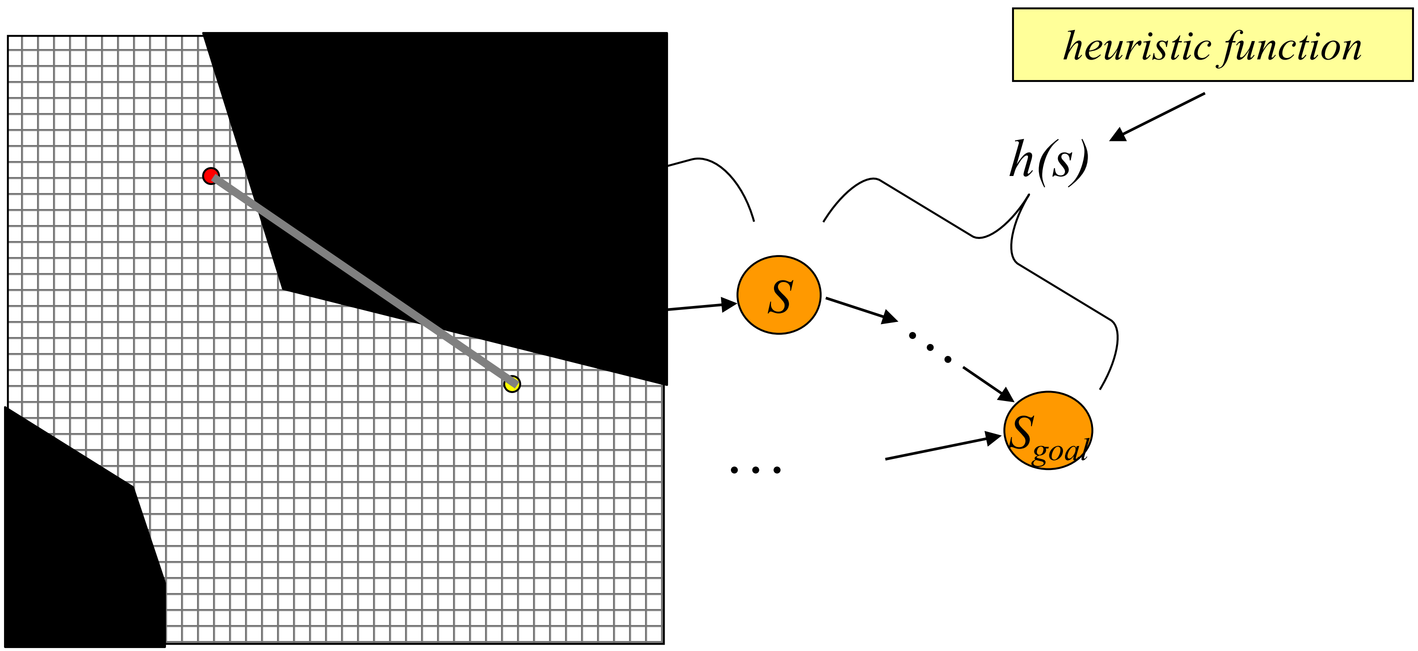


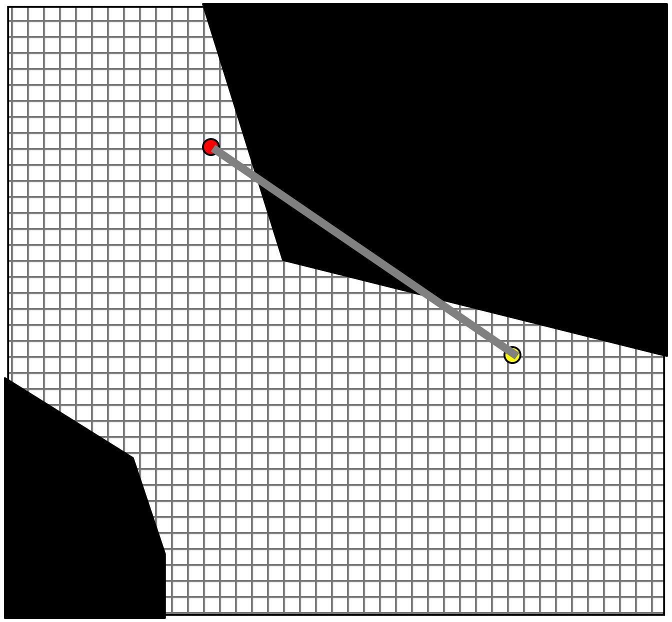
one popular heuristic function – Euclidean distance

A* Search [Hart, Nillson, Raphael, '68]

minimal cost from s to s_{goal}

- Heuristic function must be:
  - admissible: for every state s, $h(s) \leq c^*(s, s_{goal})$
  - consistent (satisfy triangle inequality):

    $h(s_{goal}, s_{goal}) = 0$ and for every $s \neq s_{goal}$, $h(s) \leq c(s, succ(s)) + h(succ(s))$
  - admissibility <u>provably</u> follows from consistency and often (<u>not always</u>) consistency follows from admissibility



A* Search [Hart, Nillson, Raphael, '68]

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  - consistent (satisfy triangle inequality). *Why triangle inequality?*

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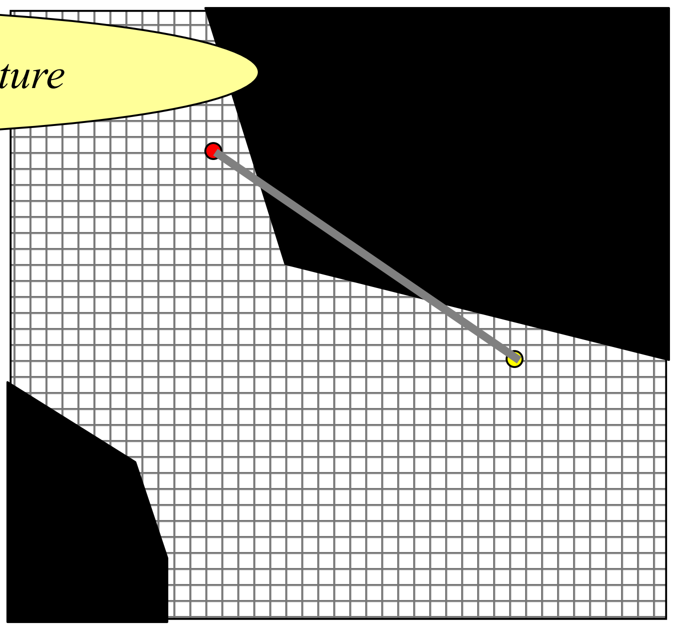
More on this in the later lecture



A*: Uninformed vs. Informed Search

- A*: expands states in the order of $\boldsymbol{f} = \boldsymbol{g}+\boldsymbol{h}$ values
- Uninformed A*: expands states in the order of $\boldsymbol{g}$ values

A* Search

- Computes optimal g-values for relevant states

Main function

$g(s_{start}) = 0;$ all other g-values are infinite; $OPEN = \{s_{start}\};$

ComputePath();

publish solution;

ComputePath function

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  remove s with the smallest ***[f(s) = g(s)+h(s)]*** from *OPEN*;

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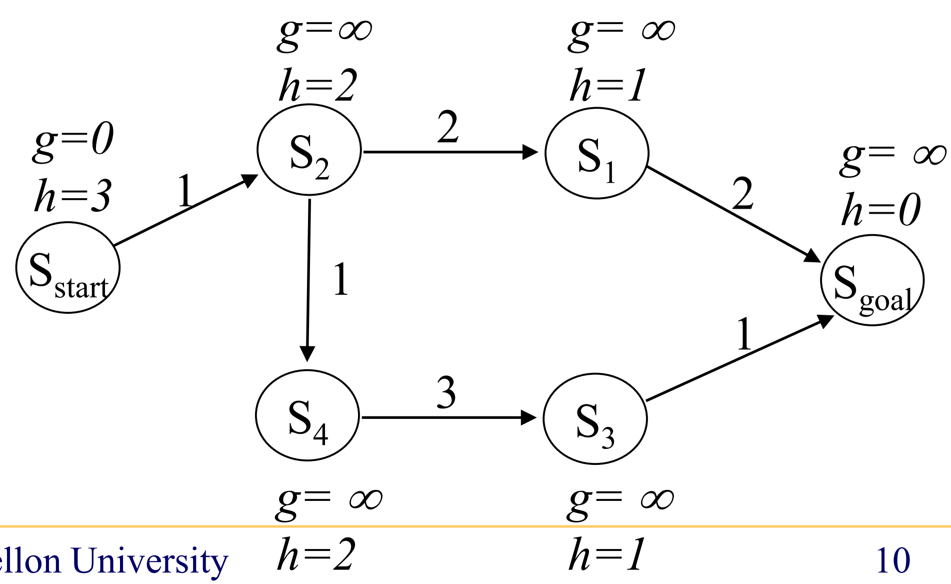

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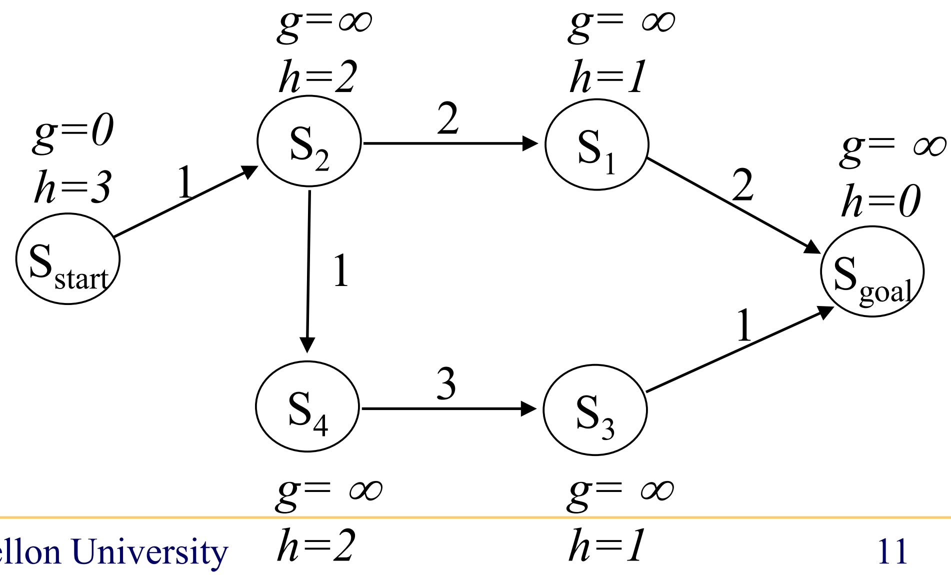

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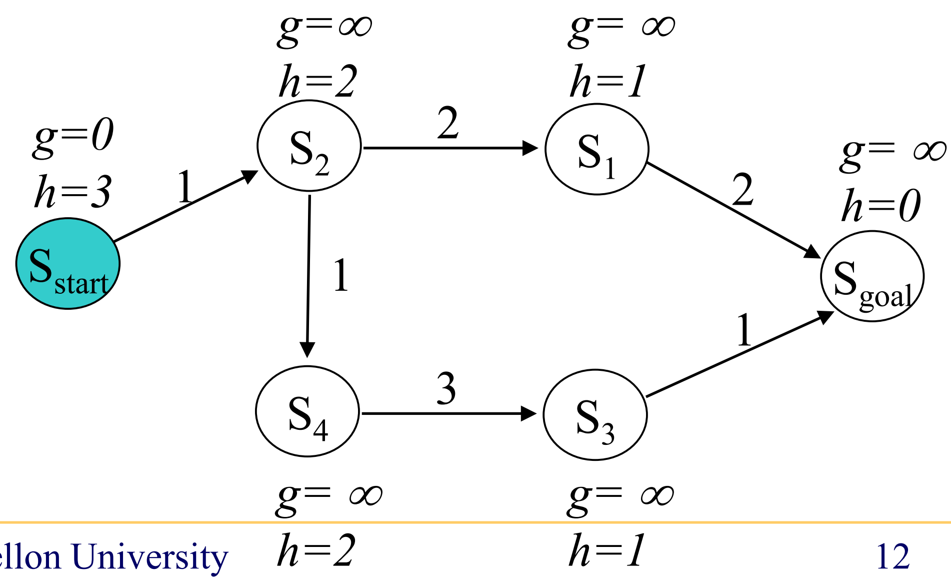
insert s' into $OPEN$;

$CLOSED = \{\}$

$OPEN = \{s_{start}\}$

next state to expand: s_{start}



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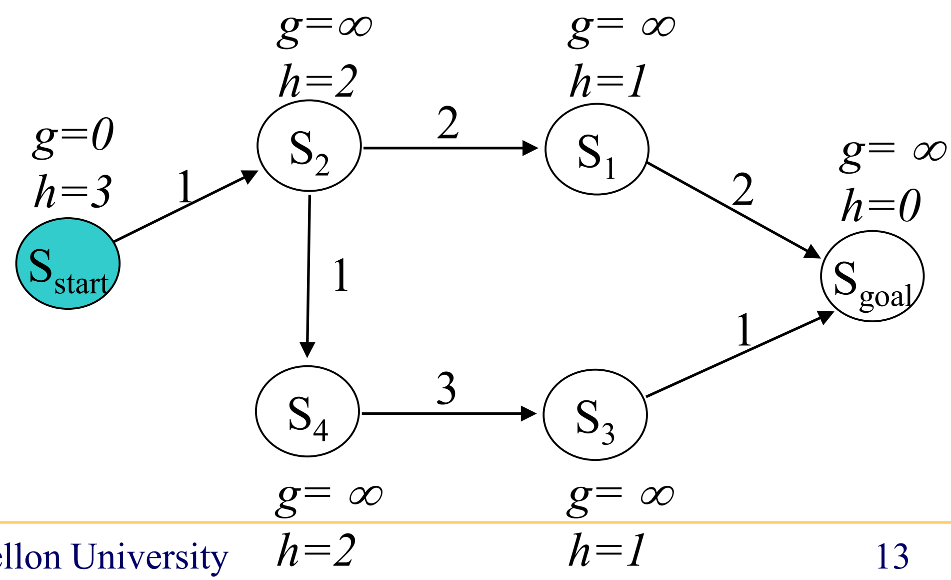
if $g(s') > g(s) + c(s,s')$

$g(s') = g(s) + c(s,s')$;

insert s' into $OPEN$;

$g(s_2) > g(s_{start}) + c(s_{start}, s_2)$



$CLOSED = \{\}$

$OPEN = \{s_{start}\}$

next state to expand: s_{start}

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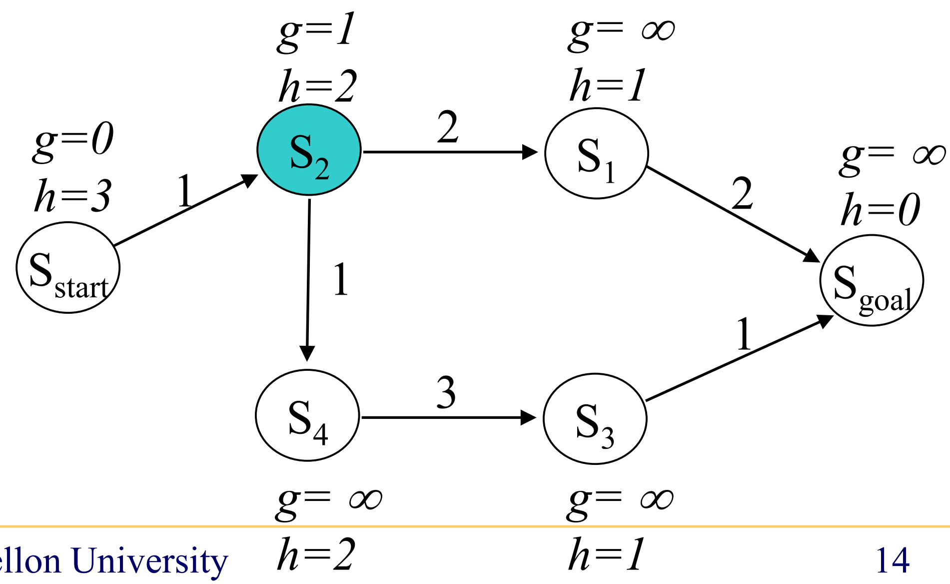

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if $g(s') > g(s) + c(s,s')$

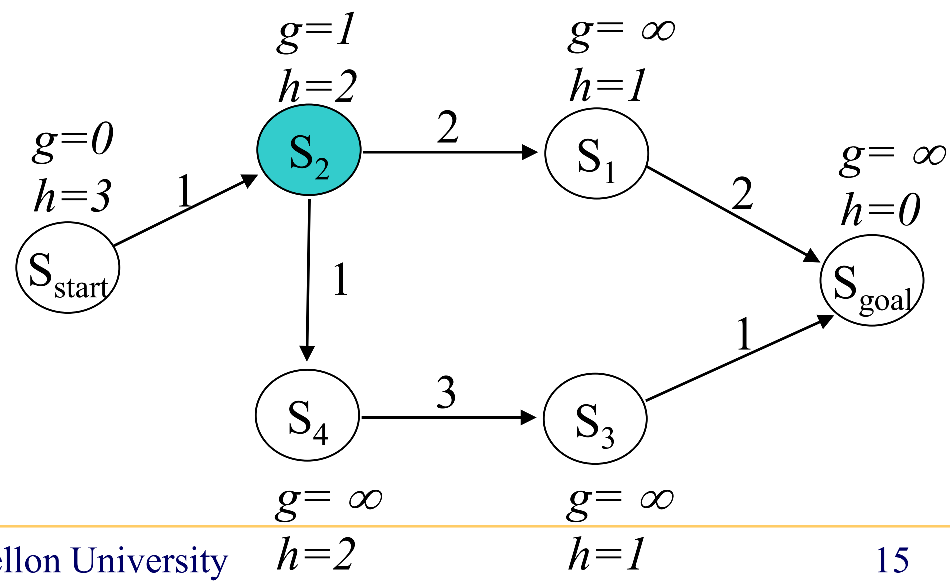
$g(s') = g(s) + c(s,s')$;

insert s' into $OPEN$;

$CLOSED = \{s_{start}\}$

$OPEN = \{s_2\}$

next state to expand: s_2



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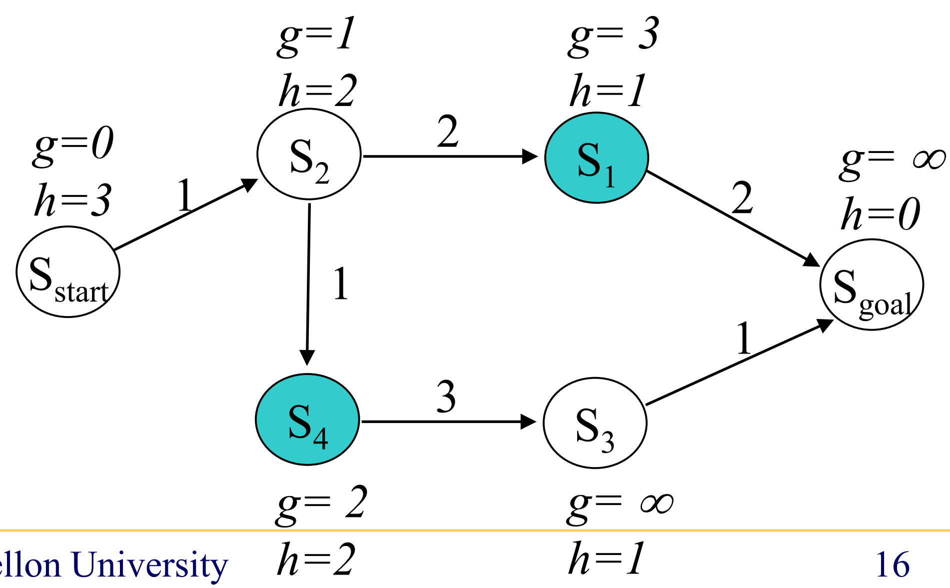
      $g(s') = g(s) + c(s,s')$;

      insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2\}$

$OPEN = \{s_1, s_4\}$

next state to expand: s_1



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if $g(s') > g(s) + c(s,s')$

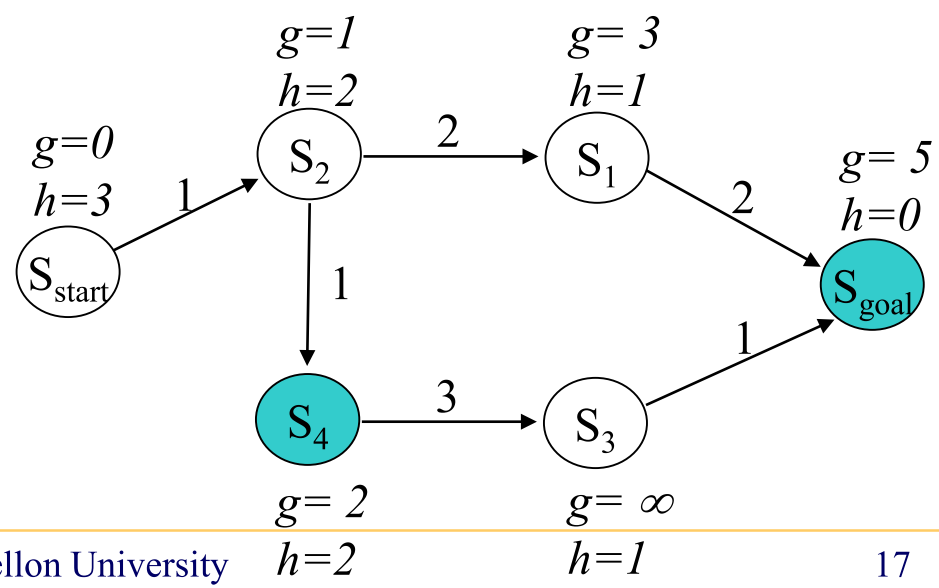
$g(s') = g(s) + c(s,s');$

insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2, s_1\}$

$OPEN = \{s_4, s_{goal}\}$

next state to expand: s_4



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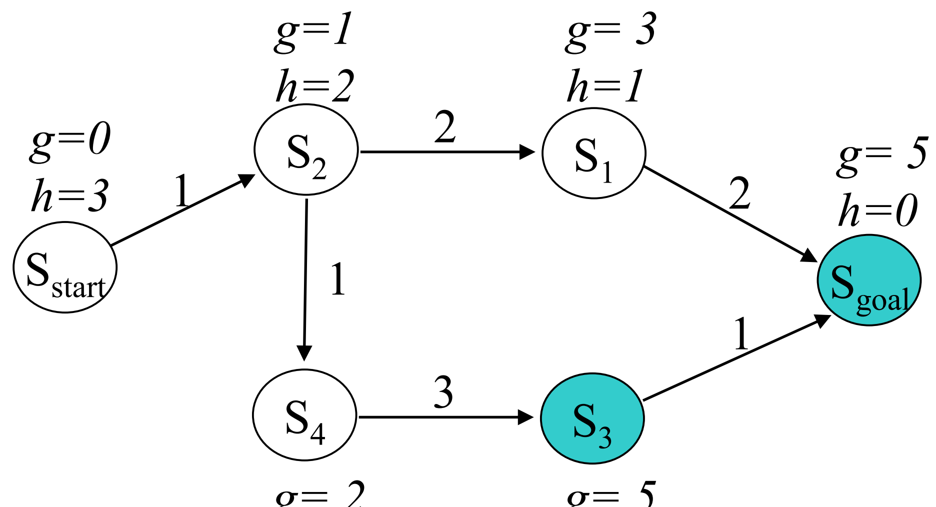
insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2, s_1, s_4\}$

$OPEN = \{s_3, s_{goal}\}$

next state to expand: s_{goal}



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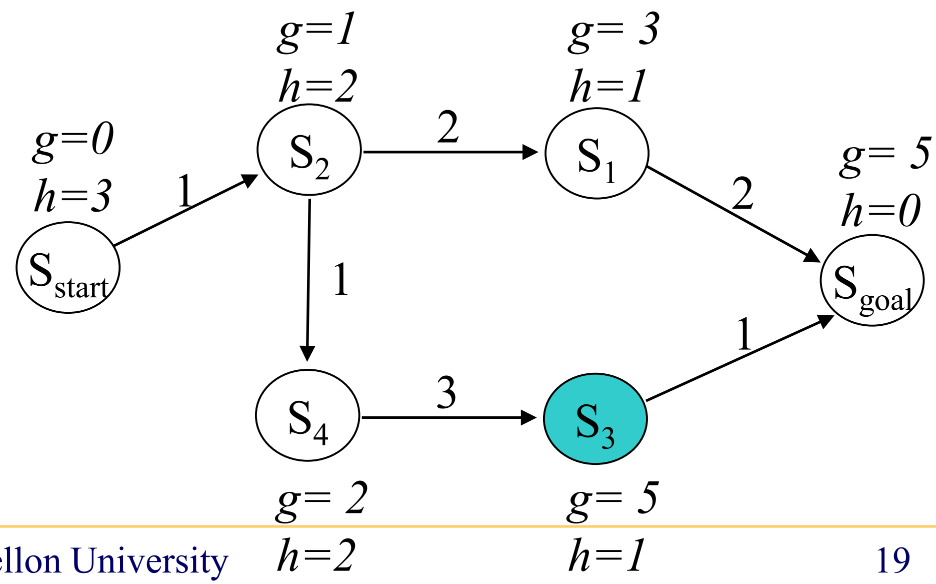
$CLOSED = \{s_{start}, s_2, s_1, s_4, s_{goal}\}$

$OPEN = \{s_3\}$

done



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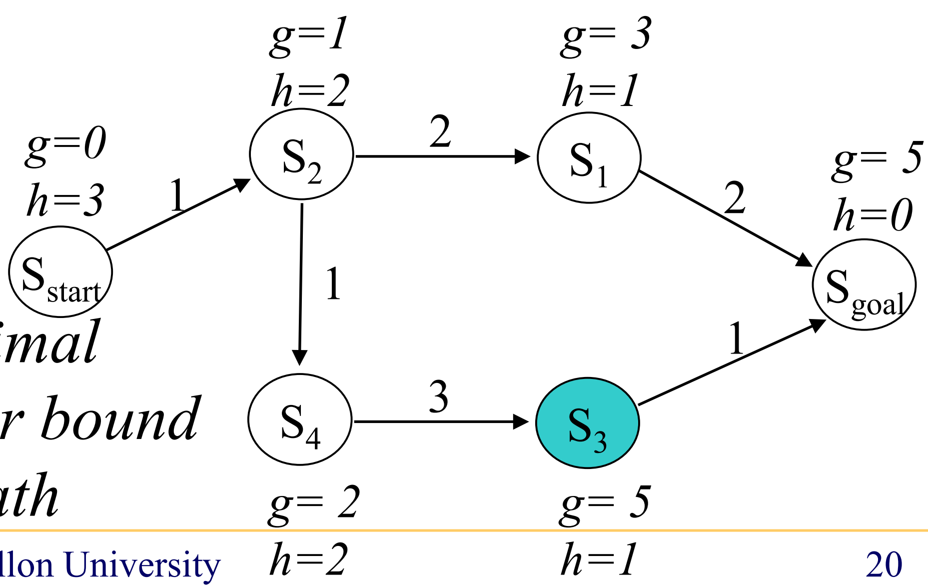

for every expanded state g(s) is optimal

for every other state g(s) is an upper bound

we can now compute a least-cost path

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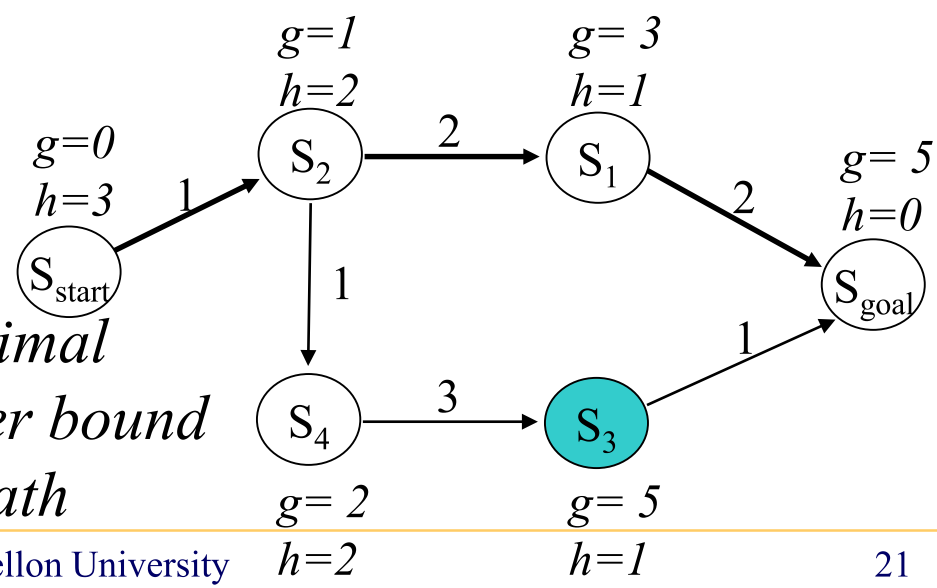

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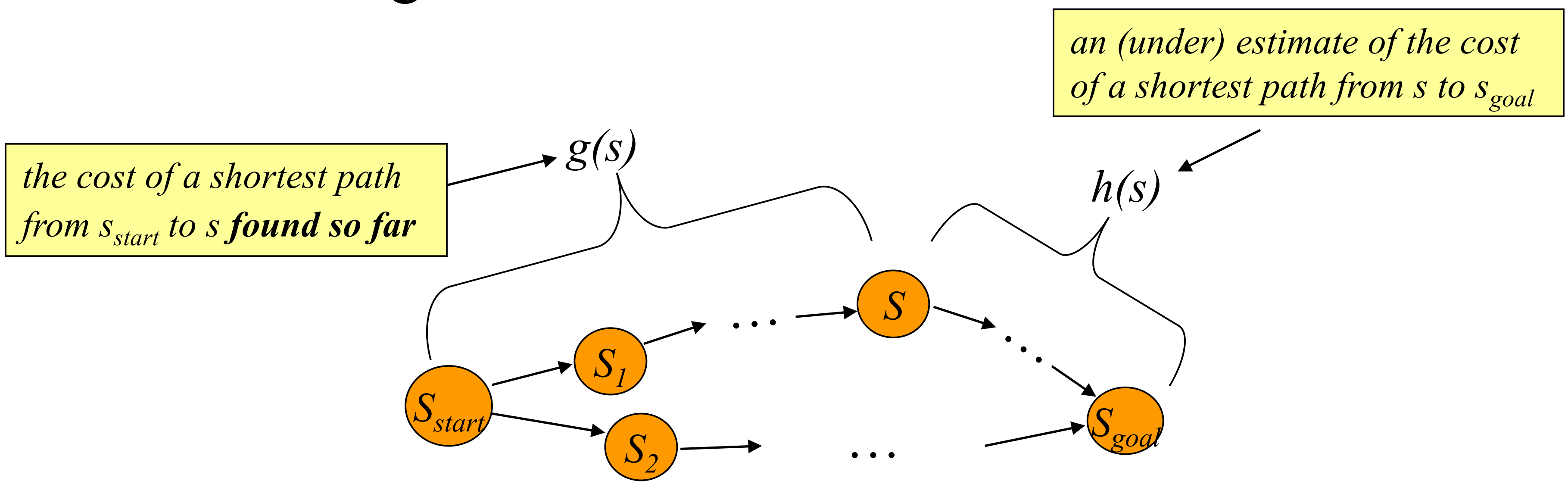
A*: Uninformed vs. Informed Search

- A*: expands states in the order of $\boldsymbol{f} = \boldsymbol{g}+\boldsymbol{h}$ values
- Uninformed A*: expands states in the order of $\boldsymbol{g}$ values

- Intuitively: $f(s)$ – estimate of the cost of a least cost path from start to goal via state s

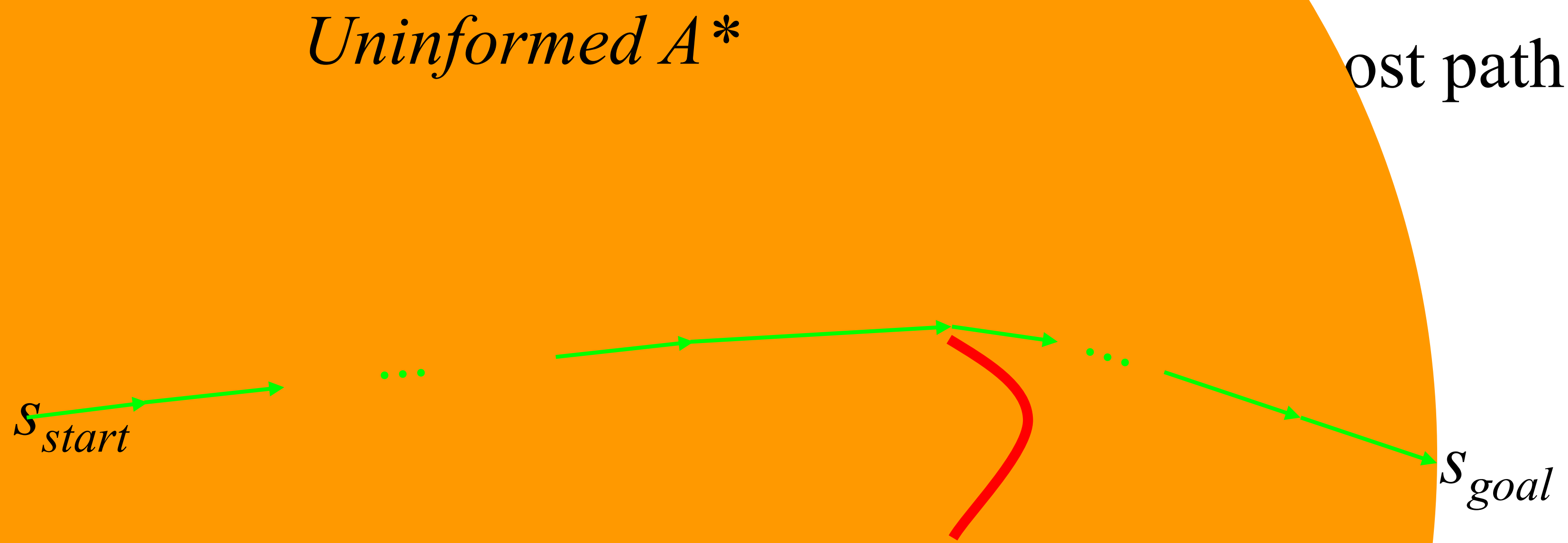


Informed Search

h values

of ***g*** values

ost path



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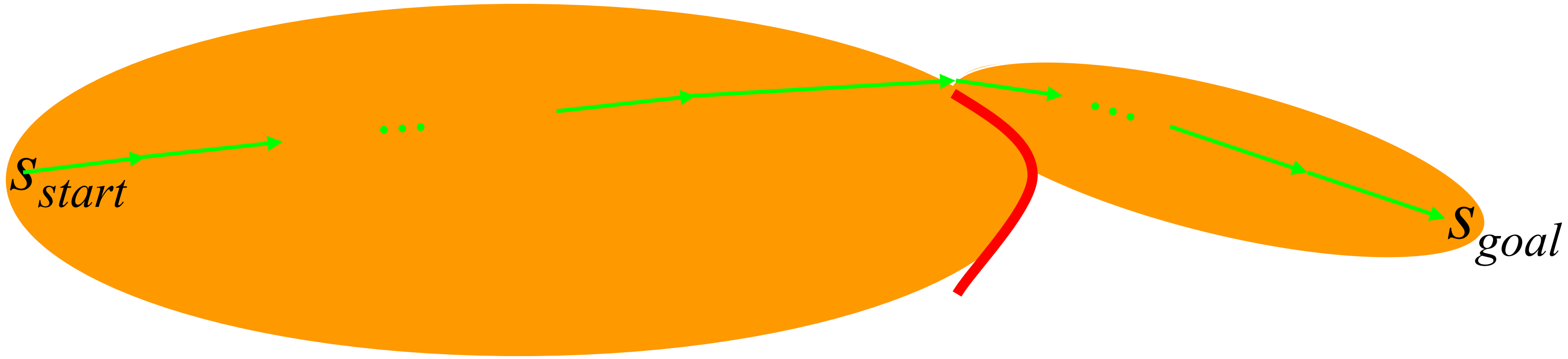


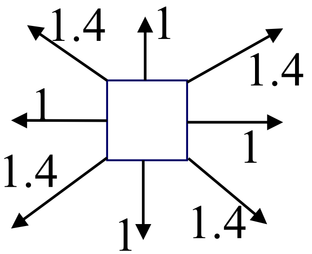
A with Heuristics=Euclidean Distance*

A* Search

- Example on a Grid-based Graph:

$$h(cell <x,y>) = max(|x-x_{goal}|, |y-y_{goal}|)$$

8-connected grid



	A	B	C	D	E	F
1	h=5	h=4	h=3	h=2	h=1	h=1
2	h=5	h=4	h=3	h=2	h=1	h=0
3	h=5	h=4			h=1	h=1
4	h=5	h=4	**h=3**	h=2	h=2	h=2

goal

robot

A* Search: Proofs

Theorem 1. For every expanded state *s*, it is guaranteed that $g(s)=g^*(s)$

Sketch of proof by induction:

- *assume all previously expanded states have optimal g-values*
- *next state to expand is s: $f(s) = g(s)+h(s)$ – min among states in OPEN*
- *assume $g(s)$ is suboptimal (we will prove that it is impossible by contradiction)*
- *then there must be at least one state s' on an optimal path from start to s such that it is in OPEN but wasn't expanded*
- $g(s') + h(s') \geq g(s)+h(s)$
- *but* $g(s') + c^*(s',s) < g(s) \Rightarrow$
- $g(s') + c^*(s',s) + h(s) < g(s) + h(s) \Rightarrow$ *(from consistency of h-values)*
- $g(s') + h(s') < g(s) + h(s) \Rightarrow$ *CONTRADICTION*
- *thus it must be the case that $g(s)$ is optimal*

A* Search: Proofs

Theorem 2. Once the search terminates, it is guaranteed that $g(s_{goal})=g^*(s_{goal})$

Sketch of proof:

Proof?

Theorem 3. Once the search terminates, the least-cost path from s_{start} to s_{goal} can be re-constructed by backtracking *(start with* s_{goal} *and from any state s backtrack to the predecessor state s' such that* $s' = \arg\min_{s'' \in pred(s)}(g(s'') + c(s'', s))$*)*

Sketch of proof:

- *every backtracking step moves to the predecessor state that continues to be on a least-cost path (because all other predecessors have g-values that are strictly larger)*

Theorem 4 (complexity). No state is expanded more than once by A*

Sketch of proof:

Proof?

A* Search: Proofs

Theorem 5. Given a graph and a heuristic function, **A* performs a minimal number of expansions to find a <u>provably</u> optimal path** *(provided goal state is always expanded first among the states with the same f-values in OPEN)*

Implementation Details of A* Search

- Computes optimal g-values for relevant states

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How to implement OPEN?

How to implement CLOSED?

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insert s' into $OPEN$;

How to implement OPEN?

Typically, a priority queue built using a binary heap

How to implement CLOSED?

Typically, each state has a Boolean flag indicating if it was already closed

A* Search with Backpointers

- After search terminates, least-cost path is given by backtracking backpointers from s_{goal} to s_{start}

Main function

$g(s_{start}) = 0;$ all other g-values are infinite; $OPEN = \{s_{start}\};$

set all backpointers bp to NULL;

ComputePath();

publish solution; //**backtrack least-cost path using backpointers *bp***

ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

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if $g(s') > g(s) + c(s,s')$

$g(s') = g(s) + c(s,s')$; **bp(s') = s;**

insert s' into $OPEN$;

Summary

- Uninformed A* is A* with heuristics h=0 for all states

- Heuristics
  – estimate of the cost to the goal
  – have profound effect on the efficiency of the algorithm
  – can be easily derived for many planning problems

- A* is an optimal algorithm:
  – Performs a minimal number of expansions to find a provably optimal path