16-782
Planning & Decision-making in Robotics

Planning under Uncertainty: Expected Formulation, Solving MDPs

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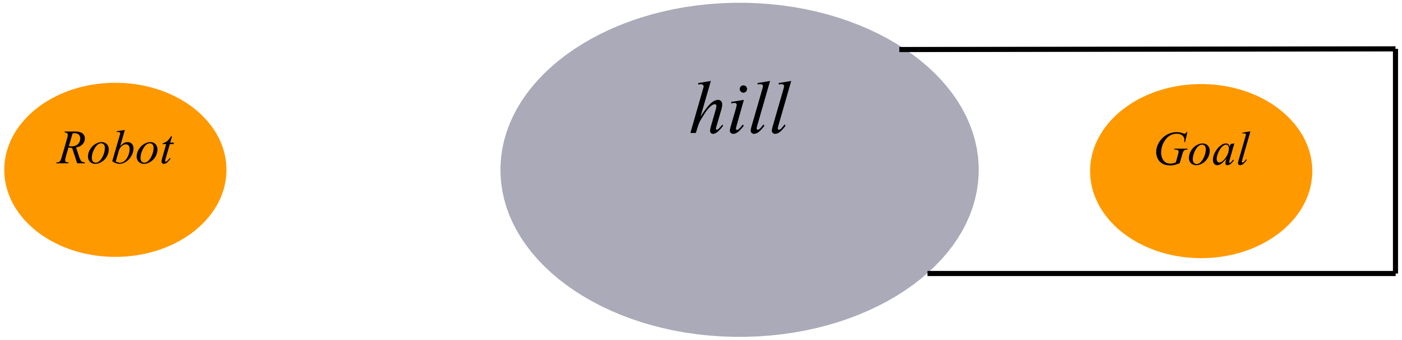
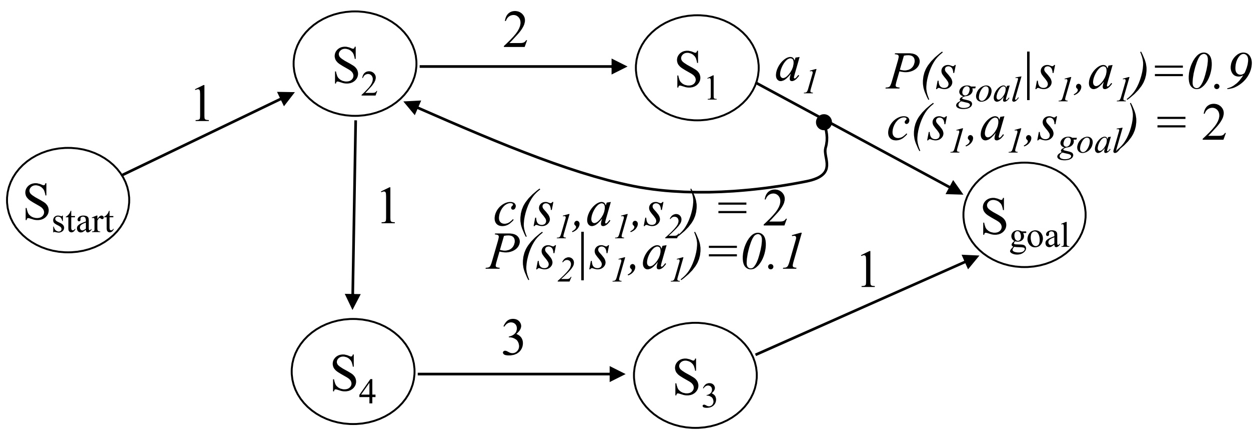
Minimax Formulation is Often Too Conservative

Example:

moving over the hill has 10% chance of slipping



Expected Cost Formulation



- Optimal policy π^*:

  minimizes the *expected* cost-to-goal

  $\pi^* = argmin_\pi E\{cost\text{-}to\text{-}goal\}$

expectation over outcomes

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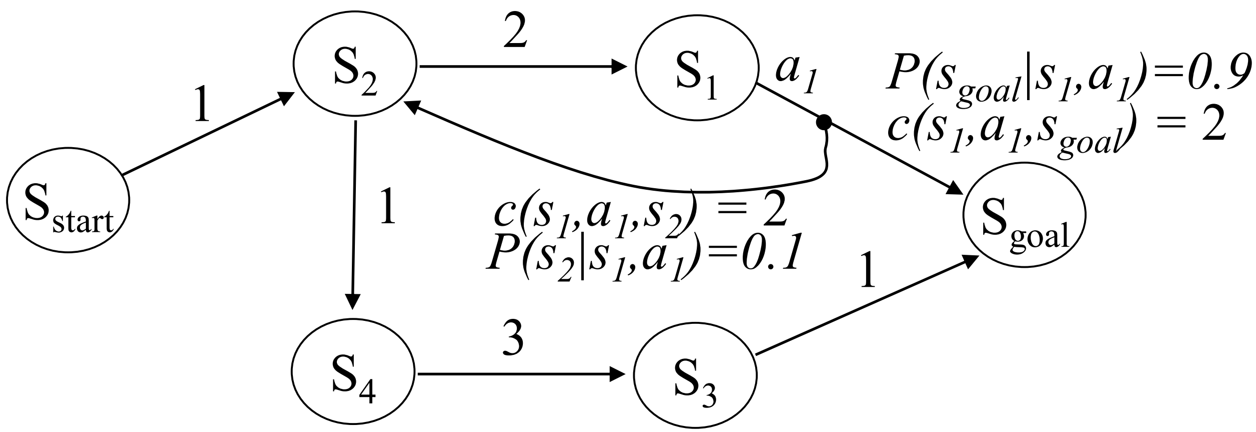

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expectation over outcomes

- expected cost-to-goal for π_1=(go through s_4) is

  1+1+3+1=6

- cost-to-goal for π_2=(try to go through s_1) is:

0.9*(1+2+2) + 0.9*0.1*(1+2+2+2+2) + 0.9*0.1*0.1*(1+2+2+2+2+2+2) + …=$5.44\overline{4}$

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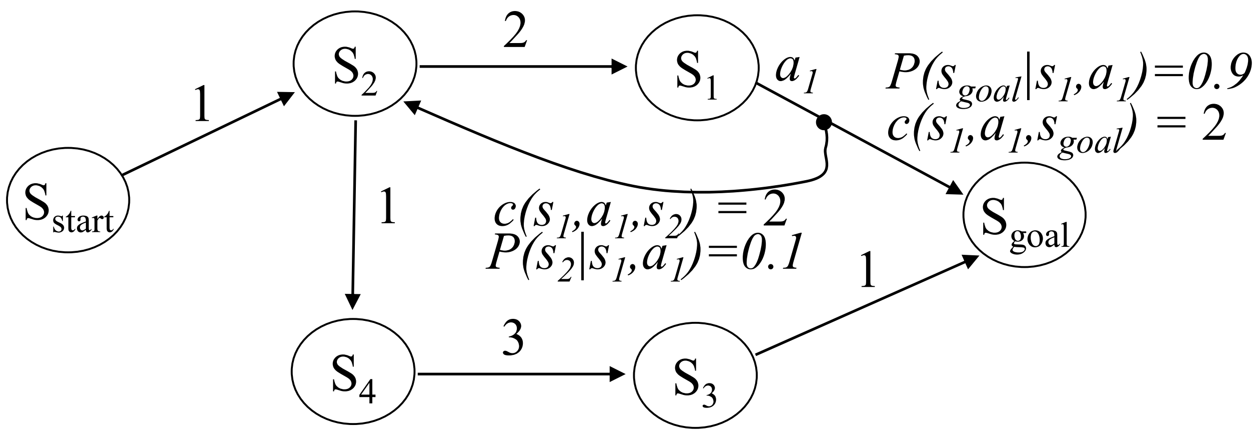

- Optimal policy π*:
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Given a policy, its value can be computed by solving a system of linear equations

expectation over outcome

How to compute it?

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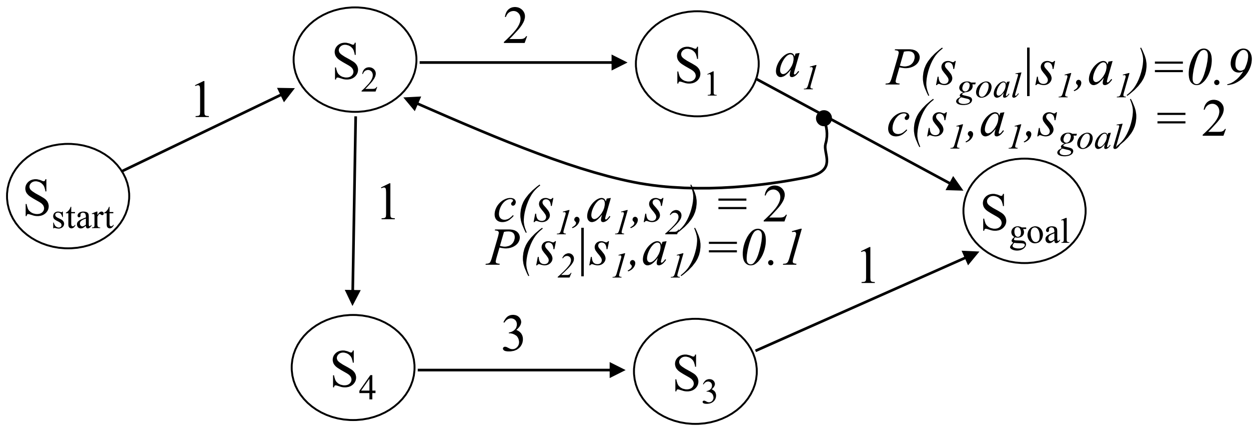

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for π_2:

$v(s_{start}) = 1+v(s_2)$

$v(s_2) = 2+v(s_1)$

$v(s_1) = 0.9*(2+v(s_{goal}))+0.1*(2+v(s_2))$

$v(s_{goal}) = 0$

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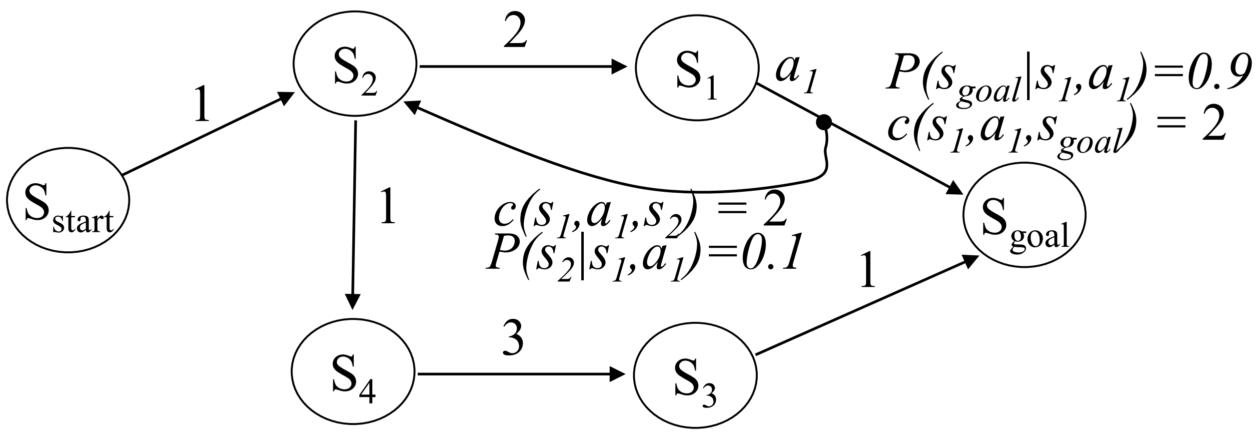

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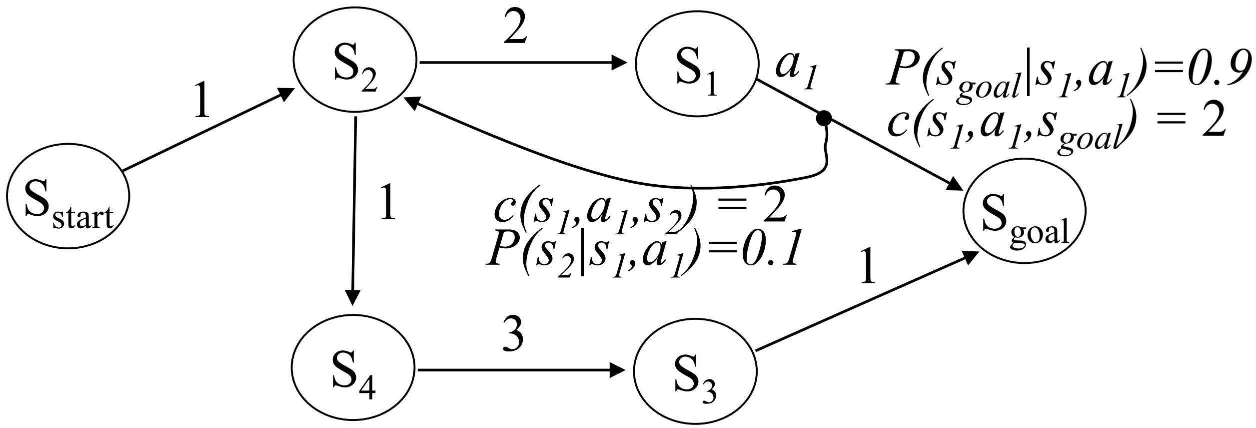

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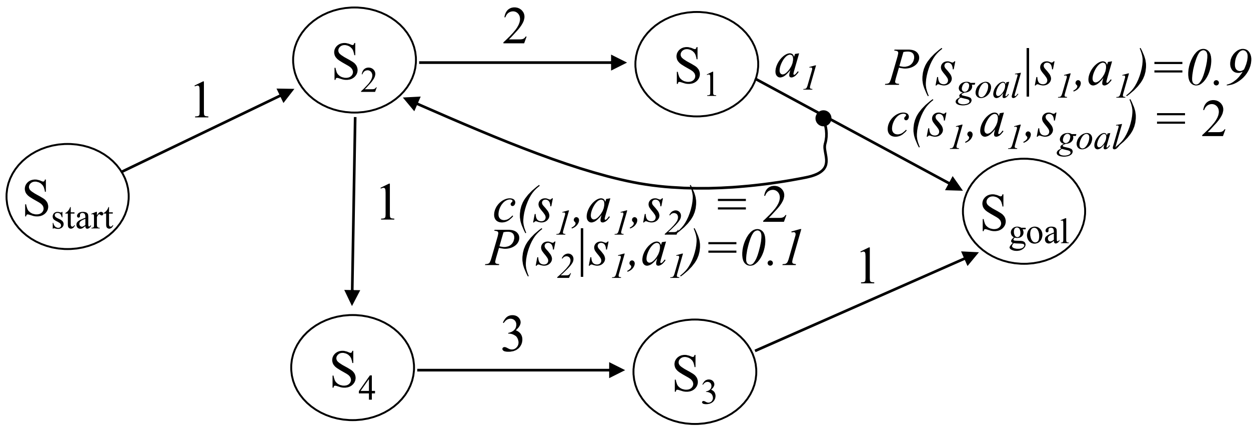
In contrast,
optimal policy for minimax formulation
was π_1=(go through s_4)

Computing Expected Cost Minimal Plans



- Optimal policy π^*:
  minimizes the *expected* cost-to-goal
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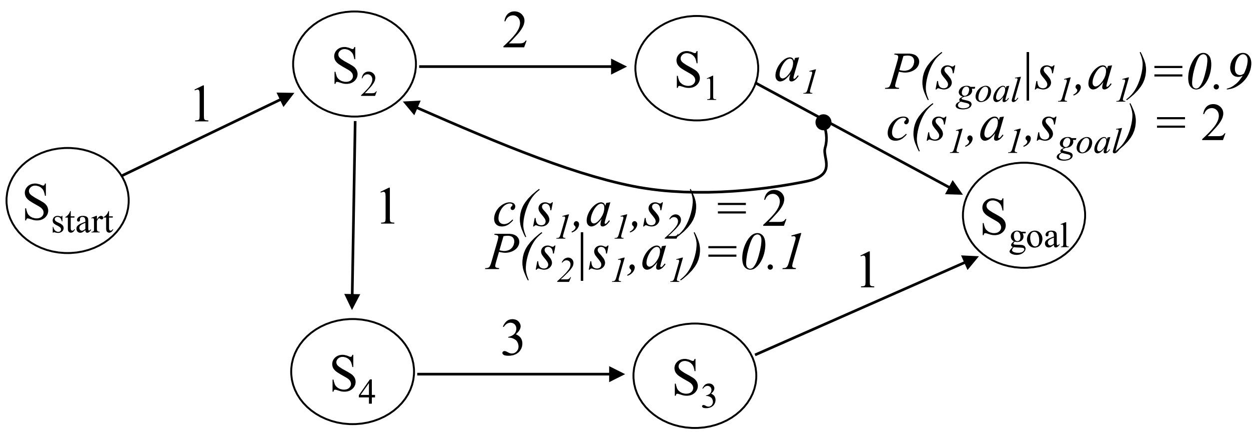
- Let $v^*(s)$ be minimal expected cost-to-goal for state s

Computing Expected Cost Minimal Plans



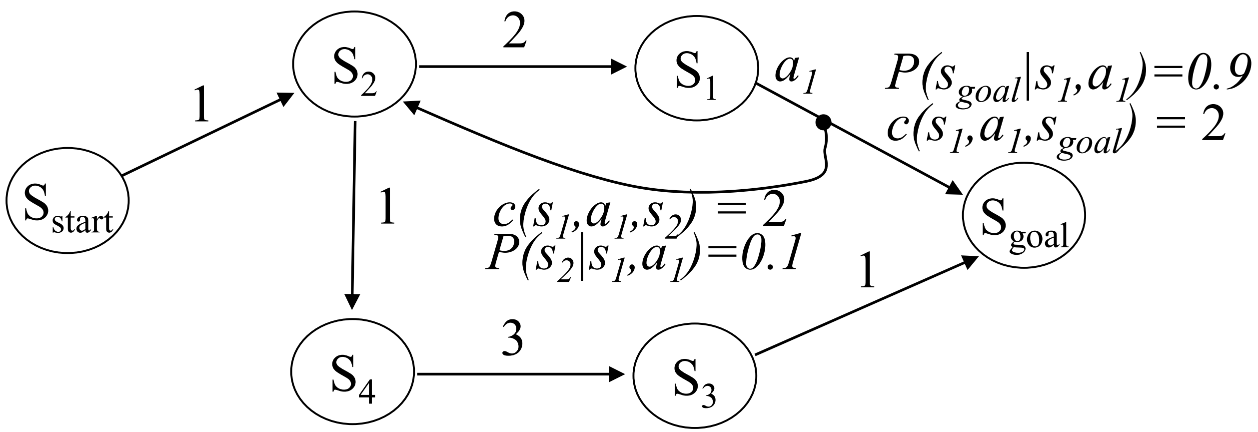
- Optimal policy π^*:

  $\pi^*(s) = \text{argmin}_a\, E\{c(s,a,s')+v^*(s')\}$

  (expectation over outcomes s' of action a executed at state s)

Why?

Computing Expected Cost Minimal Plans



- Optimal expected cost-to-goal values v^* satisfy:

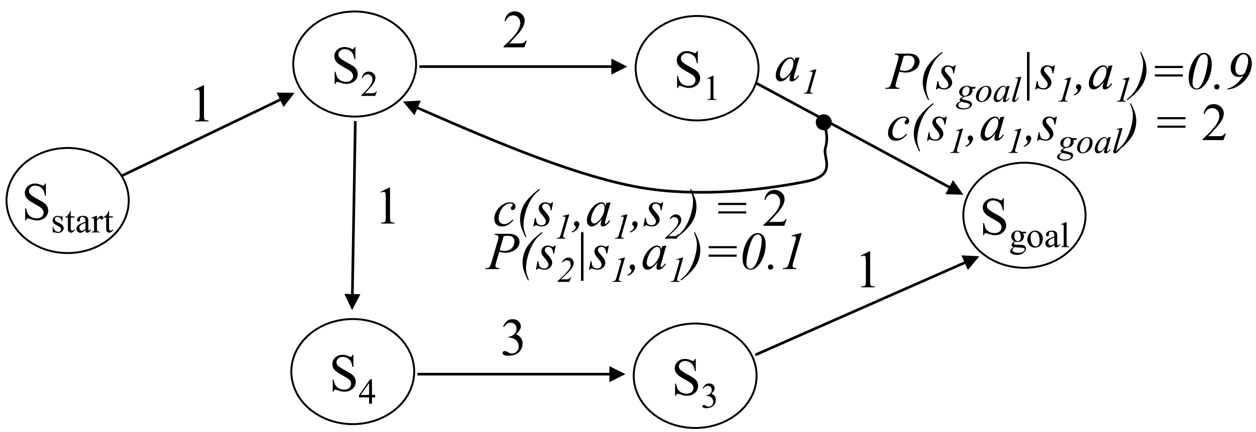
  $v^*(s_{goal})=0$

  $v^*(s) = \min_a E\{c(s,a,s')+v^*(s')\}$ *for all* $s \neq s_{goal}$

  (expectation over outcomes s' of action a executed at state s)

Bellman optimality equation

Computing Expected Cost Minimal Plans



- Value Iteration (VI):

  Initialize v-values of all states to finite values;

  Iterate over all s in MDP and re-compute until convergence:

  $v(s_{goal}) = 0$

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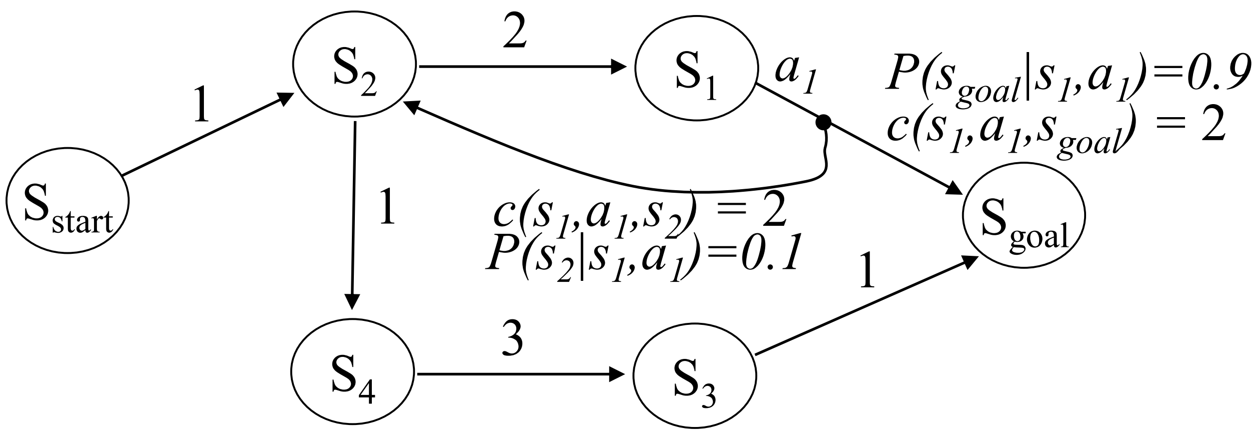
Computing Expected Cost Minimal Plans



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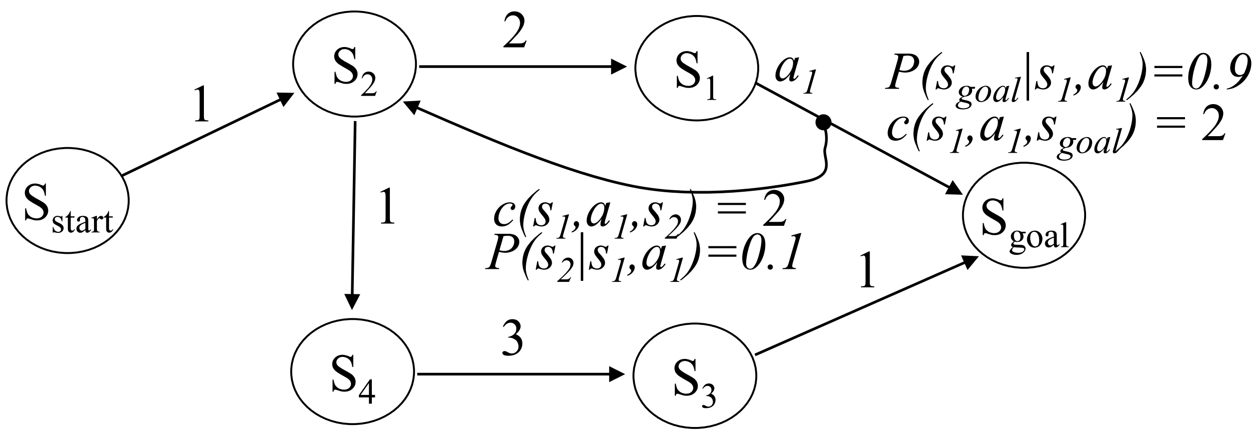
Bellman update equation (or backup)

Computing Expected Cost Minimal Plans



- Value Iteration (VI):

best to initialize to admissible values (under-estimates of the actual costs-to-goal)

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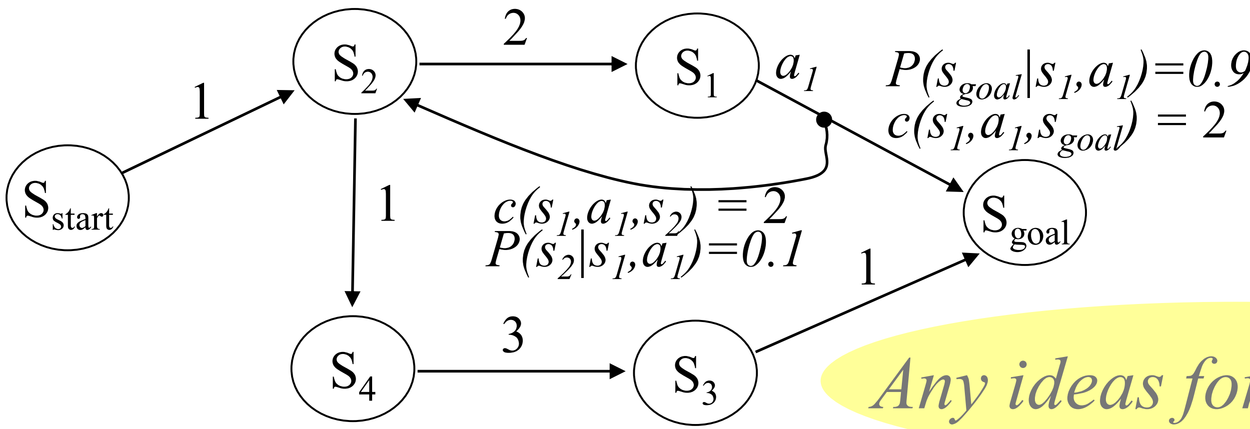
converges to an optimal value function ($v(s)=v^*(s)$ for all s) for any iteration order

the speed of convergence depends on iteration order

Computing Expected Cost Minimal Plans



Any ideas for the order?

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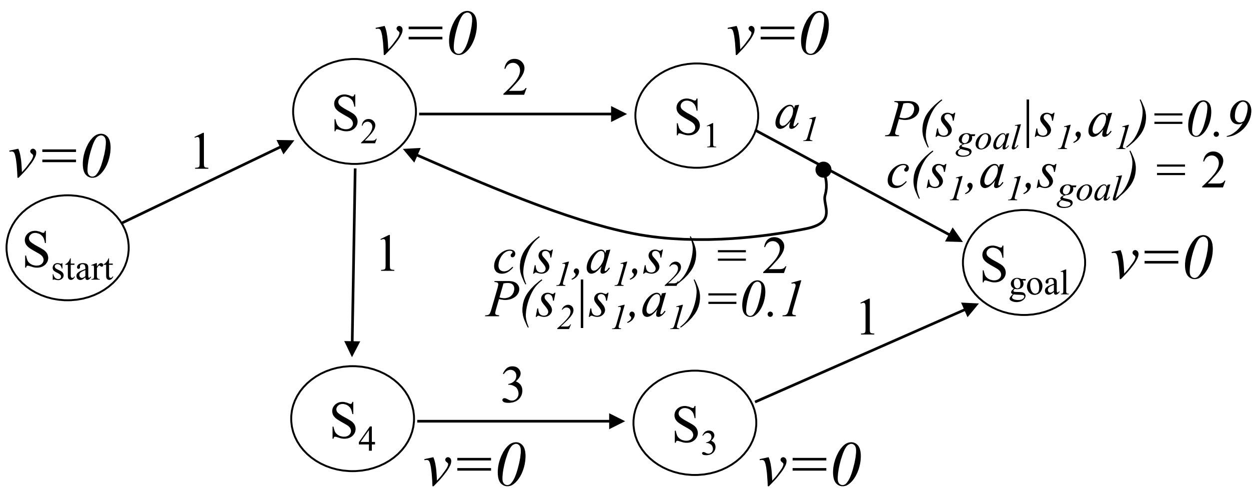

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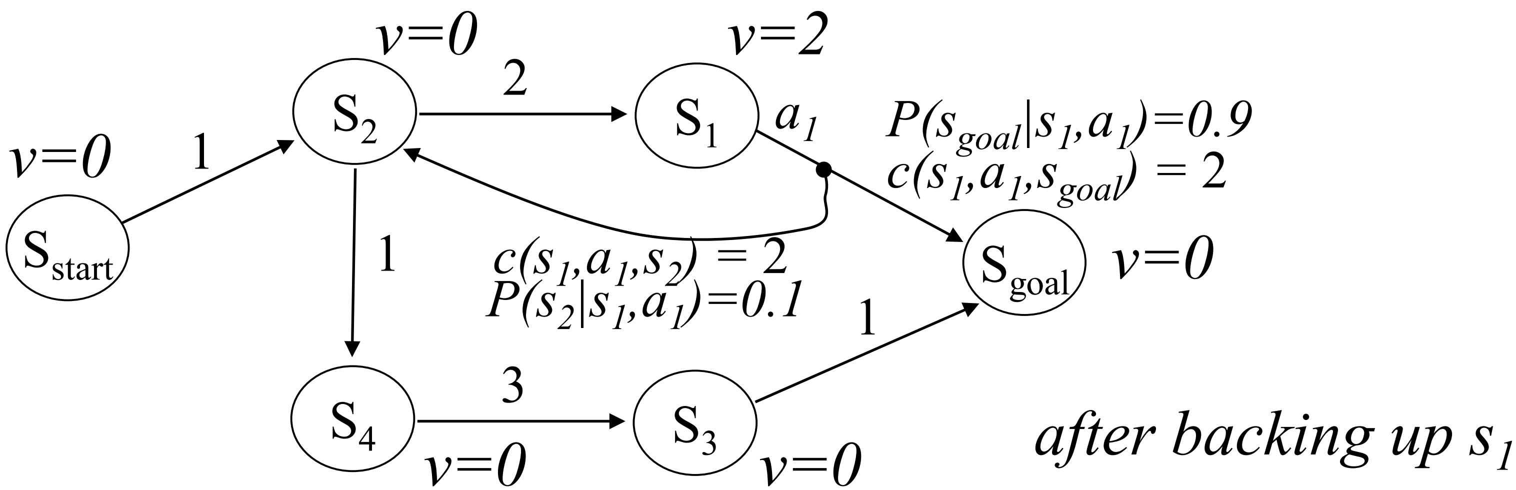

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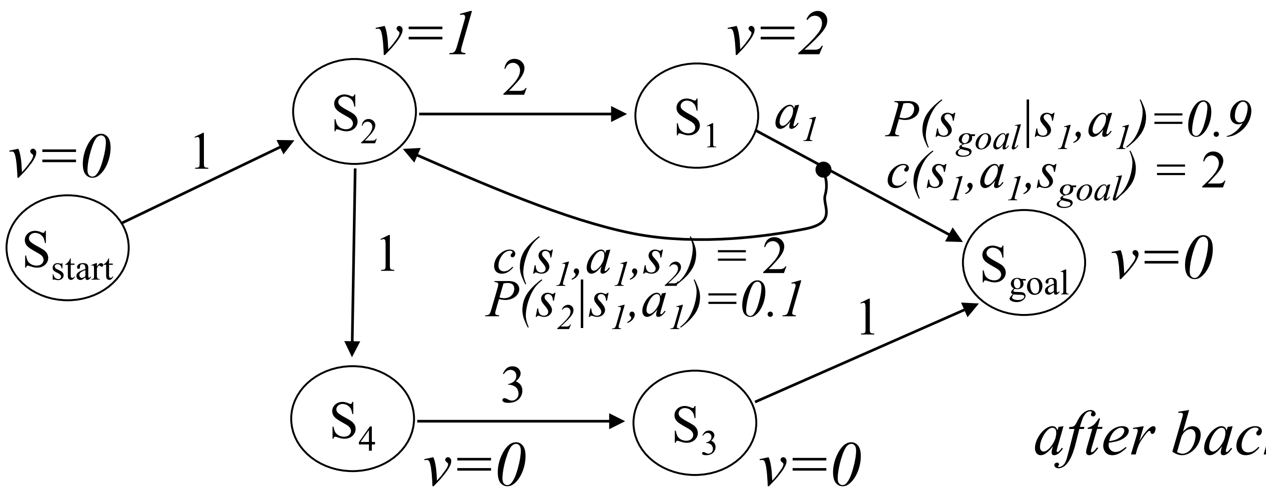

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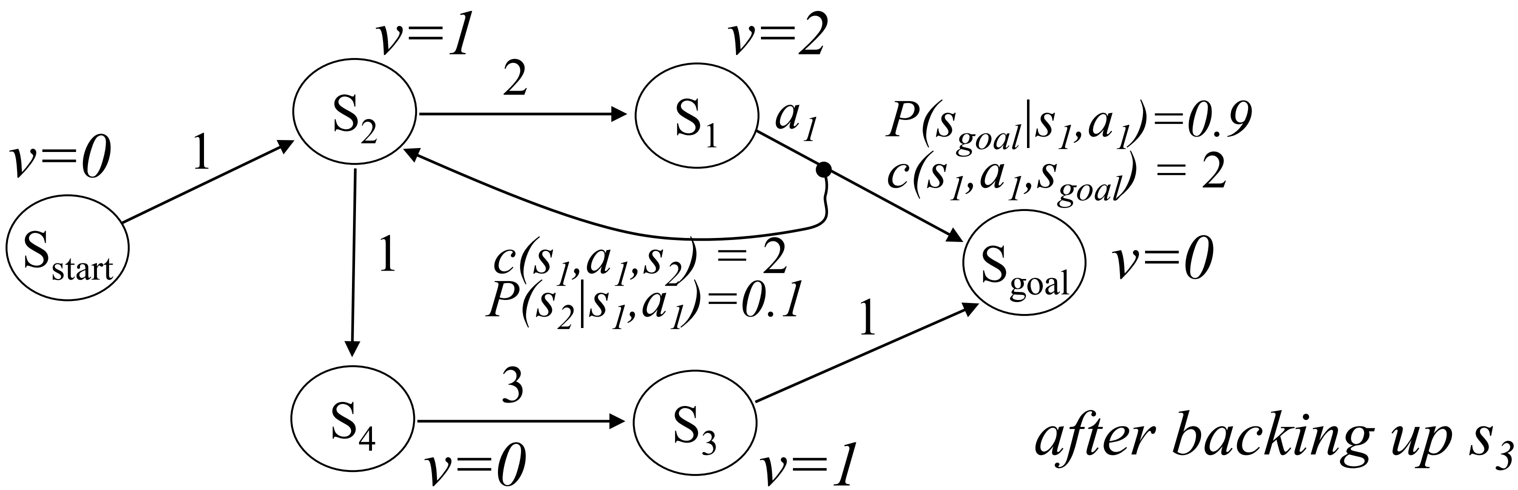

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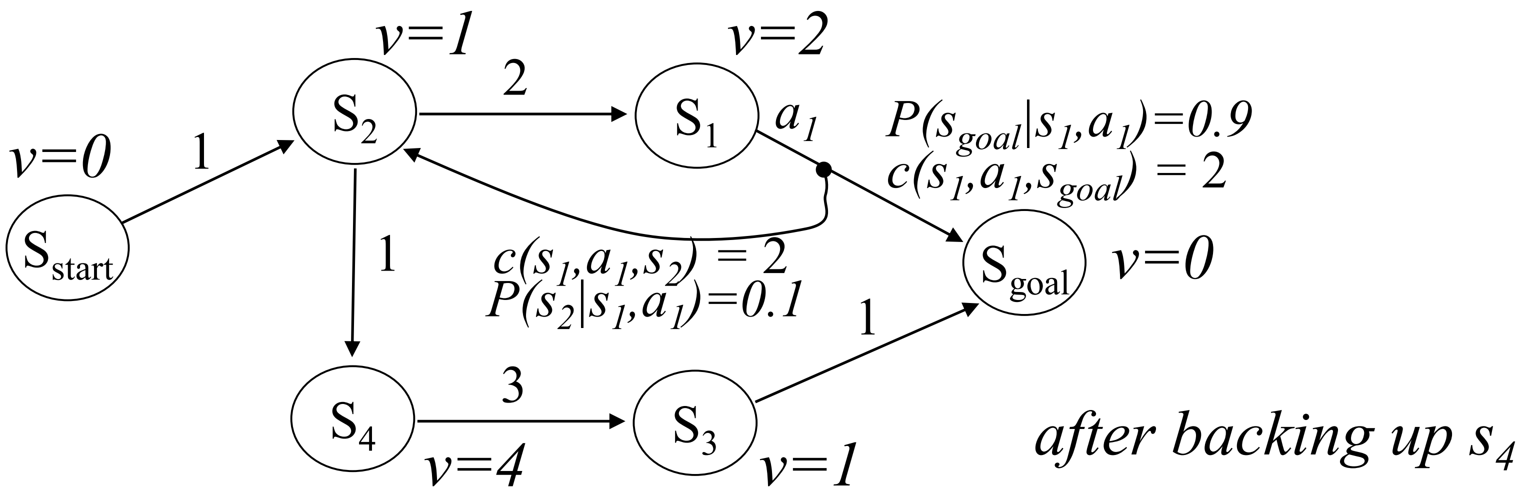

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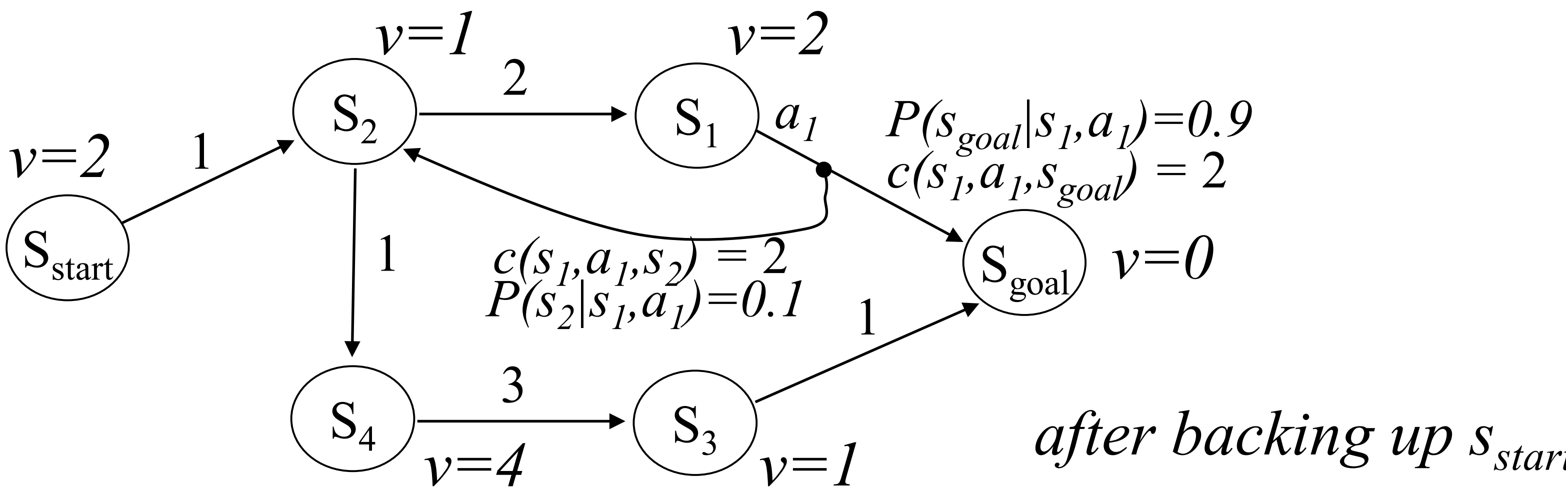

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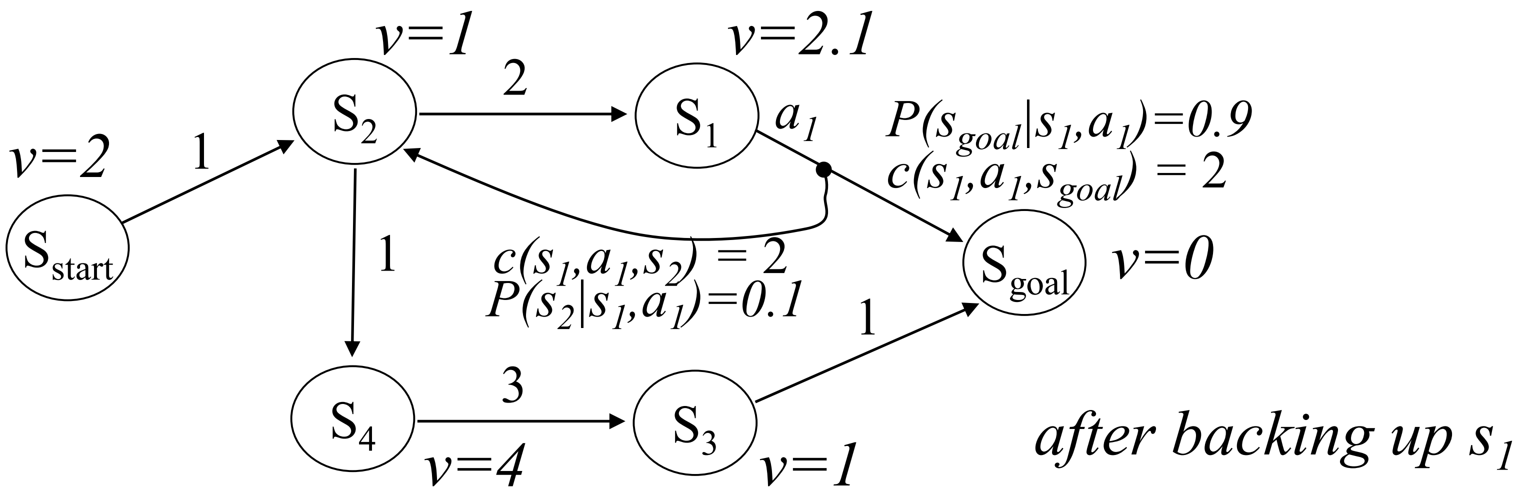

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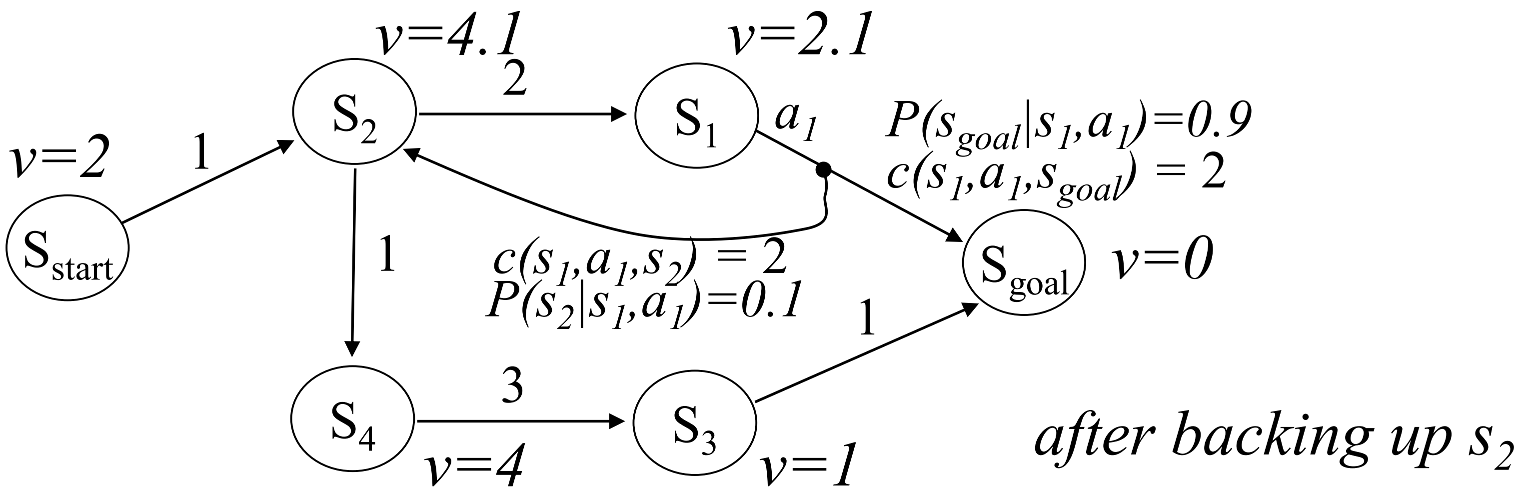

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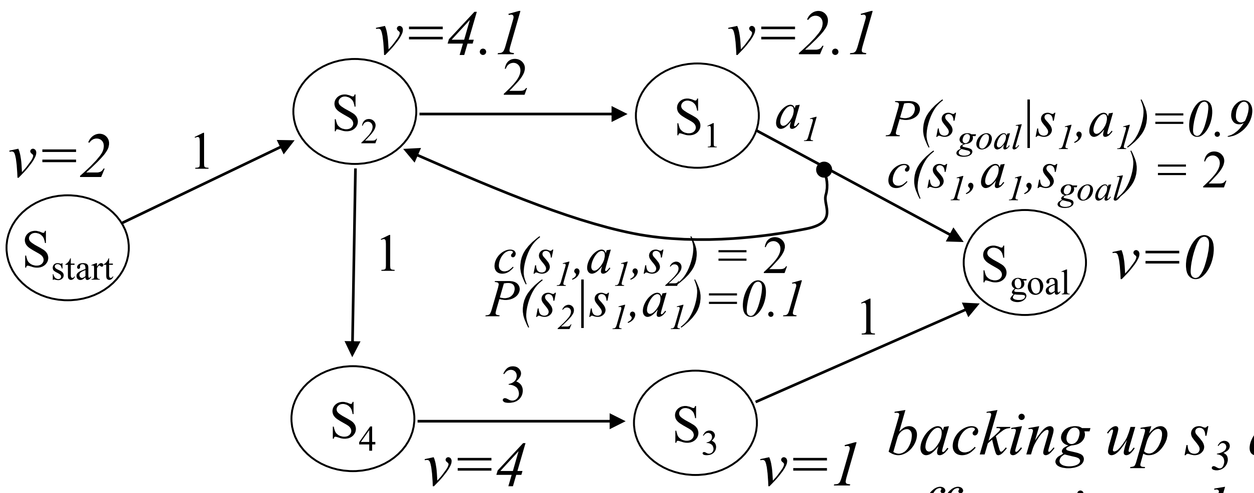

backing up s_3 and s_4 has no effect since their Bellman errors are zero

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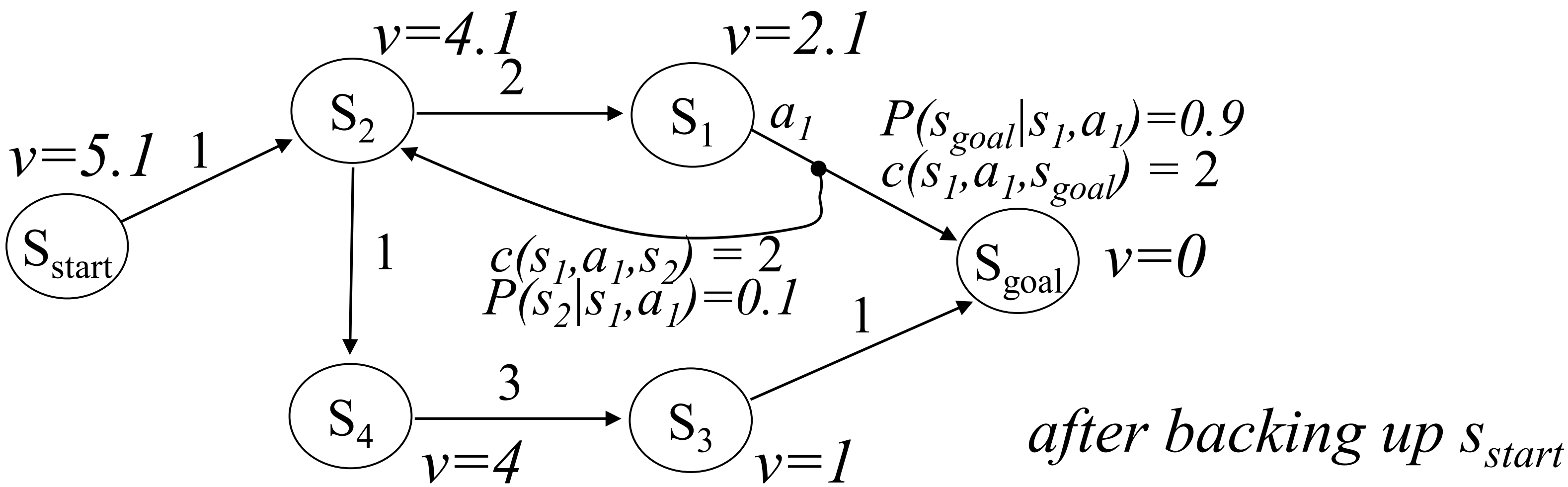

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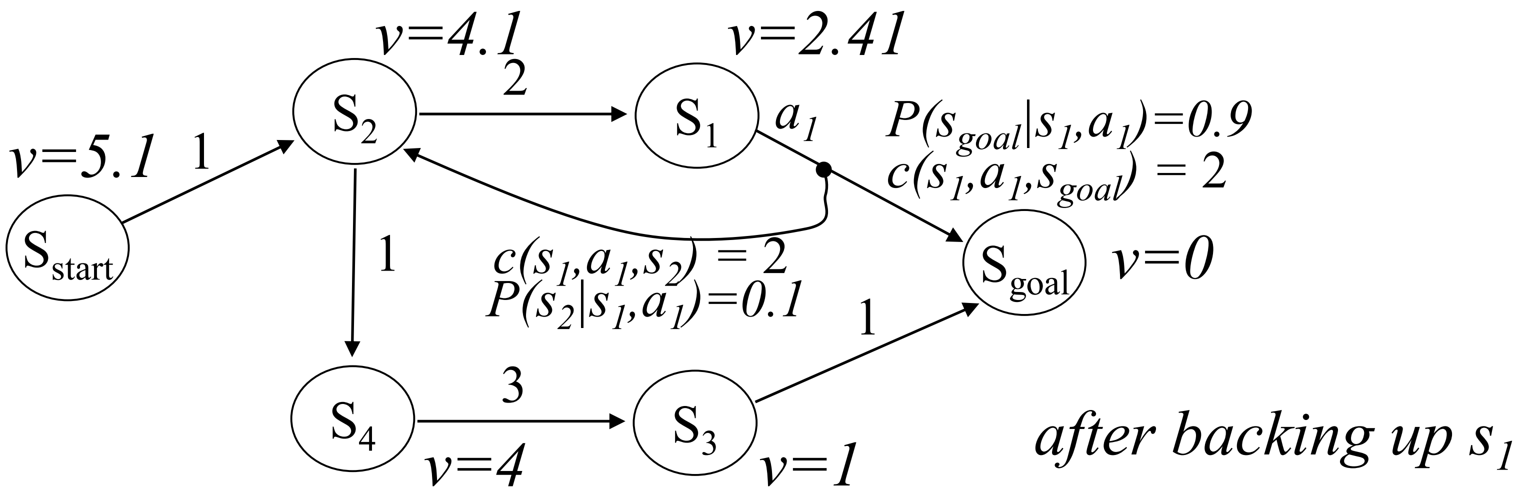

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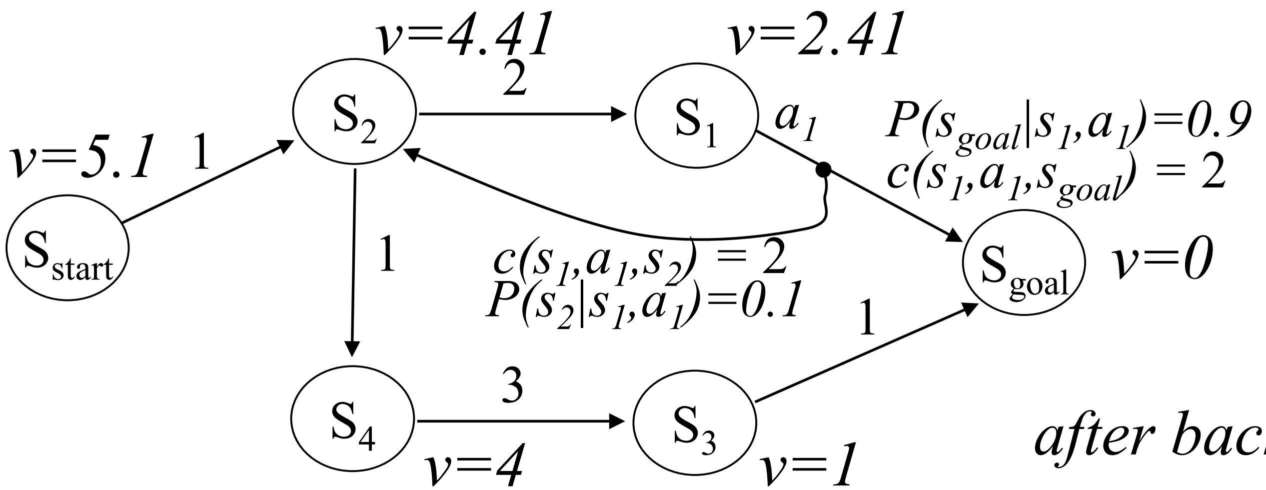
Computing Expected Cost Minimal Plans



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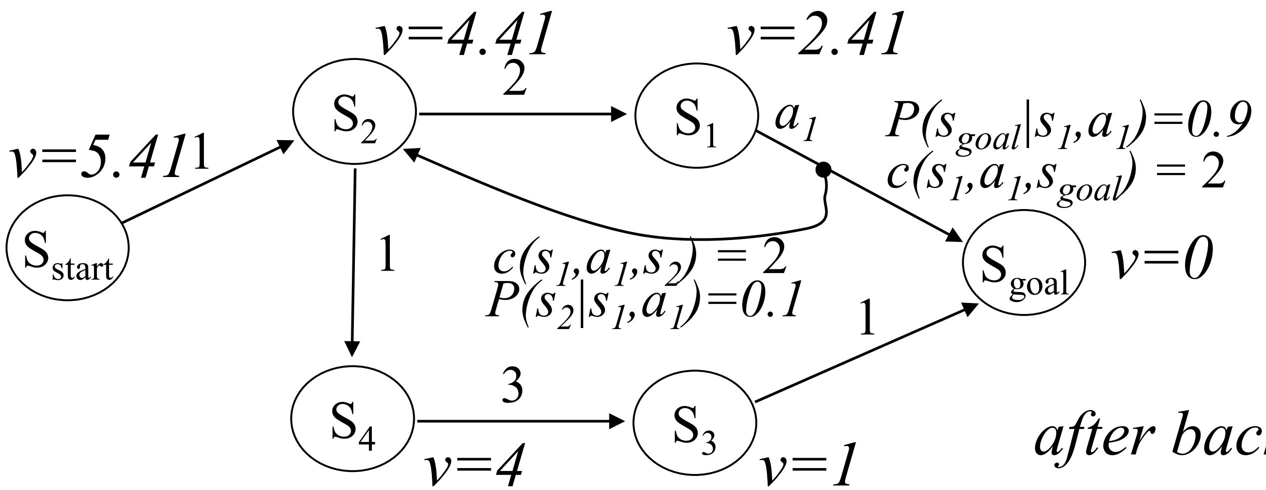
Computing Expected Cost Minimal Plans



after backing up s_{start}

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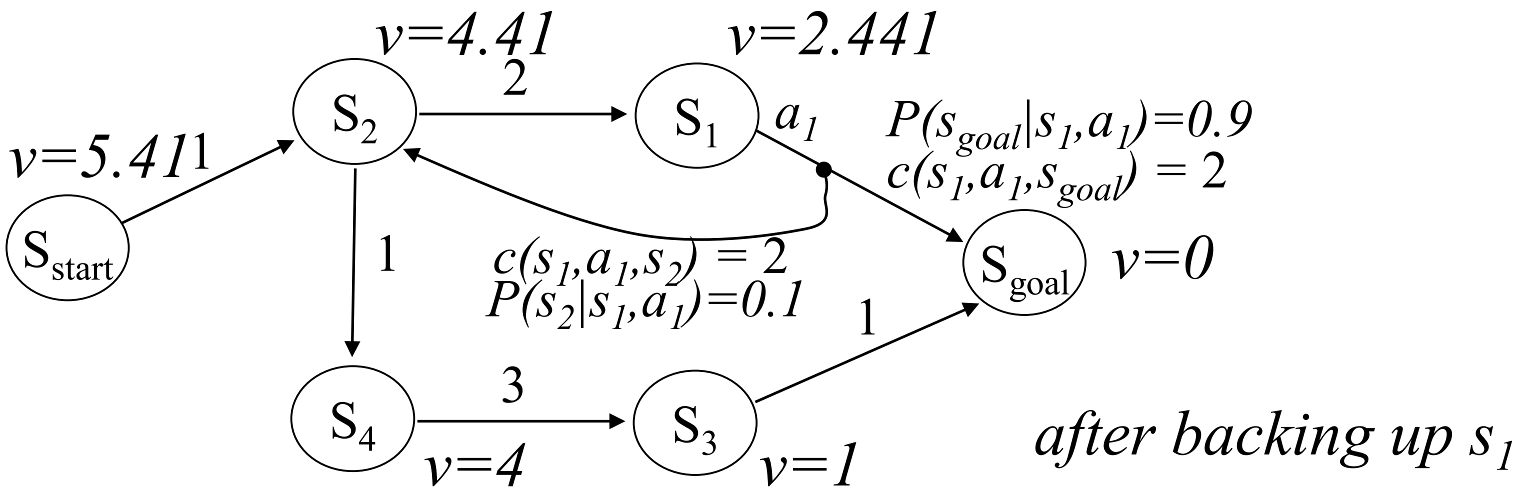

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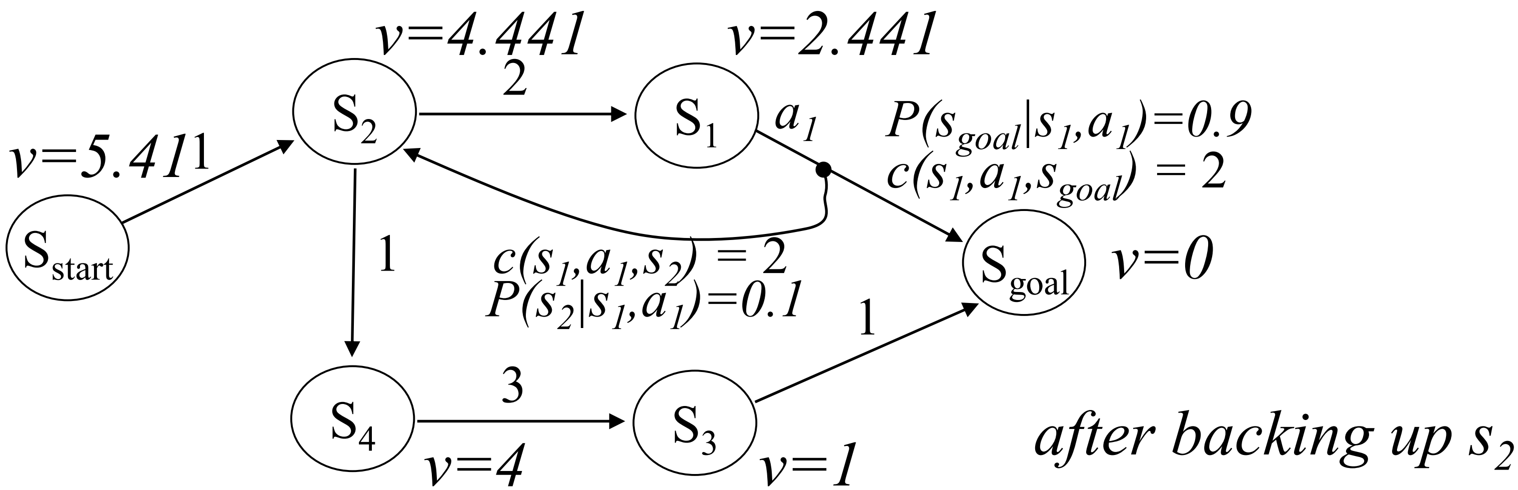

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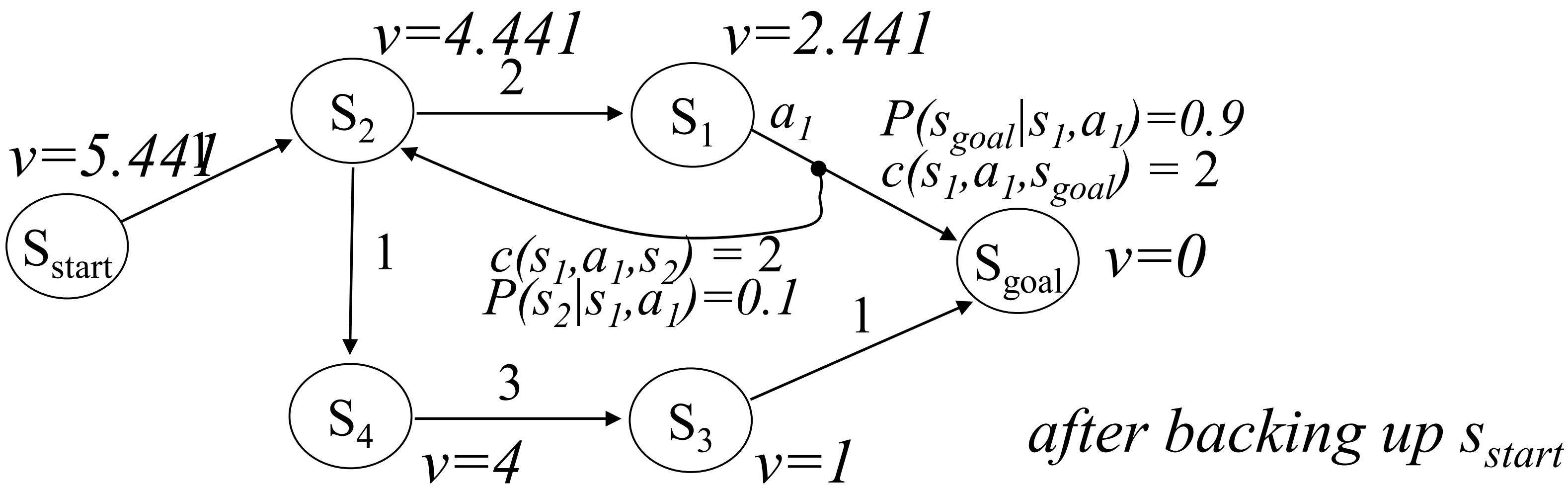

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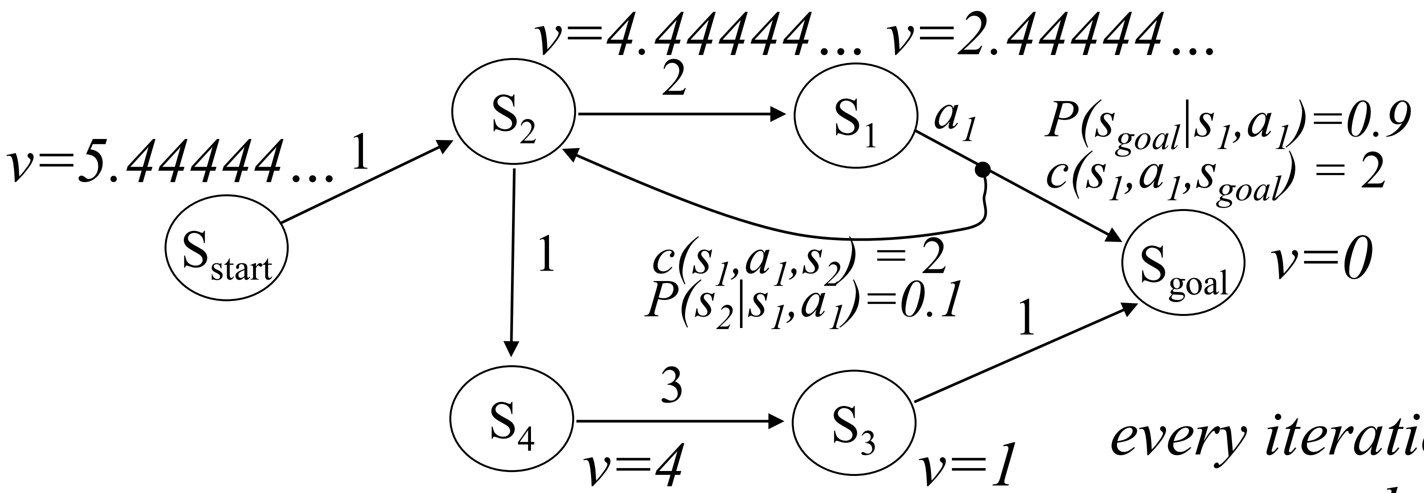

every iteration computes one more decimal point

At convergence…

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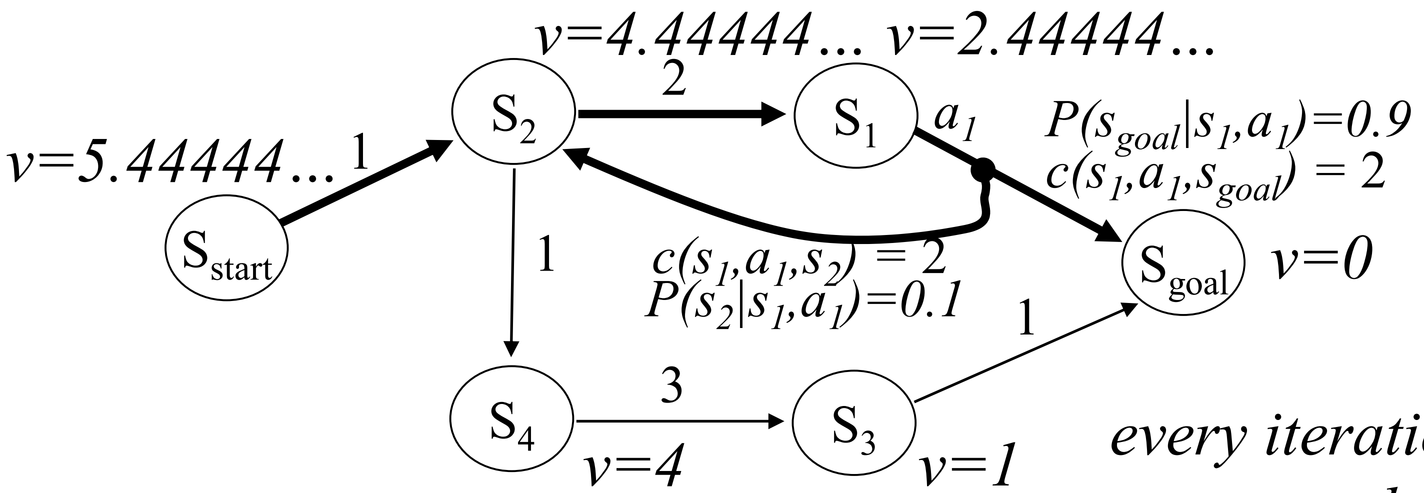

every iteration computes one more decimal point

At convergence…

optimal policy is given by greedy policy: always select an action that minimizes $E\{c(s,a,s')+v(s')\}$

- Va…

Initialize … states to finite values;

Iterate over … and re-compute until convergence:

expected cost of executing greedy policy is at most: $v^*(s_{start})c_{min}/(c_{min}-\Delta)$ where c_{min} is minimum edge cost

for any $s \neq s_{goal}$

Usual co… …ellman error over all states $< \Delta$

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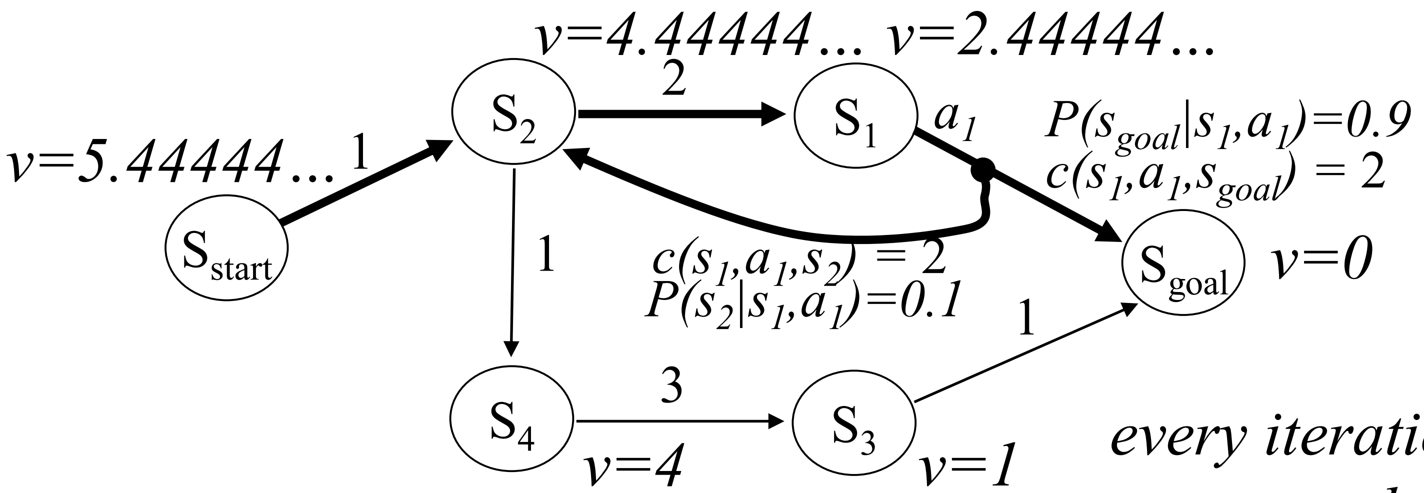

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VI converges in finite number of iterations (assuming goal is reachable from every state)

Why condition?

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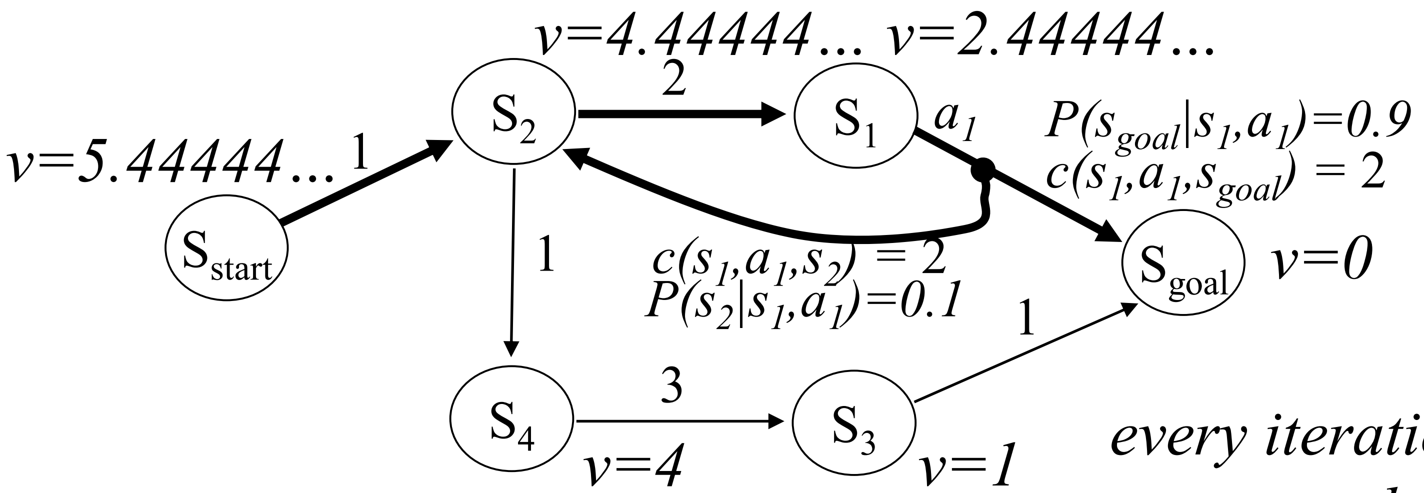
  Usual convergence condition: Bellman error over all states < Δ

  Bellman error: $|v(s) - \min_a E\{c(s,a,s')+v(s')\}|$ for any $s \neq s_{goal}$

Computing Expected Cost Minimal Plans



every iteration computes one more decimal point

VI converges in finite number of iterations (assuming goal is reachable from every state)

- Value Iteration (VI):

  Initialize v-values of all states to fin...

  Iterate over all s in MDP and r...

  $v(s_{goal}) = 0$

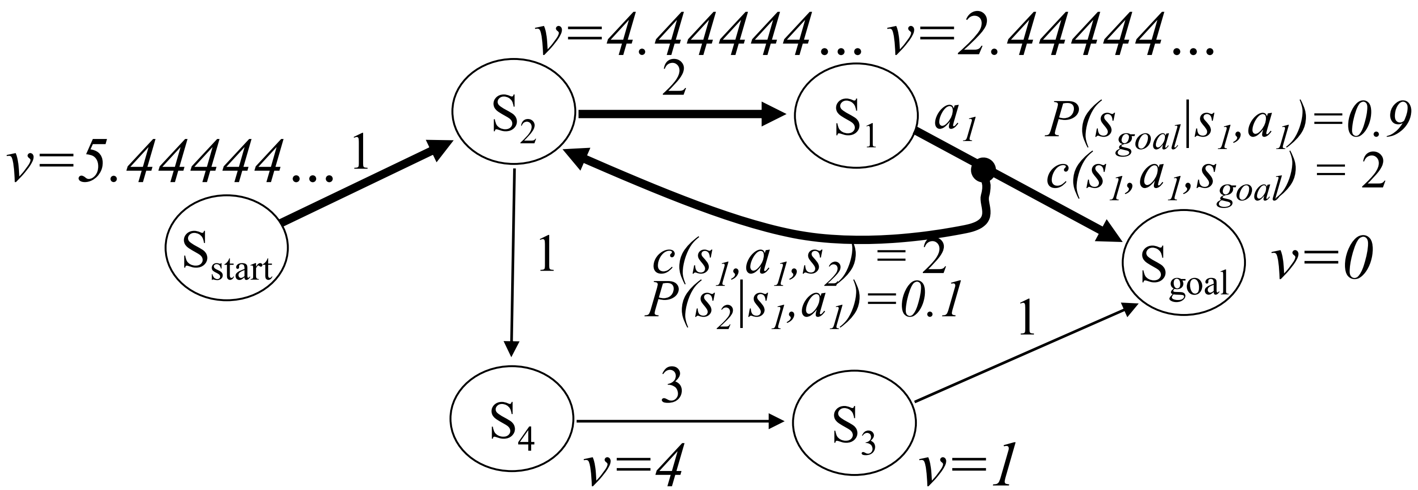
  $v(s) = \min_a E\{c(s,a,s')+v(s')\}$ for ...

How many backups required in a graph with no stochastic actions?

Usual convergence condition: Bellman error over all states < Δ

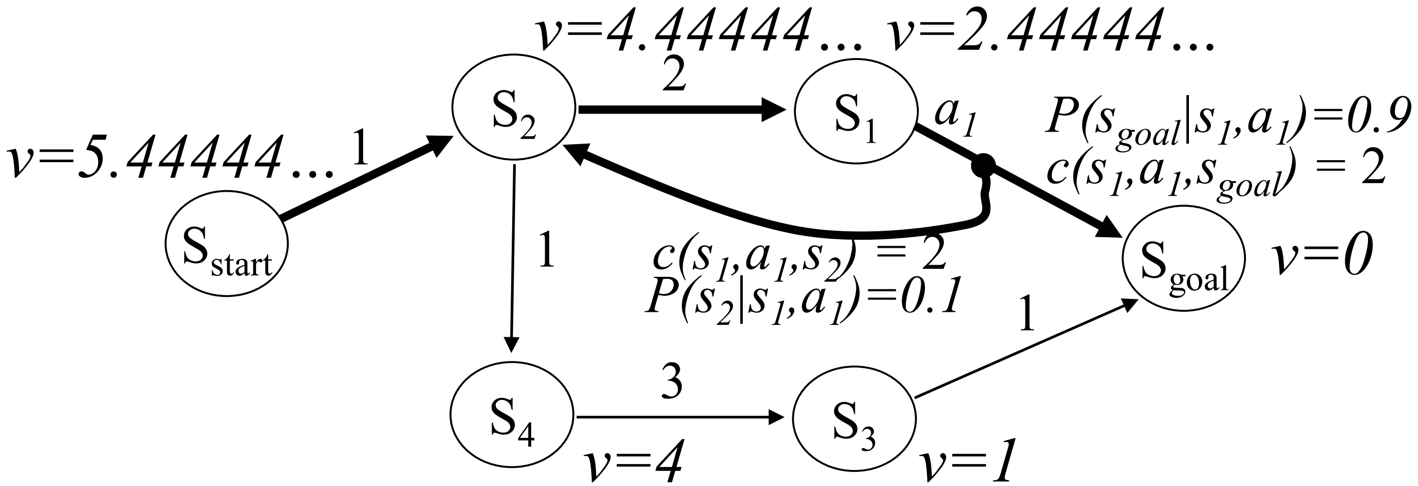
Bellman error: $|v(s) - \min_a E\{c(s,a,s')+v(s')\}|$ for any $s \neq s_{goal}$

Computing Expected Cost Minimal Plans with RTDP



- Real-time Dynamic Programming (RTDP)
  - very popular alternative to Value Iteration
  - does NOT compute values of all states
  - focusses computations on states that are relevant
  - typically, **much more efficient than Value Iteration**

Computing Expected Cost Minimal Plans with RTDP



- RTDP:

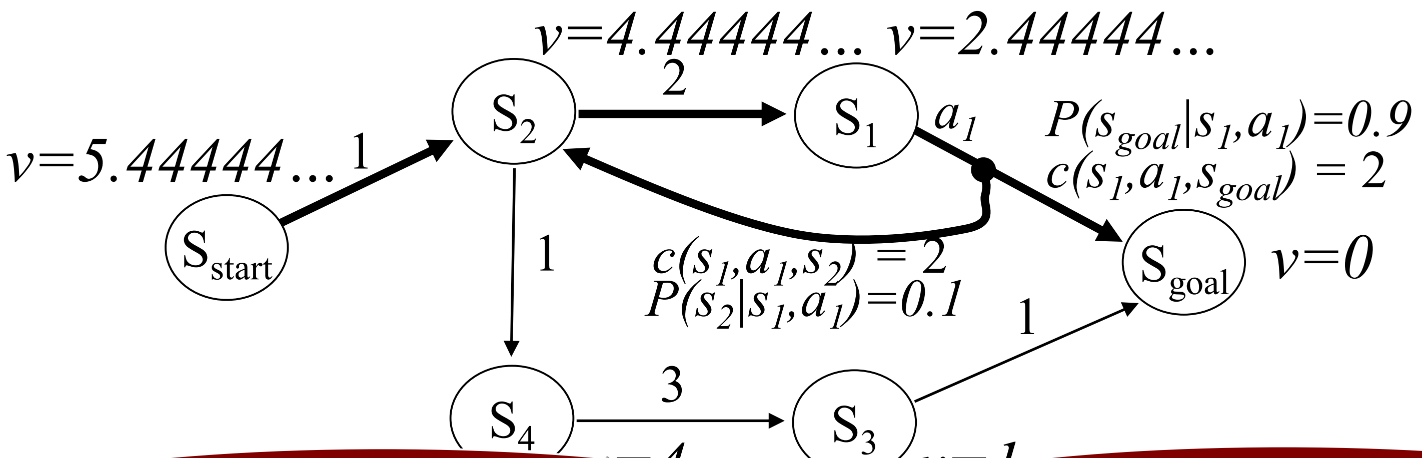
  Initialize v-values of all states to admissible values;

  1. Follow greedy policy picking outcomes at random until goal is reached;

  2. Backup all states visited on the way;

  3. Reset to s_{start} and repeat 1-3 until all states on the current greedy policy have Bellman errors < Δ;

Computing Expected Cost Minimal Plans with RTDP



For any state s, picking action a that minimizes E{c(s,a,s')+v(s')}

Picking successor state s' at random according to probability P(s'|a,s)

- RTDP:

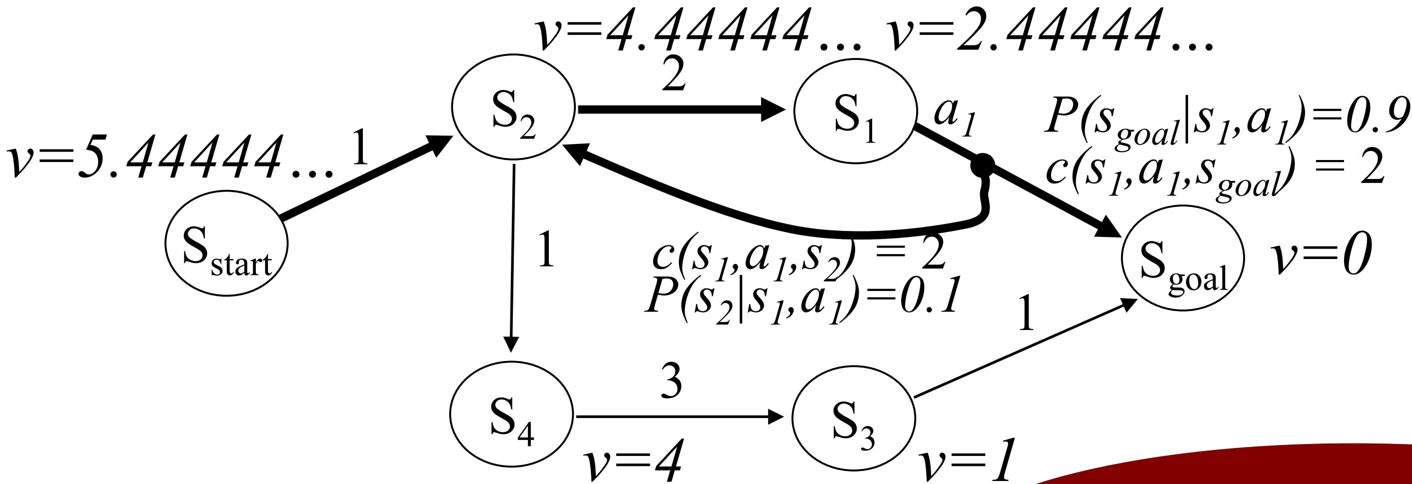
  Initialize v-values of all states to admissible values;

  1. Follow greedy policy picking outcomes at random until goal is reached;

  Updating $v(s) = \min_a E\{c(s,a,s')+v(s')\}$

  2. Backup all states visited on the way;
  3. Reset to s_{start} and repeat 1-3 until all states on the current greedy policy have Bellman errors < Δ;

Computing Expected Cost Minimal Plans with RTDP



*RTDP **focusses** its backups on what is relevant to the optimal plan rather than computing ALL state values (what VI does)*

- RTDP:

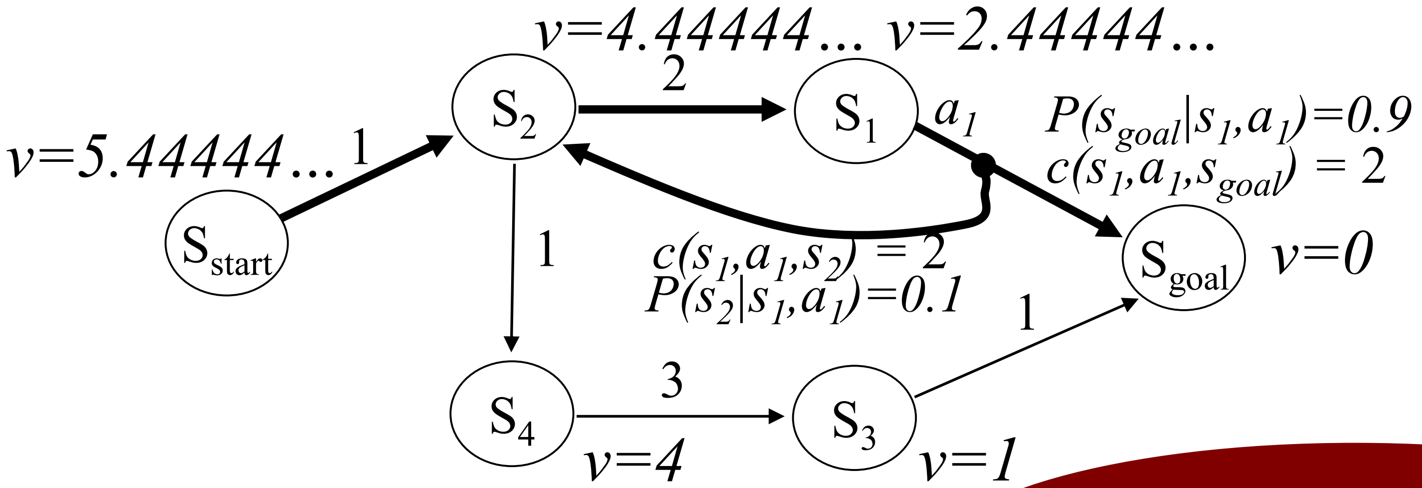
  Initialize *v*-values of all states to admissible values;

  1. Follow greedy policy picking outcomes at random until goal is reached;

  2. Backup all states visited on the way;

  3. Reset to s_{start} and repeat 1-3 until all states on the current greedy policy have Bellman errors < Δ;

Computing Expected Cost Minimal Plans with RTDP



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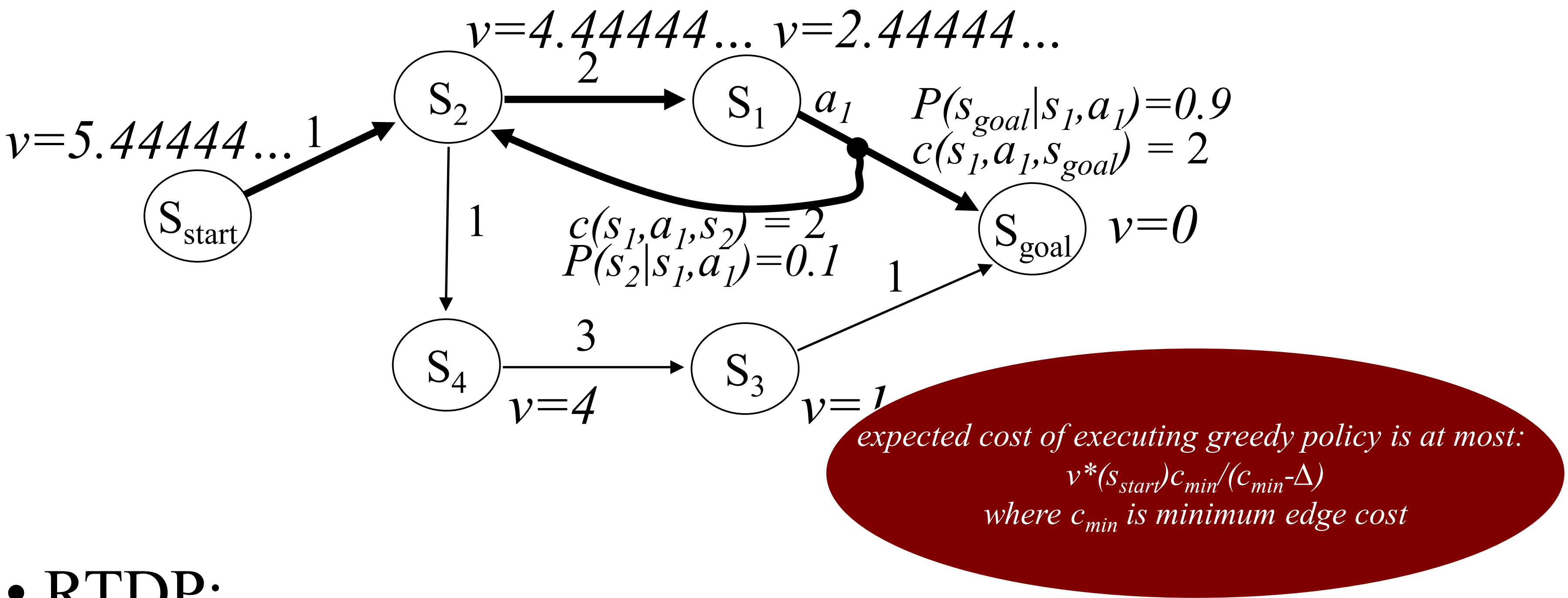
Computing Expected Cost Minimal Plans with RTDP



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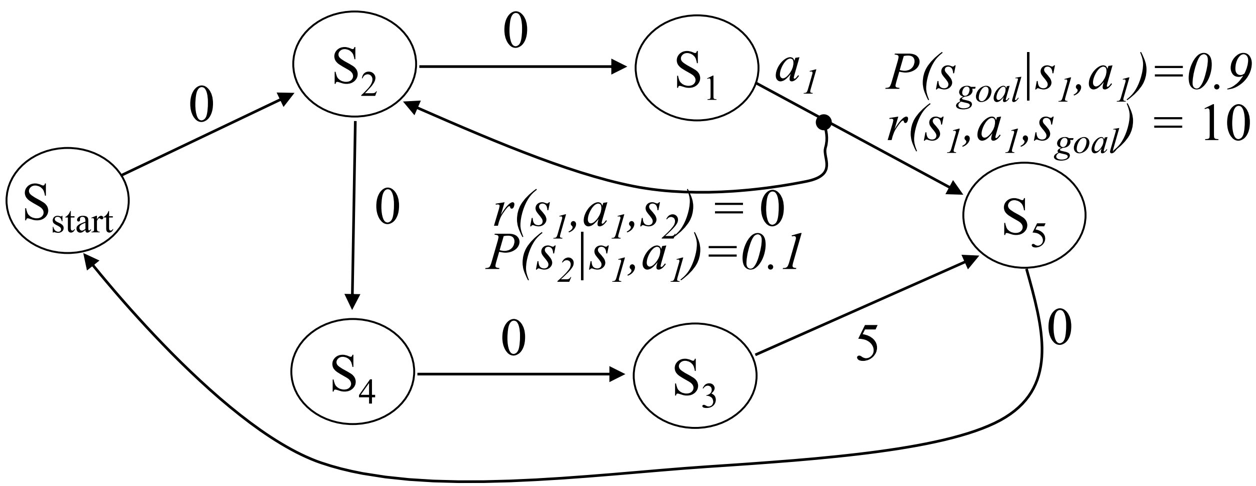
Rewards version of MDPs

- Suppose we have a Trash Collecting robot
  - its task is to go around the room and pick-up trash
  - if battery is dead, it can't move anymore
  - available actions:
    - Look for trash (takes 1 min) and discovers trash with probability 0.4
    - Pick-up trash (takes 1 min), <u>and receive reward of 100 units</u>
    - Re-charge (takes 1 min). Battery level goes back to full 3 mins if successful with probability 0.9 (there is a chance that re-charge is not successful)

Example on the board

Markov Decision Processes, REWARDS version



- Optimal expected reward values v^* satisfy:

  $v^*(s) = \boldsymbol{max}_a\, E\{r(s,a,s') + \gamma v^*(s')\}$ *for all s*

  (expectation over outcomes s' of action a executed at state s)

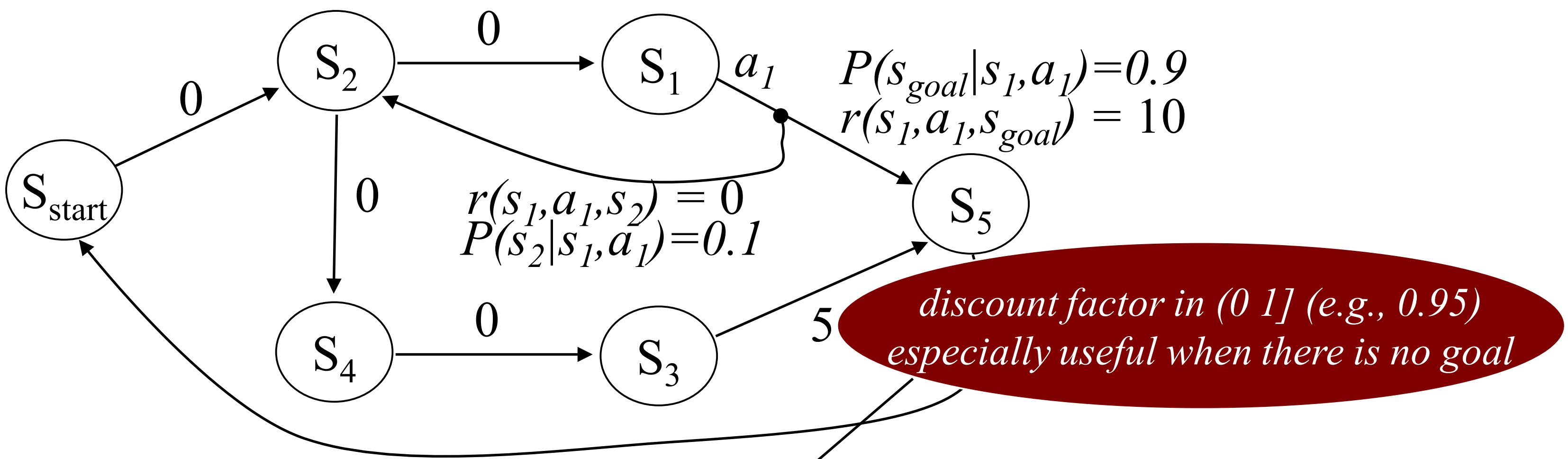
- Optimal policy π^*:

  $\pi^*(s) = arg\boldsymbol{max}_a\, E\{r(s,a,s') + \gamma v^*(s')\}$

- Computing optimal v^*-values via value iteration (VI):

  re-compute $v(s) = \boldsymbol{max}_a\, E\{r(s,a,s') + \gamma v(s')\}$ *until convergence*

Markov Decision Processes, REWARDS version



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- Computing optimal v^*-values via value iteration (VI):

  re-compute $v(s) = \boldsymbol{max}_a\, E\{r(s,a,s')+ \gamma v(s')\}$ *until convergence*

What You Should Know…

- Pros and Cons of solving Expected Cost formulation (rather than Minimax formulation)
- The operation of Value Iteration
- The operation of RTDP
- Rewards formulation of MDPs and when it should be used