# Grasp Analysis as Linear Matrix Inequality Problems *

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## Abstract

Three important problems in the study of grasping and manipulation by multifingered robotic hands are: (a) Given a grasp characterized by a set of contact points and the associated contact models, determine if the grasp has force closure; (b) If the grasp does not have force closure, determine if the fingers are able to apply a specified resultant wrench on the object; and (c) Compute "optimal" contact forces if the answer to problem (b) is affirmative. In this paper, based on an early result by Buss, Hashimoto and Moore, which transforms the nonlinear friction cone constraints into positive definiteness of certain symmetric matrices, we further cast the friction cone constraints into *linear matrix inequalities (LMIs)* and formulate all three of the problems stated above as a set of *convex optimization problems involving LMIs.* The latter problems have been extensively studied in optimization and control community and highly efficient algorithms with *polynomial time* complexity are now available for their solutions. We perform simulation studies to show the simplicity and efficiency of the LMI formulation to the three problems.

## 1 Introduction

Grasping and manipulation by multifingered robotic hands have been active areas of research in robotics over the last two decades, see [8, 9, 12, 13, 14, 17, 18, 19, 20, 21] and references therein for further details. Three important problems in the study of grasping and multifingered manipulation are: (a) *Given a grasp which is characterized by a set of contact points and the associated contact models, determine if the grasp has force closure*; (b) *If the grasp does not have force closure, determine if the fingers are able to apply a specified resultant wrench on the object*; and (c) *Compute "optimal" contact forces if the answer to problem (b) is affirmative.* These three problems will collectively be referred to as *grasp analysis* problems. One may note that these problems also arise in the study of foot-step planning and force distribution by multilegged robots [6]. Other applications of these problems can be found in fixturing, cell manipulation by multiple laser probes, and the control of satellites with multiple unidirectional thrusters.

One major difficulty associated with these problems has been the *nonlinear* constraints of the contact friction models. The most commonly used contact friction models are: (a) point contact with friction (PCWF) and (b) soft-finger contact (SFC). Analytical (quadratic) models for these contact types have been obtained and experimentally verified [8, 12]. Due to the difficulty of handling the nonlinear models, the problem of analyzing and synthesizing force closure grasps was first studied for the simplified frictionless models [13]. While simplifying the analysis, ignoring friction forces, however, leads to grasps with seven or more contacts. Such contacts make control more difficult and require a mechanically complex hands. As for frictional grasps, the force closure theorems [17, 20] have been expressed in geometric terms such as antipodal positions and specialized for the grasps characterized by the number of contact points and the associated contact models.

The problem of grasping force optimization [6, 9] has mainly been studied by linearizing the friction cone constraints and then applying linear programming techniques. The drawbacks of this approach are: (1) the friction cone must be approximated conservatively, (2) the orientation of the tangent plane directions in the contact frame affect the results of grasp analysis (which violates the usual assumption of isotropic Coulomb friction), (3) increasing the ac-

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curacy of the linearized friction model increases the running time unacceptably for real-time applications. Nonlinear programming approaches[15] have also been proposed for the grasping force optimization problem. However, current computing resources can only allow off-line analysis.

One major progress in the study of grasping force optimization was made by Buss, Hashimoto and Moore (BHM) [3]. They made the important observation that the nonlinear friction cone constraints are equivalent to positive definiteness of certain symmetric matrices. Consequently, the grasping force optimization problem was formulated as an optimization problem on the Riemannian manifold of linearly constrained symmetric positive definite matrices and solved by projected gradient flow methods [3, 2]. Various experimental studies [4, 5, 10] showed the efficiency of this approach. The optimization algorithm, however, needs valid contact forces, which satisfy the friction cone constraints and generate the specified object wrench, as the starting point. The initial contact forces are not easy to compute for general grasps and make their algorithm not applicable for solving force closure and grasping force existence problems.

In this paper, based on the BHM observation and a detailed analysis of the structure of the symmetric positive definite matrices arising from the friction cone constraints, we further cast the friction cone constraints into *linear matrix inequalities (LMIs)* and formulate the basic grasp analysis problems as a set of *convex optimization problems involving LMIs* [1]. The latter problems have been extensively studied in optimization and control community and highly efficient algorithms with *polynomial time* complexity [16, 1], are now available for their solutions. We perform simulation studies to show the simplicity and efficiency of the LMI formulation to the three problems.

## 2 Problem Review

Consider an object grasped by a $k$-fingered robotic hand. The grasp map, $G \in \mathrm{R}^{6\times m}$, transforms applied finger forces expressed in local contact frames to resultant object wrenches

$$F = Gx \tag{1}$$

where $x = [x_1^T \dots x_i^T \dots x_k^T]^T \in \mathcal{F} \subset \mathrm{R}^m$ is the contact wrench of the grasp, $x_i \in \mathrm{R}^{m_i}$ the independent wrench intensity vector of finger $i$ which is constrained to the friction cone $\mathcal{F}_i \subset \mathrm{R}^{m_i}$ of the respective contact model, $m = \sum_{i=1}^k m_i$ the dimension of total independent contact wrenches of the grasp and $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_k \subset \mathrm{R}^m$ the friction cone of the grasp.

For a PCWF contact, we have $m_i = 3$ and

$$\mathcal{F}_i = \left\{ x_i \in \mathrm{R}^3 \mid x_{i3} \geq 0, \;\; \frac{1}{\mu_i^2}(x_{i1}^2 + x_{i2}^2) \leq x_{i3}^2 \right\} \tag{2}$$

where $x_{i3}$ is the normal force component at the point of contact, $x_{i1}, x_{i2}$ the tangential components and $\mu_i$ is the coefficient of Coulomb friction.

For a soft-finger contact, we have $m_i = 4$ and

$$\mathcal{F}_i = \left\{ x_i \in \mathrm{R}^4 \mid x_{i3} \geq 0, \frac{1}{\mu_i}(x_{i1}^2 + x_{i2}^2) + \frac{1}{\mu_{it}} x_{i4}^2 \leq x_{i3}^2 \right\} \tag{3}$$

where $x_{i4}$ is the component of moment about the contact normal and $\mu_{it}$ a proportionality constant between the torsion and shear limits. Note that the friction cone constraints given in equation (3) is based on an elliptic approximation [8]. A linearized version of the above model which has also been investigated through experimental studies [8] is given by:

$$\mathcal{F}_i = \left\{ x_i \in \mathrm{R}^4 \mid x_{i3} \geq 0, \frac{1}{\mu_i}\sqrt{(x_{i1}^2 + x_{i2}^2)} + \frac{1}{\mu_{it}'}|x_{i4}| \leq x_{i3} \right\} \tag{4}$$

where $\mu_{it}'$ still models the relation between the torsion and shear limits at the point of contact, but differs in value from $\mu_{it}$ in the elliptic model.

The grasping force $x$ is exerted by hand joint effort $\tau_h \in \mathrm{R}^n$ and their relation is given by

$$J^T x + g_{ext} = \tau_h$$

where $J^T$ is the transpose of the hand Jocobian[14] $J \in \mathrm{R}^{m\times n}$ and $g_{ext} \in \mathrm{R}^n$ is the vector of generalized forces experienced by the joints due to external loads such as gravity (and Coriolis, centripetal, and inertial loads if the hand is moving).

Assume that the elements of the joint effort vector are bounded by known constants: the lower bound $\tau^L \in \mathrm{R}^n$ and the upper bound $\tau^U \in \mathrm{R}^n$, i.e.

$$\tau^L \leq \tau_h \leq \tau^U \tag{5}$$

Then the admissible grasping force $x$ must satisfy *joint effort constraints*

$$\mathcal{T} = \{x \in \mathrm{R}^m | \tau^L \leq J^T x + g_{ext} \leq \tau^U\} \tag{6}$$

Collectively, the friction cone constraints $\mathcal{F}$ and joint effort constraints $\mathcal{T}$ will be referred to as the *grasping force constraints.*

**Problem 1** Force closure problem
*Given a grasp $(G, \mathcal{F})$, determine if it has force closure, i.e., $G(\mathcal{F}) = \mathrm{R}^6$.*

**Problem 2** Grasping force existence problem
*Given a grasp $(G, \mathcal{F})$, joint effort constraints $\mathcal{T}$ and an object wrench $F = (f, \tau) \in \mathrm{R}^6$, determine if there exists a grasping force $x \in \mathcal{F} \cap \mathcal{T}$ such that $Gx = F$.*

**Problem 3** Grasping force optimization problem
*Given a grasp $(G, \mathcal{F})$, joint effort constraints $\mathcal{T}$ and an object wrench $F = (f, \tau) \in \mathrm{R}^6$, find an optimal grasping force $x \in \mathcal{F} \cap \mathcal{T}$ such that $Gx = F$.*

## 3 Formulating Grasping Force Constraints as LMIs

First, let us recall the important observation made by Buss, Hashimoto and Moore [3] that the friction cone constraints $\mathcal{F}$ with contact models specified in (2), (3) and (4) are equivalent to positive semi-definiteness of the matrix

$$P = \mathrm{Blockdiag}(P_1, \cdots, P_{i-1}, P_i, P_{i+1}, \cdots, P_k),$$

where for a PCWF contact

$$P_i = \begin{bmatrix} \mu_i x_{i3} & 0 & x_{i1} \\ 0 & \mu_i x_{i3} & x_{i2} \\ x_{i1} & x_{i2} & \mu_i x_{i3} \end{bmatrix} \quad (7)$$

and for a soft-finger contact with elliptic approximation (SFCE),

$$P_i = \begin{bmatrix} x_{i3} & 0 & 0 & \alpha_i x_{i1} \\ 0 & x_{i3} & 0 & \alpha_i x_{i2} \\ 0 & 0 & x_{i3} & \beta_i x_{i4} \\ \alpha_i x_{i1} & \alpha_i x_{i2} & \beta_i x_{i4} & x_{i3} \end{bmatrix} \quad (8)$$

where $\alpha_i = \sqrt{\frac{1}{\mu_i}}, \beta_i = \sqrt{\frac{1}{\mu_{it}}}$. For a soft-finger contact with linear approximation (SFCL), we have

$$P_i = \begin{bmatrix} x_{i3} & 0 & 0 & 0 & & & \\ 0 & \delta_i & 0 & x_{i1} & & & \\ 0 & 0 & \delta_i & x_{i2} & & 0 & \\ 0 & x_{i1} & x_{i2} & \delta_i & & & \\ & & & & \sigma_i & 0 & x_{i1} \\ & & 0 & & 0 & \sigma_i & x_{i2} \\ & & & & x_{i1} & x_{i2} & \sigma_i \end{bmatrix} \quad (9)$$

where $\delta_i = \mu_i(x_{i3} + \frac{1}{\mu'_{it}} x_{i4}), \sigma_i = \mu_i(x_{i3} - \frac{1}{\mu'_{it}} x_{i4})$.

We make another important observation that the matrix $P_i$ in (7), (8) or (9) is in fact linear in the contact wrench vector $x_i \in \mathrm{R}^{m_i}$ and has the form

$$\boxed{P_i = \sum_{j=1}^{m_i} x_{ij} S_{ij} = x_{i1} S_{i1} + \cdots + x_{im_i} S_{im_i}} \quad (10)$$

where $S_{ij}, j = 1, \cdots, m_i$, are real *symmetric* constant matrices. For example, the $S'_{ij}$s for a PCWF contact are

$$\begin{aligned} S_{i1} &= E^3_{13} + E^3_{31} \\ S_{i2} &= E^3_{23} + E^3_{32} \\ S_{i3} &= \mu_i(E^3_{11} + E^3_{22} + E^3_{33}) \end{aligned} \quad (11)$$

where $E^a_{bc}$ stands for a square matrix of dimension $a$ with element $(b, c)$ to be 1 and all others to be zero. Note that $E^a_{bc} = (E^a_{cb})^T$. Likewise, for a SFCE contact, the $S'_{ij}$s are given by

$$\begin{aligned} S_{i1} &= \alpha_i(E^4_{14} + E^4_{41}) \\ S_{i2} &= \alpha_i(E^4_{24} + E^4_{42}) \\ S_{i3} &= E^4_{11} + E^4_{22} + E^4_{33} + E^4_{44} \\ S_{i4} &= \beta_i(E^4_{34} + E^4_{43}) \end{aligned} \quad (12)$$

and for a SFCL contact by

$$\begin{aligned} S_{i1} &= E^7_{24} + E^7_{42} + E^7_{57} + E^7_{75} \\ S_{i2} &= E^7_{34} + E^7_{43} + E^7_{67} + E^7_{76} \\ S_{i3} &= E^7_{11} + \mu_i \sum_{j=2}^{7} E^7_{jj} \\ S_{i4} &= \frac{\mu_i}{\mu'_{it}}(\sum_{j=2}^{4} E^7_{jj} - \sum_{j=5}^{7} E^7_{jj}) \end{aligned} \quad (13)$$

Since P is block diagonal with the $P'_i s$ on the main diagonal, it can be written as:

$$\begin{aligned} P(x) &= \mathrm{Blockdiag}(P_1, \cdots, P_{i-1}, P_i, P_{i+1}, \cdots, P_k) \\ &= \sum_{i=1}^{k} \mathrm{Blockdiag}(0, \cdots, 0, P_i, 0, \cdots, 0) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{m_i} x_{ij} \mathrm{Blockdiag}(0, \cdots, 0, S_{ij}, 0, \cdots, 0) \\ &= \sum_{l=1}^{m} x_l S_l \end{aligned} \quad (14)$$

where the double-indexed $x_{ij}$ is simplified to $x_l$, $l(i, j) = \sum_{b=1}^{i-1} m_b + j$ and $S_l = \mathrm{Blockdiag}(0, \cdots, 0, S_{ij}, 0, \cdots, 0)$, $l = 1, \cdots, m$. Note that the $S'_l s$ remain symmetric.

In summary, the friction cone constraints are equivalent to the positive definiteness of a linear combination of constant symmetric matrices, i.e.,

$$\boxed{P(x) = \sum_{l=1}^{m} x_l S_l \geq 0}. \quad (15)$$

where $S_l, l = 1, \cdots m$, are constant and symmetric. Replacing $\geq$ in equation (15) by $>$ defines the constraint of the interior of the friction cone, $int(\mathcal{F})$.

An inequality of the form (15) is a special case, with $S_0$ being zero, of what is called a nonstrict *linear matrix inequality(LMI)* [1]

$$Q(x) = S_0 + \sum_{l=1}^{m} x_l S_l \geq 0 \quad (16)$$

where $x \in \mathrm{R}^m$ is the variable and the constant matrices $S_l's$ are symmetric. The interior friction cone constraint corresponds to a strict linear matrix inequality whose general form is

$$Q(x) = S_0 + \sum_{l=1}^{m} x_l S_l > 0 \quad (17)$$

Recall that a set $\mathcal{A} \subset \mathrm{R}^m$ is convex if $\forall a_1, a_2 \in \mathcal{A}, \lambda \in [0,1]$, $\lambda a_1 + (1-\lambda) a_2$ is also in $\mathcal{A}$. One key property of LMIs is that both nonstrict LMIs (16) and strict LMIs (17) are convex constraints on $x$, i.e., their feasible sets are convex, as shown in the following proposition[7].

**Proposition 1** *Given $Q(x) = S_0 + \sum_{l=1}^{m} x_l S_l$, where $S_l = S_l^T, l = 0, \cdots, m$. The sets $\mathcal{A}_n = \{x \in \mathrm{R}^m \mid Q(x) \geq 0\}$ and $\mathcal{A}_s = \{x \in \mathrm{R}^m \mid Q(x) > 0\}$ are convex.*

In general, LMIs can be viewed as an extension of linear inequality constraints where the componentwise inequalities between vectors are replaced by matrix inequalities. It is shown in [1] that the LMIs(16) (17) can represent a wide class of convex constrains on $x$ such as linear inequalities, (convex) quadratic inequalities and matrix norm inequalities. Here we will only take a linear inequality constraint as an example. Consider

$$Ax + b \geq 0 \quad (18)$$

where $A = [a_1 \cdots a_m] \in \mathrm{R}^{n \times m}$ and $b \in \mathrm{R}^n$. Since a vector $y \geq 0$ (componentwise) if and only if the matrix $\mathrm{diag}(y)$ (the diagonal matrix with the components of $y$ on its diagonal) is positive semi-definite, the linear inequality constraint(18) can be cast into a nonstrict LMI with $Q(x) = \mathrm{diag}(Ax + b)$, i.e.,

$$S_0 = \mathrm{diag}(b), S_i = \mathrm{diag}(a_i), i = 1, \cdots, m. \quad (19)$$

As a direct application of this example, partition the joint effort constraints $\mathcal{T}$ in (6) into two linear inequality constraints:

$$J^T x + g_{ext} - \tau^L \geq 0, \quad -J^T x - g_{ext} + \tau^U \geq 0 \quad (20)$$

and formulate the corresponding LMIs as:

$$C^L(x) = \mathrm{diag}(J^T x + g_{ext} - \tau^L) = C_0^L + \sum_{l=1}^{m} x_l C_l^L \geq 0$$

$$C^U(x) = \mathrm{diag}(-J^T x - g_{ext} + \tau^U) = C_0^U + \sum_{l=1}^{m} x_l C_l^U \geq 0$$

where $C_0^L = \mathrm{diag}(g_{ext} - \tau^L), C_0^U = \mathrm{diag}(-g_{ext} + \tau^U), C_l^L = \mathrm{diag}(J_l^T), C_l^U = \mathrm{diag}(-J_l^T), l = 1, \ldots, m$, and $J_l^T$ is the $l^{th}$ column of matrix $J^T$.

Therefore, the joint effort constraints(6) can also be cast into one LMI constraint:

$$\boxed{C(x) = \mathrm{Blockdiag}(C^L(x), C^U(x)) = C_0 + \textstyle\sum_{l=1}^{m} x_l C_l \geq 0} \quad (21)$$

where $C_l = \mathrm{Blockdiag}(C_l^L, C_l^U), l = 0, \ldots, m$.

## 4 Grasp Analysis Problems

Based on the LMI formulation of grasping force constraints, we now reformulate the grasp analysis problems as follows:

**Problem 1** Force Closure Problem
*Given a grasp $(G, \mathcal{F})$, determine if for every $F \in \mathrm{R}^6$, $\exists\ x \in \mathrm{R}^m$, such that $P(x) \geq 0$ and $Gx = F$.*

**Problem 2** Grasping Force Existence Problem
*Given a grasp $(G, \mathcal{F})$, joint effort constraints $\mathcal{T}$ and an object wrench $F \in \mathrm{R}^6$, determine if $\exists\ x \in \mathrm{R}^m$, such that $P(x) \geq 0$, $C(x) \geq 0$ and $Gx = F$.*

**Problem 3** Grasping Force Optimization Problem
*Given a grasp $(G, \mathcal{F})$, joint effort constraints $\mathcal{T}$ and an object wrench $F \in \mathrm{R}^6$, find an "optimal" grasping force $x \in \mathrm{R}^m$ satisfying $P(x) \geq 0$, $C(x) \geq 0$ and $Gx = F$.*

In this section, we will analyze these problems and transform them into standard *convex optimization problems involving LMIs.* The resulting problems can be efficiently solved in *polynomial time* using recently developed interior-point methods [16, 1].

### 4.1 Force Closure Problem

It is shown that a grasp has force closure if and only if the grasp map $G$ has full row rank and there exists a strictly internal grasping force[14]. In other words, the following two conditions are simultaneously satisfied:

1. $\mathrm{rank}(G) = 6$; and
2. $\exists x_{int} \in \mathrm{R}^m$, s.t. $P(x_{int}) > 0$ and $Gx_{int} = 0$.

While verifying the first condition is straightforward, the second condition, i.e. *the existence of a strictly internal force*, is difficult due to the nonlinear friction constraints. To resolve this problem, note that $x_{int}$ lies in the null space of $G$ and since the rank condition is satisfied, there exists $z \in \mathrm{R}^{m-6}$ such that

$$x_{int} = Vz \quad (22)$$

where the columns of $V \in \mathrm{R}^{m\times(m-6)}$ is a basis for the null space of grasp map $G$.

Substituting equation (22) into the LMI $P(x) > 0$, we obtain an equivalent LMI in terms of $z$, which encodes the null space and the friction cone constraints for strictly internal forces:

$$\boxed{\tilde{P}(z) := P(Vz) = \sum_{l=1}^{m-6} z_l \tilde{S}_l > 0} \qquad (23)$$

$\tilde{P}(z)$ is indeed a LMI since LMI structure is preserved under affine transformations as shown in the following proposition[7].

**Proposition 2** *Given $Q(x) = S_0 + \sum_{l=1}^{m} x_l S_l$, where $S_l = S_l^T, l = 0, \cdots, m$. Let $x = Az + b$, where $A \in \mathrm{R}^{m\times n}$, $b \in \mathrm{R}^m$ and $z \in \mathrm{R}^n$ is the new variable. Then $\tilde{Q}(z) := Q(Az+b)$ has LMI structure, i.e., $\tilde{Q}(z) = \tilde{S}_0 + \sum_{l=1}^{n} z_l \tilde{S}_l$, and $\tilde{S}_l = \tilde{S}_l^T, l = 0, \cdots, n$.*

In summary, the force closure problem is determined by first checking the rank of $G$ and, if it is onto, then determining if there exists a $z \in \mathrm{R}^{m-6}$ such that (23) holds. The latter problem is a standard *LMI feasibility problem* [1] and efficient algorithms exist for its solution.

### 4.2 Grasping Force Existence Problem

The *grasping force existence problem* is very similar to the *internal force existence problem* and can be solved using a similar approach: First, determine if there exists a solution $x_0 \in \mathrm{R}^m$ for the linear equation

$$Gx_0 = F \qquad (24)$$

Here, $x_0 \in \mathrm{R}^m$ need not satisfy the grasping force constraints. Thus, a simple choice is the least-square solution:

$$x_0 = G^{\#} F \qquad (25)$$

where $G^{\#}$ is the generalized inverse of $G$. The solution $x_0$ is exact if $F \in \mathrm{Range}(G)$. Otherwise, the answer to the grasping force existence problem is negative. For the case that $F \in \mathrm{Range}(G)$, the general solution of equation(24) has the form

$$x = x_0 + Vz = G^{\#}F + Vz \qquad (26)$$

where the columns of $V \in \mathrm{R}^{m\times(m-r)}$ is a basis for the null space of $G$ and $r$ is the rank of $G$.

Thus, the answer to the grasping force existence problem is positive if and only if $F \in \mathrm{Range}(G)$ and there exists $z \in \mathrm{R}^{m-r}$ such that the LMIs hold:

$$\boxed{\tilde{P}(z) := P(x_0 + Vz) = \tilde{S}_0 + \sum_{l=1}^{m-r} z_l \tilde{S}_l \geq 0}$$
$$\boxed{\tilde{C}(z) := C(x_0 + Vz) = \tilde{C}_0 + \sum_{l=1}^{m-r} z_l \tilde{C}_l \geq 0} \qquad (27)$$

Again, the latter one is a *LMI feasibility problem.*

### 4.3 Grasping Force Optimization Problem

Given a grasp $(G, \mathcal{F})$, joint effort constraints $\mathcal{T}$ and an object wrench $F$, the *grasping force optimization problem* amounts to finding an optimal grasping force $x$ in the feasible set

$$\mathcal{A}_x = \{x \in \mathrm{R}^m | P(x) \geq 0, C(x) \geq 0, Gx = F\}. \qquad (28)$$

Here, we only consider the nontrivial case when the feasible set $\mathcal{A}_x$ is nonempty. This is true if and only if the answer to the corresponding *grasping force existence problem* is affirmative. In this case, there exists a feasible set for $z$:

$$\mathcal{A}_z = \{z \in \mathrm{R}^{m-r} | \tilde{P}(z) \geq 0, \tilde{C}(z) \geq 0\} \qquad (29)$$

where $\tilde{P}(z)$ and $\tilde{C}(z)$ are defined in (27).

We define a measure of optimality for grasping forces by

$$\Psi(x) = w^T x + \ln det P^{-1}(x) \qquad (30)$$

where the vector $w = [w_1^T \cdots w_i^T \cdots w_k^T]^T \in \mathrm{R}^m$ is used to weight the normal components of the grasping force $x$, for a PCWF contact $w_i$=[0 0 $d_i]^T$ and for a SFC contact $w_i$=[0 0 $d_i$ $0]^T$, $d_i \geq 0$. This objective function (30) is very similar to the self-concordant one proposed in [2] and can be interpreted as follows: the first term grows with the magnitudes of the normal components of the contact forces; and the second term is a barrier term which tends to infinity as any contact force approaches the boundary of its friction cone. More discussions on the weight can be found in [3, 5].

The *grasping force optimization* problem can therefore be stated as follows

$$\mathrm{argmin}_{x\in\mathcal{A}_x} \Psi(x) \qquad (31)$$

Let $x = G^{\#}F + Vz$ and

$$\tilde{\Psi}(z) := \Psi(G^{\#}F + Vz).$$

The problem can be transformed into a problem of

$$\mathrm{argmin}_{z\in\mathcal{A}_z} \tilde{\Psi}(z) \qquad (32)$$

Recall that a function $f(x)$ is strictly convex if

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2),$$

$\forall\lambda \in (0,1)$ and $\forall x_1, x_2 \in$ domain(f) $\subset \mathrm{R}^m$. In order to apply the convex optimization algorithms in [16, 1] to the above problems we need to show that the objective function and the domain of the function are both convex[7].

**Proposition 3** *The constraint domains $\mathcal{A}_x$ in (28) and $\mathcal{A}_z$ in (29) are convex.*

**Proposition 4** *The functions $\Psi(x)$ and $\tilde{\Psi}(z)$ are strictly convex functions on the sets $\mathcal{A}_x$ and $\mathcal{A}_z$, respectively.*

Problems (31) and (32) are thus shown to be convex optimization problems involving LMIs. Problem (32) is also in the standard form of *determinant maximization problem with LMI constraints* [23] and can be efficiently solved using existing software package.

**Remark 1.** The barrier term $\ln detP^{-1}(x)$ in the objective function (30), in fact, requires $P(x) > 0$. Another way to define a convex objective function is to only include the linear term $w^T x$. Then the whole friction cone $P(x) \geq 0$ will be used in the optimization. Such a linear objective function will lead the grasping force optimization problem to a standard *semi-definite programming* problem [22] or a standard *second order cone programming* problem [11], depending on the friction cone constraints being cast into LMIs(7, 8, 9) or the conventional cone constraints (2,3, 4). Paper[11] includes a brief discussion on the grasping force optimization problem as an engineering application of the second order cone progamming.

## 5 Simulation Results

In this section, we discuss the simulation results obtained from applying software package *maxdet* by Wu, Vandenberghe and Boyd [24] to various grasp analysis problems. We downloaded the source code of *maxdet* (written in ANSI C) from http://www.stanford.edu/~boyd/MAXDET.html and extended it to record the contact forces and objective values during the optimization procedure. Please refer to paper[23] and manual[24] for the information of the algorithm and the software package.

To simplify the presentation, we assumed lower and upper bounds for the contact force components as a simplified way to incorporate joint effort constraints. This exempted us from presenting a long description of the Jocobian matrix $J$ and allowed our presentation to focus on the properties of various grasps. All simulations presented here used -10 and 10 as lower and upper bounds for all contact wrenches. Other simulation parameters can be found in our technical report[7].

All our simulations were done within matlab5 on a Sun SPARCstation 4. For each example, besides the convergence of the normal wrenches, we will also report, where applicable, the *feasibility time, optimization time and computation time*, which are defined, respectively, as the time to determine the feasibility of the LMI, the time to optimize the feasible optimization problem and the total time used to solve the given problem, including preparing the LMI constraints, determining the feasibility and optimizing the objectives. All simulations observed monotone decreasing objective values during the optimization procedure and we will only show the trend of the objective function for one example. More simulation results can be found in our technical report [7].

### 5.1 A Four-Fingered Grasp

Here we considered the same numerical example as given in paper [2]. It has four hard fingers grasping a rectangular prism with PCWF contacts. Using the grasp map, object wrench and valid initial contact forces given in paper[2], our LMI simulation results of normal contact wrenches are shown in figure(1), similar to what was observed by Buss *et. al.* (see figures 4 and 6 in paper [2]). It was reported in paper [2] that the computation times using Matlab on a SUN SPARCstation 20 for three continuous gradient flow methods (without Dikin's algorithm) were 9sec, 38sec and 60sec. Our implementation of the discrete version of their fastest algorithm for this particular example used 1.45sec for optimization and 3.42sec overall on our platform (different from the one reported in paper [2]). By contrast, the optimization and total computation times of the LMI simulation on our platform were 0.27sec and 0.87sec, respectively.

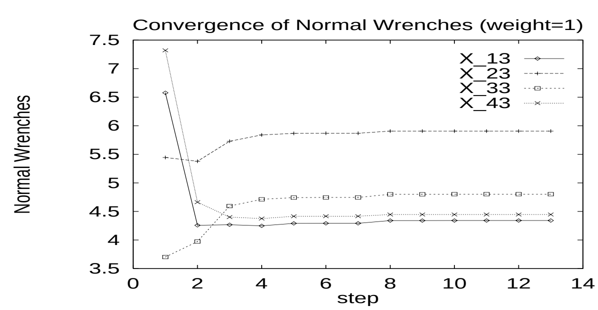


Figure 1: Four-Fingered PCWF Grasp

### 5.2 Two-Fingered Grasps

Consider the case that two fingers grasp a unit sphere at its south pole and north pole. We know that such a grasp has force closure if the contacts are soft finger contacts, but it does not if they are point contacts. The infeasibility of force closure under the PCWF model is determined by the rank of the corresponding grasp map. As for the soft finger contact, *maxdet* needs to be used to determine the existence of a strictly internal force. When the weights $d_i's$ in the objective function were all set to 0.1, 1 and 10, the corresponding feasibility times were 0.34sec, 0.35sec and 0.35sec for SFCL contacts. The strictly internal forces found in this case were the normal forces with equal magnitudes for both fingers. Figure(2) shows the optimization of the normal forces for different weights. As observed by Buss, Hashimoto and Moore [3], a smaller weight resulted in larger normal wrenches and therefore a tighter grasp on the object. Similar trends were observed for soft finger contacts with elliptic approximations[7].

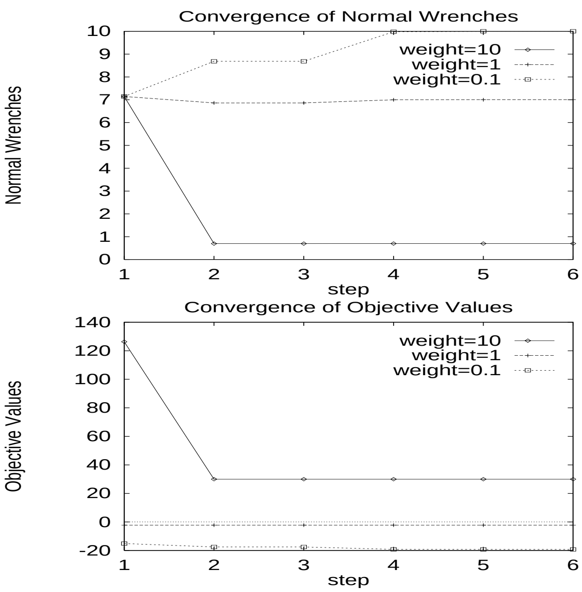


Figure 2: Two-Fingered SFCL Antipodal Grasp

### 5.3 Three-Fingered Grasps

Consider two three-fingered grasps of a unit sphere on its equator. The first scenario is that two fingers, say, finger 2 and 3, are close to each other and are both close to the antipodal point of finger 1. The second grasp has three fingers that are $120^o$ apart from each other, *i.e.*, the contact points form a equilateral triangle. In this section, these two grasps will be referred to as the antipodal grasp and the equilateral grasp. It is easy to prove that both grasps have force closure under our simulation parameters [7], no matter what contact models are used. However, because of the constraints on the contact wrenches, they cannot generate arbitrary object wrenches. Figure(3) shows the simulation results for generating object wrench (17.000,-0.5472,0,0,0,0) by the three-fingered PCWF antipodal grasp. The feasibility, optimization and computation times were 0.32sec, 0.21sec and 0.82 sec, respectively. It took *maxdet* 0.31sec to determine the infeasibility of the given object wrench for the PCWF equilateral grasp. This conclusion can be easily verified by straightforward computation [7].

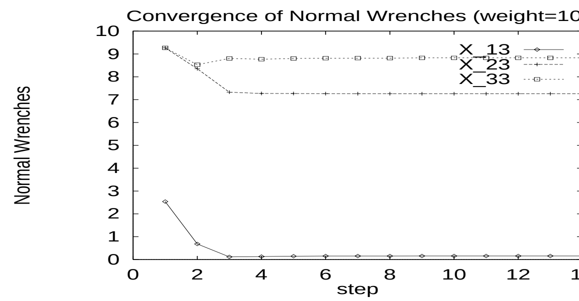


Figure 3: Three-Fingered PCWF Antipodal Grasp

**Remark 2.** The simulation in this subsection (in fact, all simulations we performed) required more time to determine feasibility than to optimize the wrenches. This implies that feasibility is a harder problem than optimization. Nonetheless, we have demonstrated that the tools and algorithms of LMI theory can be used to solve this problem quickly. More generally, our new LMI formulations of the three fundamental grasp analysis problems (with any number of contacts, each of any type) can be solved readily by combining algorithms from LMI theory and convex optimization.

## 6 Conclusion

Grasp analysis is of fundamental importance in robotics, yet despite many years of research effort, efficient solutions to general formulations of some of the basic problems, such as grasp feasibility, have not previously been developed. The major stumbling block has been the *nonlinear* friction cone constraints imposed by the contact models. In this paper, based on the important observation by Buss, Hashimoto and Moore [3], that the nonlinear friction cone constraints are equivalent to the constraint that certain symmetric matrices be positive definite, we further cast the friction cone constraints into *linear matrix inequalities (LMIs)* and formulate the basic grasp analysis problems as a set of *convex optimization problems involving LMIs.* The resulting problems can be solved in *polynomial time* by highly efficient algorithms. Our simulation results showed the simplicity and efficiency of this approach.

Convex optimization has found wide applications in various areas such as control and system theory, combinatorial optimization, statistics, computational geometry and pattern recognition. It can efficiently solve problems involving nonlinear and nondifferentible functions, which would be considered to be very difficult in a standard treatment of optimization.

Backed with its natural application to grasp analysis problems, it appears that convex optimization will play an active role in solving complicated mathematical and engineering problems in robotics.

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