When Some Tricks Are Trickier Than Others

A Collection of Probability Tricks

Maverick Woo

My voice is too boring?

Both Theory Lunch and Theory Workshop
are looking for speakers!

Prosper

This is not a PowerPoint talk.

Google for “propser”.
Let me show you the source of this presentation.

References

- Mathematics of Chance
- Fifty Challenging Problems in Probability
- Concrete Mathematics
- Some IMO books

Principle of Reflection

Scenario

The year is 1970 and you are working for Coca-Cola.

Automatic coke machine has just been invented and Coca-Cola has only one product: Coke.

Each coke costs 5 cents.

Due to engineering issues, the coke machine can only take either a nickel (5 cents) or a dime (10 cents).

The Nickel Bag

If the customer inserts a nickel, a coke will be dispensed and the nickel will be put into a bag inside the machine.

If the customer inserts a dime, a coke will also be dispensed (*duh!*) and a nickel from the bag will be returned to the customer.

But what if a dime is inserted and yet the bag is empty?

Customer Relationship

We will pretend there is no coke left and return the dime.

Alas, another engineering issue comes up: the nickel dispenser will be confused by the empty bag and effectively hangs the whole coke machine.

So, if a dime is inserted when the nickel bag is empty, the machine will block until an engineer fixes it.

It's all about $$$. . .

We want to maximize the time that the machine stays up.

So when we load the machine, we put a nickels into the bag to start.

Assuming that all sequences of customers are equally likely, what happens when we vary the value of a?

Realistically

We don’t want to provide 100% guarantee.

How many nickels are sufficient for a 95% probability to ensure that the coke machine will not get blocked?

An Easy Version

Assume we know the queue structure of customers.

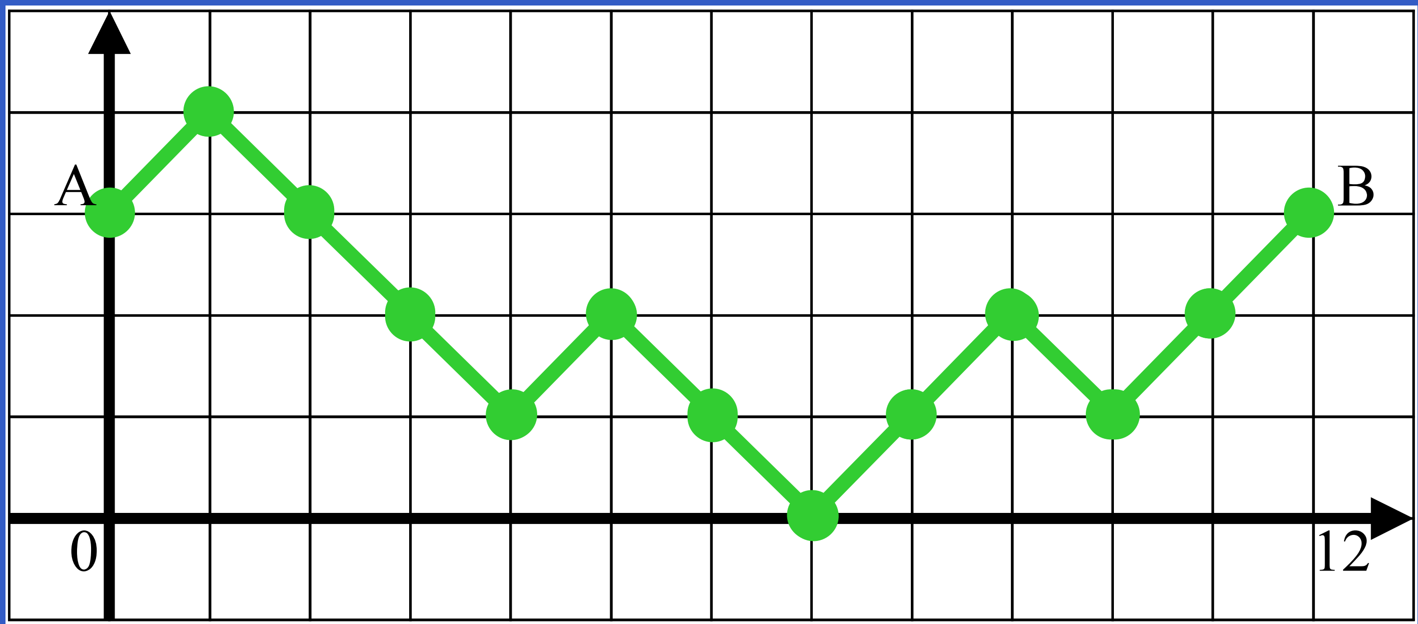
Say we will have N customers (fixed).

m of them have nickels. So, $n = N - m$ of them have dime.

Trajectory

Represent the number of nickels in the bag graphically.

Example: $N = 12, m = 6, n = 6$



Starts at $A = (0, a)$ and ends at $B = (N, a + m - n)$.

Observations

Total number of trajectories: $\binom{N}{m}$

All trajectories are equally likely.

Those that fall under the horizontal x-axis corresponds to blocking.

Two cases

If $a + m - n < 0$, then there is nothing we can do.

Now only need to consider $a + m - n \geq 0$.

$$\begin{aligned} m &\geq n - a \\ &= (N - m) - a \\ &= \frac{N - a}{2} \end{aligned}$$

The Trick

Call the line $y = -1$ the threshold line t. As soon as the trajectory touches line t, keep track of the reflection of the trajectory with respect to t.

Principle of Reflection

Since the original line ends at $B = (N, a+m+n)$, this auxiliary graph terminates at $C = (N, -a-m+n-2)$.

The Principle:

There exists exactly one trajectory from A to C
corresponding to a trajectory from A to B
that falls at least once under the horizontal axis.

Analysis

If a trajectory from A to C has u segments up and d segments down, then

$$\begin{aligned} u + d &= N \\ a + u - d &= -a - m + n - 2 \end{aligned}$$

Solving this yields

$$u = n - a - 1, \quad d = m + a + 1.$$

So, the number of trajectories from A to C is just $\binom{N}{m+a+1}$.

Done!

The probability of NOT blocking is just

$$P_{a,p} = 1 - \frac{\binom{N}{m+a+1}}{\binom{N}{m}}.$$

For 95% confidence,

N	m	a	$P_a(N,m)$
30	15	6	0.962
60	30	9	0.965
120	60	13	0.962
30	10	13	0.980
60	20	23	0.964
120	40	43	0.953

A Harder Version

We still know the number of customers is N.
(Call them “nickel customers”.)

A customer has a nickel with probability p.

Note that this is already quite realistic.

Total Probability: $\Pr(A) = \sum_{B_i} \Pr(A|B_i)\Pr(B_i)$

Let B_m denotes the event that there are precisely m nickel customers among the N customers.

We have already computed $\Pr(\text{Block}|B_m)$.

Under our assumption, the number of nickel customers is a random variable with binomial distribution,

$$\Pr(B_m) = \binom{N}{m} p^m (1-p)^{N-m},$$

for $m = 0, 1, \ldots, N$.

Summing Up

The probability that the machine will NOT block is

$$P_{a,p}(N) = \sum_{m=\lceil \frac{N-a}{2} \rceil}^{N} \left[1 - \frac{\binom{N}{m+a+1}}{\binom{N}{m}} \right] \binom{N}{m} p^m (1-p)^{N-m}.$$

For 95% confidence,

N	p	a	$P_{a,p}(N)$
30	$\frac{1}{2}$	10	0.957
60	$\frac{1}{2}$	15	0.960
120	$\frac{1}{2}$	21	0.955
30	$\frac{1}{3}$	18	0.952
60	$\frac{1}{3}$	32	0.956
120	$\frac{1}{3}$	58	0.962

Comparison

Knowing the exact number of customers helps quite a bit.

For 95% confidence,

N	m	a	$P_a(N,m)$	p	a	$P_{a,p}(N)$
30	15	6	0.962	$\frac{1}{2}$	10	0.957
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30	10	13	0.980	$\frac{1}{3}$	18	0.952
60	20	23	0.964	$\frac{1}{3}$	32	0.956
120	40	43	0.953	$\frac{1}{3}$	58	0.962

An Even Harder Version

Let N customers occur with probability f_N and customers come independently. Let f denotes $(f_0, f_1, \ldots)$

Still, a customer has a nickel with probability p.

What is $P_{a,p,f}$?

An Even Harder Version

Let N customers occur with probability f_N and customers come independently. Let f denotes $(f_0, f_1, \ldots)$

Still, a customer has a nickel with probability p.

What is $P_{a,p,f}$?
Remember that we know $P_{a,p}$.
Again, by the Theorem of Total Probability

$$P_{a,p,f} = \sum_{N=0}^{\infty} P_{a,p}(N) f_N$$

Presidential Election

The year is 2005. Presidential election is just over.
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Let B_k and G_k be the number of votes for B and G respectively.

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Let B_k and G_k be the number of votes for B and G respectively.

Define $X_k = |B_k - G_k|$. Find EX_k.

Breaking Records

Record

Consider a series of real numbers $X_1, X_2, \ldots, X_n$.

X_1 is defined to be a record. For later X_i's, X_i is a record iff

$$X_i > \max(X_1, \ldots, X_{i-1}).$$

Assume that $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables.

Also assume that the distribution of X_i's are continuous. (No ties.)

Expected Number of Records

There are $n!$ ways to order the n values.

Suppose X_n is a record, then the remaining $n-1$ values can be ordered in $(n-1)!$ different ways.

The probability p_i that X_i is a record is

$$p_i = \frac{(i-1)!}{i!} = \frac{1}{i}.$$

Indicator Variables

This is *THE* Trick of the Trade.

Let $Y_1, Y_2, \ldots, Y_n$ be auxiliary variables such that

$$Y_i = \begin{cases} 1 & \text{if } X_i \text{ is a record,} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots, n$.

Their power comes from

$$EY_i^n = 1 \times p_i + 0 \times (1 - p_i) = p_i.$$

Linearity of Expectation

This is the Trade.

The total number of records R_n is

$$R_n = Y_1 + \cdots + Y_n.$$

By linearity of expectation,

$$ER_n = EY_1 + \cdots + EY_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

This is H_n, the n-th Harmonic number.

Cold Winters

H_n grows very slowly.

Smallest n such that $H_n \geq N$:

N	2	3	4	5	6	7	8	9	10
n	4	11	31	83	227	616	1674	4550	12367

But temperature is not exactly independent.

Probability of r Records

Let $p_{r,n}$ denotes the probability that there are exactly r records in $X_1, \ldots, X_n$.

We already know $p_{1,n} = \Pr(R_n = 1) = \frac{1}{n}$.

How about $p_{n,n}$?

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How about $p_{n,n}$?

$$p_{n,n} = \frac{1}{n!}$$

A Useful Theorem

If n and r are arbitrary positive integers such that $r \leq n$, then

$$p_{r,n} = \frac{n-1}{n} p_{r,n-1} + \frac{1}{n} p_{r-1,n-1},$$

where $p_{1,1} = 1$ and $p_{r,0} = 0$.

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Proof.
Let $A_{i,j}$ be the event that there will be exactly i records in the prefix $X_1, \ldots, X_j$.
Let B_n be the event that X_n will be a record.

Proof

By Partition Theorem,

$$p_{r,n} = \Pr(R_n = r) = \Pr(A_{r,n-1} \cap B_n^c) + \Pr(A_{r-1,n-1} \cap B_n),$$

By Bayes Theorem,

$$\Pr(A_{r,n-1} \cap B_n^c) = \Pr(A_{r,n-1} | B_n^c) \times \Pr(B_n^c).$$

We already know that $\Pr(B_n) = \frac{1}{n}$, so $\Pr(B_n^c) = \frac{n-1}{n}$.

Proof

The probability that there will be r records in series $X_1, \ldots, X_{n-1}$ is *not* influenced by the fact that X_n is not a record.

Unconditioning, we have

$$\Pr(A_{r,n-1}|B_n^c) = \Pr(A_{r,n-1}) = p_{r,n-1}.$$

Plugging everything back in, we have

$$\Pr(A_{r,n-1} \cap B_n^c) = \frac{n-1}{n} p_{r,n-1}.$$

Proof

Analogously,

$$\begin{aligned}\Pr(A_{r-1,n-1} \cap B_n) &= \Pr(A_{r-1,n-1}|B_n) \times \Pr(B_n) \\ &= \Pr(A_{r-1,n-1}) \times \Pr(B_n) \\ &= \frac{1}{n} p_{r-1,n-1}.\end{aligned}$$

QED

Take Away

If n and r are arbitrary positive integers such that $r \leq n$, then

$$p_{r,n} = \frac{n-1}{n} p_{r,n-1} + \frac{1}{n} p_{r-1,n-1},$$

where $p_{1,1} = 1$ and $p_{r,0} = 0$.

Asymptotically speaking,

$$p_{r,n} \approx \frac{1}{(r-1)!n} (\ln n + \gamma)^{r-1}.$$

A Closer Look At Our Indicators

Let $Y_1, Y_2, \ldots, Y_n$ be auxiliary variables such that

$$Y_i = \begin{cases} 1 & \text{if } X_i \text{ is a record,} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots, n$.

Theorem. If $i \neq j$, then Y_i and Y_j are independent.

Proof

Let $1 \le i < j \le j$. We have

$$\begin{aligned}
& \Pr(Y_i = 1, Y_j = 1) \\
= & \Pr(X_i = \max(X_1, \ldots, X_i), X_j = \max(X_1, \ldots, X_j)) \\
= & \Pr(X_i = \max(X_1, \ldots, X_i) < X_j = \max(X_{i+1}, \ldots, X_j)) \\
= & \Pr(X_i = \max(X_1, \ldots, X_i)) \\
& \times \Pr(\max(X_1, \ldots, X_i) < \max(X_{i+1}, \ldots, X_j)) \\
& \times \Pr(X_j = \max(X_{i+1}, \ldots, X_j)) \\
= & \frac{1}{i} \frac{j-i}{j} \frac{1}{j-i} \\
= & \Pr(Y_i = 1) \Pr(Y_j = 1).
\end{aligned}$$

Proof

Since $\Pr(Y_i = 0, Y_j = 1) + \Pr(Y_i = 1, Y_j = 1) = \Pr(Y_j = 1)$, we also get

$$\begin{aligned} & \Pr(Y_i = 0, Y_j = 1) \\ = \; & \Pr(Y_j = 1) - \Pr(Y_i = 1, Y_j = 1) \\ = \; & \Pr(Y_j = 1) - \Pr(Y_i = 1) \Pr(Y_j = 1) \\ = \; & [1 - \Pr(Y_i = 1)] \Pr(Y_j = 1) \\ = \; & \Pr(Y_i = 0) \Pr(Y_j = 1). \end{aligned}$$

You can similarly verify the two other cases.

Product of Expectations

If X and Y are independent, then

$$\begin{aligned}
E(XY) &= \sum_{X=x,Y=y} \Pr(X = x, Y = y)xy \\
&= \sum_{X=x,Y=y} \Pr(X = x)\Pr(Y = y)xy \\
&= (\sum_{X=x} \Pr(X = x)x)(\sum_{Y=y} \Pr(Y = y)y) \\
&= E(X)E(Y).
\end{aligned}$$

Variance of R_n

We use the power of the indicator variables again to obtain

$$varY_i = EY_i^2 - (EY_i)^2 = \frac{1}{i} - \frac{1}{i^2}.$$

So,

$$\begin{aligned} varR_n &= var\sum_i Y_i \\ &= \sum_i varY_i + \sum_{i \neq j}(EY_iY_j - EY_iEY_j) \\ &= \sum_i varY_i = \sum_i \frac{1}{i} - \sum_i \frac{1}{i^2}. \end{aligned}$$

More Take Away

Since

$$\sum_i \frac{1}{i^2} \to \frac{\pi^2}{6},$$

we have

$$var R_n \approx \ln n + \gamma - \frac{\pi^2}{6}.$$

It can also be proved that Y_i's are independent variables.

Just For Curiosity

Also, since

$$\frac{varY_n}{(ER_n)^2} \approx \frac{1}{n(\ln n)^2},$$

we have

$$\sum_{n=1}^{\infty} \frac{varY_n}{(\sum_{i=1}^{n} EY_i)^2} < \infty.$$

By Kolomogorov convergence criterion, $R_n/ER_n \to 1$ with probability 1. Thus, $R_n \to \infty$ with probability 1 as $n \to \infty$ in the same speed as $\ln n$.

Some Facts

We can also compute N_r, the index of the variable that creates the r-th record.

It turns out that the second record occurs in finite time with probability 1, but the expected value of N_2 actually diverges.

We can also compute $W_r = N_{r+1} - N_r$, the waiting time between records. But you will spare me.

Applications

Famous example: the secretary hiring problem.

Industrial application:
If we want to know under which force a board breaks, we must apply increasing stress until it really breaks.
What if we have 100 boards and we want to find the weakest of them?

$$ER_{100} = H_{100} \approx 5.19$$

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Computer Science application: Treaps :P

The d-min property

Let χ be a finite set of random variables and d be some constant.
χ has the d-min property iff there is some constant c so that for any enumeration $X_1, X_2, \ldots, X_m$ of the elements of any subset of χ

$$\Pr(X_1 < X_2 < \cdots < X_d < \{X_{d+1}, \ldots, X_m\}) \leq \frac{c}{m^{\underline{d}}}$$

Let χ be a set of n random variables, each uniformly distributed over a common integer range of size at least n.
χ has the d-min property if its random variables are $(3d+2)$-wise independence.

Expectation Recurrence

A m-face Dice Game

We throw a fair dice with m faces until a number appears k times consecutively. What is the expected number of throws?

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Let E_k be the expected number of throws. Then for $k \geq 2$,

$$E_k = E_{k-1} + \frac{1}{m} + \frac{m-1}{m} E_k.$$

Simplifying we have $E_k = mE_{k-1} + 1$ for $k \geq 2$.

Since $E_1 = 1$, we have

$$E_k = 1 + m + m^2 + \cdots + m^{k-1} = \frac{m^k - 1}{m - 1}.$$

A Simple Length of Sum

Numbers are drawn randomly with replacement from the set $\{1, 2, \ldots, n\}$ until their sum first exceeds k, for $0 \leq k \leq n$. Find the expected number of draws E_k.

Assume $n \geq 2$ to be interesting.

Clearly, $E_0 = 1$.

Recurrence

Let the outcome of the first draw be i.

If $0 \leq i \leq k$, then we need the expected number of draws E_{k-i} so that the sum will exceed k. If $i > k$, we stop.

$$E_k = 1 + \frac{E_0}{n} + \frac{E_1}{n} + \cdots + \frac{E_{k-1}}{n}$$

It's easy to show by strong induction that

$$E_k = \left(1 + \frac{1}{n}\right)^k.$$

A More Difficult Length of Sum

We will play the same game, except now the domain is $\{0, 1, \ldots, n-1\}$. Any take?

A More Difficult Length of Sum

We will play the same game, except now the domain is $\{0, 1, \ldots, n-1\}$.

Analogous to the previous case, we obtain

$$E_k = 1 + \frac{1}{n} \sum_{i=0}^{k} E_i.$$

And for $k = 0$, we have

$$E_0 = 1 + \frac{1}{n} E_0.$$

If we knew E_0 is finite, then of course $E_0 = \frac{n}{n-1}$.

The Proof

If the sum first exceeds zero in the m-th draw, then zero must be selected $m-1$ times in a row to start, followed by a non-zero. This has probability

$$\left(\frac{1}{n}\right)^{m-1}\frac{n-1}{n}.$$

Thus

$$E_0 = \sum_{m=1}^{\infty}\left(\frac{1}{n}\right)^{m-1}\frac{n-1}{n}m = (n-1)\sum_{m=1}^{\infty}\frac{m}{n^m}.$$

Recognize Me?

$$(n-1)\sum_{m=1}^{\infty}\frac{m}{n^m} = \frac{n}{n-1}$$

Since $E_k \le (k+1)E_0$, we see that $E_1, \ldots, E_{n-1}$ are also finite. Now,

$$E_k = \frac{n}{n-1}\left(1 + \frac{1}{n}\sum_{i=0}^{k-1} E_i\right).$$

Using strong induction, we have

$$E_k = \left(\frac{n}{n-1}\right)^{k+1},$$

for $k = 0, 1, \ldots, n-1$.

Indicator Variables

Expected Value Of The Smallest Element

Consider the set $\{1, 2, \ldots, n\}$. Let $1 \leq r \leq n$.

Pick a subset uniformly at random.
(There are $\binom{n}{r}$ of them.)

Let Z be the smallest number in this subset. Find EZ.

Life Without Indicator Variables

It is painful. Recall the "well-known" identity

$$\sum_{v=0}^{s} \binom{v+k-1}{k-1} = \binom{s+k}{k}.$$

Let $1 \le i \le n - r + 1$.

The number of subsets with r elements such that the smallest element is i equals to

$$\binom{n-i}{r-1}.$$

Pain

Let's sum the smallest elements of those subsets with exactly r elements.

$$
\begin{aligned}
S &= \sum_{i=1}^{n-r+1} i \binom{n-i}{r-1} \\
&= \sum_{i=0}^{n-r} (n+1-r-i) \binom{i+r-1}{r-1} \\
&= (n+1) \sum_{i=0}^{n-r} \binom{i+r-1}{r-1} - \sum_{i=0}^{n-r} (i+r) \binom{i+r-1}{r-1} \\
&= \cdots
\end{aligned}
$$

More Pain

$$
\begin{aligned}
\cdots &= (n+1)\sum_{i=0}^{n-r}\binom{i+r-1}{r-1} - \sum_{i=0}^{n-r}(i+r)\binom{i+r-1}{r-1} \\
&= (n+1)\binom{n}{r} - r\sum_{i=0}^{n-r}\binom{i+r}{r} \\
&= (n+1)\binom{n}{r} - r\binom{n+1}{r+1} = \binom{n+1}{r+1}
\end{aligned}
$$

Since the number of subsets of size r is $\binom{n}{r}$, we have

$$
EZ = \frac{\binom{n+1}{r+1}}{\binom{n}{r}} = \frac{n+1}{r+1}.
$$

Life With Indicator Variables

Define $M = \binom{n+1}{r+1}$ and $N = \binom{n}{r}$.

Let $X_1, \ldots, X_M$ be all subsets with $r+1$ elements from $\{0, 1, \ldots, n\}$.

Let $Y_1, \ldots, Y_N$ be all subsets with r elements from $\{1, 2, \ldots, n\}$.

What is the Matrix?

Introduce a matrix of indicator variables $A_{M\times N} = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if we get } Y_j \text{ after removing} \\ & \quad \text{the smallest element from } X_i \\ 0 & \text{otherwise.} \end{cases}$$

Observations:

- On each row precisely one element is 1.
- The number of 1s in the j-th column is equal to the smallest element of Y_j.

Check

Let $X_1, \ldots, X_M$ be all subsets with $r+1$ elements from $\{0, 1, \ldots, n\}$.

Let $Y_1, \ldots, Y_N$ be all subsets with r elements from $\{1, 2, \ldots, n\}$.

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The Same Sum, Easily

- On each row precisely one element is 1.
- The number of 1s in the j-th column is equal to the smallest element of Y_j.

The sum of the smallest elements of all sets $Y_1, \ldots, Y_N$ is th same as the number of rows of A. Thus,

$$EZ = \frac{M}{N} = \frac{\binom{n+1}{r+1}}{\binom{n}{r}} = \frac{n+1}{r+1}.$$

Indicator Variable Aside

Suppose we have a pack of well-shuffled playing cards.
We flip the cards one by one until we hit the first Ace.

What is the number of cards we expect to flip?

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Suppose we have a pack of well-shuffled playing cards. We flip the cards one by one until we hit the first Ace.

What is the number of cards we expect to flip?

There are $\binom{52}{4}$ ways the four Aces can occupy. We want the minimum index. From our result, this is

$$\frac{52+1}{4+1}.$$

More Generally

Q: You have a dart board with n slots. You throw r darts, disallowing repeats, into the board. What is the index of the smallest dart?

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A:

$$n - \frac{n+1}{r+1} + 1$$

A Continuous Version

Q: On a closed interval from 0 to 1, we throw r darts. What is the expected value of the minimum?

A: $\frac{1}{r+1}$

The Principle of Symmetry:

When r darts are dropped at random on an interval, the lengths of the $r + 1$ line segments have identical distributions.

A Sketch

Imagine $r+1$ darts are being dropped onto a *circle* whose circumference has length 1.

You expect them to spread evenly.

Imagine the last dart, being too sharp, actually breaks the circle. . .

The End

Binary Search With Random Pivot

Imagine we have an sorted array of integers $A[1:n]$ and we want to do a binary search for x.

At each step, given sub-array $A[i:k]$, instead of picking the middle element, we uniformly pick a random index j of the sub-array and compare $A[j]$ with x.

Depending on the result, we either stop, recurse on $A[i:j-1]$ or $A[j+1:k]$.

Let $f(n)$ be the number of calls needed. Find $Ef(n)$.

Maximum of Two

The Art of Waiting

Suppose two players A and B are simultaneously conducting independent trials with probability of success p.

Each player will repeat until success. Let X and Y be the number of trials for A and B, respectively.

The game ends as soon as both players have stopped.

Let Z be the length of the game, i.e. $Z = \max(X, Y)$. Find EZ.

Step 1

Let $q = 1 - p, X' = X - 1, Y' = Y - 1$, and $Z' = \max(X', Y')$. Note that now X' and Y' have distribution $Ge(p)$.

Since $Z' \leq j$ iff $X' \leq j$ and at the same time $Y' \leq j$, we have

$$
\begin{aligned}
\Pr(Z' \leq j) &= \Pr(X' \leq j) \Pr(Y' \leq j) \\
&= \left(\sum_{i=0}^{j} pq^i \right) \left(\sum_{i=0}^{j} pq^i \right) \\
&= (1 - q^{j+1})^2
\end{aligned}
$$

for $j = 0, 1, 2, \ldots$

Step 2

If $j \neq 0$, then

$$\begin{aligned} \Pr(Z' = j) &= \Pr(Z' \leq j) - \Pr(Z' \leq j-1) \\ &= (1 - q^{j+1})^2 - (1 - q^j)^2 \\ &= pq^j(2 - q^j - q^{j+1}). \end{aligned}$$

For $j = 0$, $\Pr(Z' = 0) = \Pr(Z' \leq 0) = p^2$.

$$EZ' = \sum_{j=0}^{\infty} j \Pr(Z' = j) = \frac{q(2+q)}{p(1+q)}.$$

and so

$$EZ = 1 + EZ' = \frac{1+2q}{1-q^2}.$$