15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

Chomsky Normal Form
and
TURING MACHINES

TUESDAY Feb 4

CHOMSKY NORMAL FORM

A context-free grammar is in Chomsky normal form if every rule is of the form:

$A \to BC$	**B and C aren't start variables**
$A \to a$	**a is a terminal**
$S \to \varepsilon$	**S is the start variable**

Any variable A that is not the start variable can only generate strings of length > 0

CHOMSKY NORMAL FORM

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$S \to \varepsilon$	S is the start variable

$S \to 0S1$
$S \to TT$
$T \to \varepsilon$

$S_0 \to TU \mid TV \mid \varepsilon$
$T \to 0$
$U \to SV$
$S \to TU \mid TV$
$V \to 1$

Theorem: If G is in CNF, $w \in L(G)$ and $|w| > 0$, then any derivation of w in G has length $2|w| - 1$

Proof (by induction on $|w|$):

Base Case: If $|w| = 1$, then any derivation of w must have length 1

Inductive Step: Assume true for any string of length at most $k \geq 1$, and let $|w| = k+1$

Since $|w| > 1$, derivation starts with $A \rightarrow BC$

So $w = xy$ where $B \Rightarrow^* x$, $|x| > 0$ and $C \Rightarrow^* y$, $|y| > 0$

By the inductive hypothesis, the length of any derivation of w must be

$$1 + (2|x| - 1) + (2|y| - 1) = 2(|x| + |y|) - 1$$

Theorem: Any context-free language can be generated by a context-free grammar in Chomsky normal form

“Can transform any CFG into Chomsky normal form”

Theorem: **Any context-free language can be generated by a context-free grammar in Chomsky normal form**

Proof Idea:

1. Add a new start variable
2. Eliminate all $A \rightarrow \varepsilon$ rules. Repair grammar
3. Eliminate all $A \rightarrow B$ rules. Repair

4. Convert $A \rightarrow u_1 u_2 \dots u_k$ to $A \rightarrow u_1 A_1, A_1 \rightarrow u_2 A_2, \dots$

If u_i is a terminal, replace u_i with U_i and add $U_i \rightarrow u_i$

1. Add a new start variable S_0 and add the rule $S_0 \rightarrow S$

$S \rightarrow 0S1$

$S \rightarrow T\#T$

$S \rightarrow T$

$T \rightarrow \varepsilon$

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$S_0 \rightarrow S$

$S \rightarrow 0S1$

$S \rightarrow T\#T$

$S \rightarrow T$

$T \rightarrow \varepsilon$

2. Remove all $A \to \varepsilon$ rules (where A is not S_0)

For each occurrence of A on right hand side of a rule, add a new rule with the occurrence deleted

If we have the rule $B \to A$, add $B \to \varepsilon$, unless we have previously removed $B \to \varepsilon$

$S_0 \to S$
$S \to 0S1$
$S \to T\#T$
$S \to T$
$T \to \varepsilon$
$S \to T\#$
$S \to \#T$
$S \to \#$

$S \to 01$
$S_0 \to \varepsilon$

2. Remove all $A \rightarrow \varepsilon$ rules (where A is not S_0)

For each occurrence of A on right hand side of a rule, add a new rule with the occurrence deleted

If we have the rule $B \rightarrow A$, add $B \rightarrow \varepsilon$, unless we have previously removed $B \rightarrow \varepsilon$

3. Remove unit rules $A \rightarrow B$

Whenever $B \rightarrow w$ appears, add the rule $A \rightarrow w$ unless this was a unit rule previously removed

$S_0 \rightarrow S$

$S \rightarrow 0S1$

$S \rightarrow T\#T$

$S \rightarrow T$

$T \rightarrow \varepsilon$

$S \rightarrow T\#$

$S \rightarrow \#T$

$S \rightarrow \#$

$S \rightarrow 01$

$S_0 \rightarrow \varepsilon$

$S_0 \rightarrow 0S1$

4. Convert all remaining rules into the proper form:

$S_0 \to 0S1$

$S_0 \to A_1A_2$

$A_1 \to 0$

$A_2 \to SA_3$

$A_3 \to 1$

$S_0 \to 01$

$S_0 \to A_1A_3$

$S \to 01$

$S \to A_1A_3$

$S_0 \to \varepsilon$

$S_0 \to 0S1$

$S_0 \to T\#T$

$S_0 \to T\#$

$S_0 \to \#T$

$S_0 \to \#$

$S_0 \to 01$

$S \to 0S1$

$S \to T\#T$

$S \to T\#$

$S \to \#T$

$S \to \#$

$S \to 01$

Convert the following into Chomsky normal form:

$A \to BAB \mid B \mid \varepsilon$

$B \to 00 \mid \varepsilon$

$S_0 \to A$

$A \to BAB \mid B \mid \varepsilon$

$B \to 00 \mid \varepsilon$

$S_0 \to A \mid \varepsilon$

$A \to BAB \mid B \mid BB \mid AB \mid BA$

$B \to 00$

$S_0 \to BAB \mid 00 \mid BB \mid AB \mid BA \mid \varepsilon$

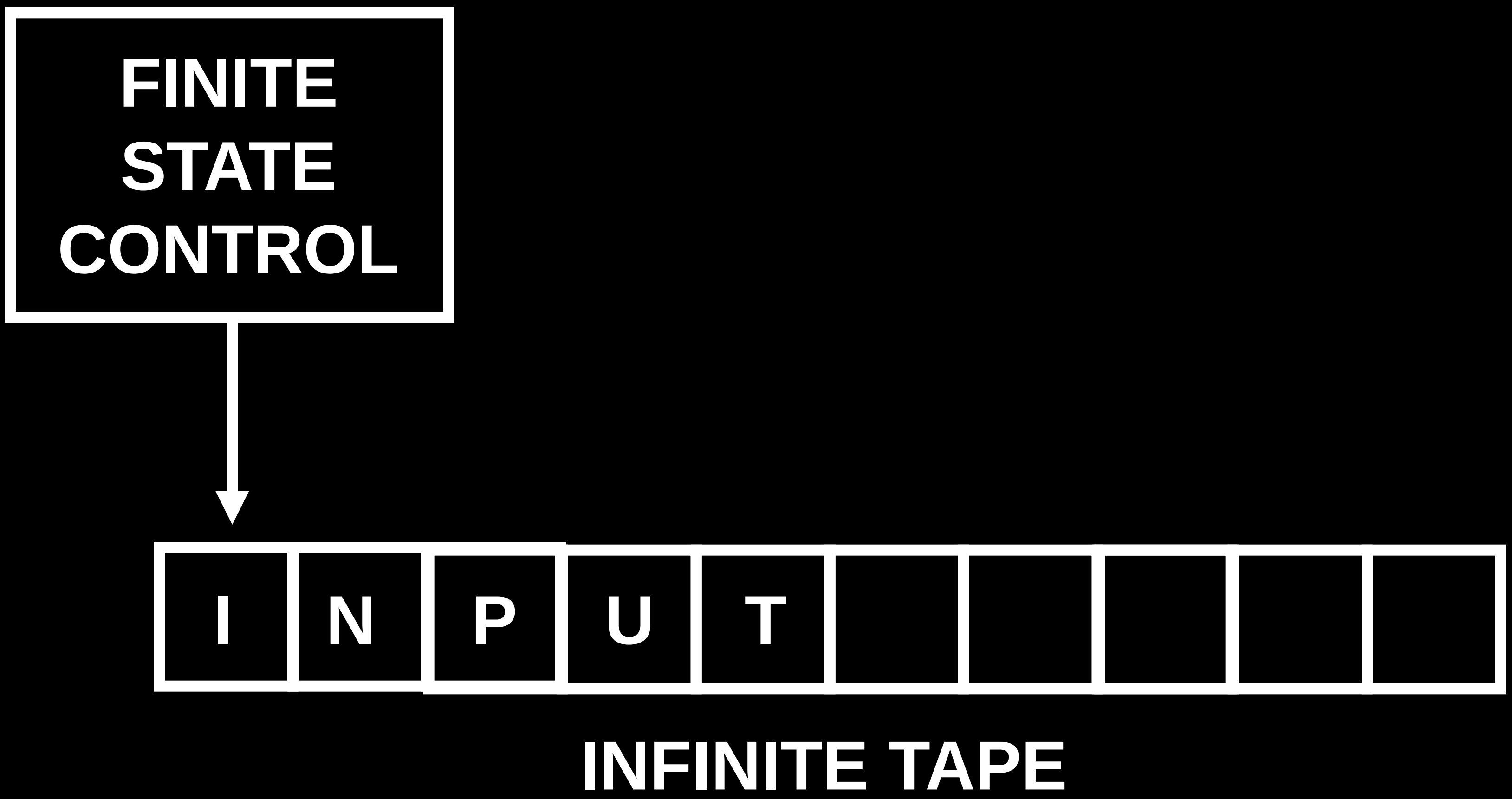
$A \to BAB \mid 00 \mid BB \mid AB \mid BA$

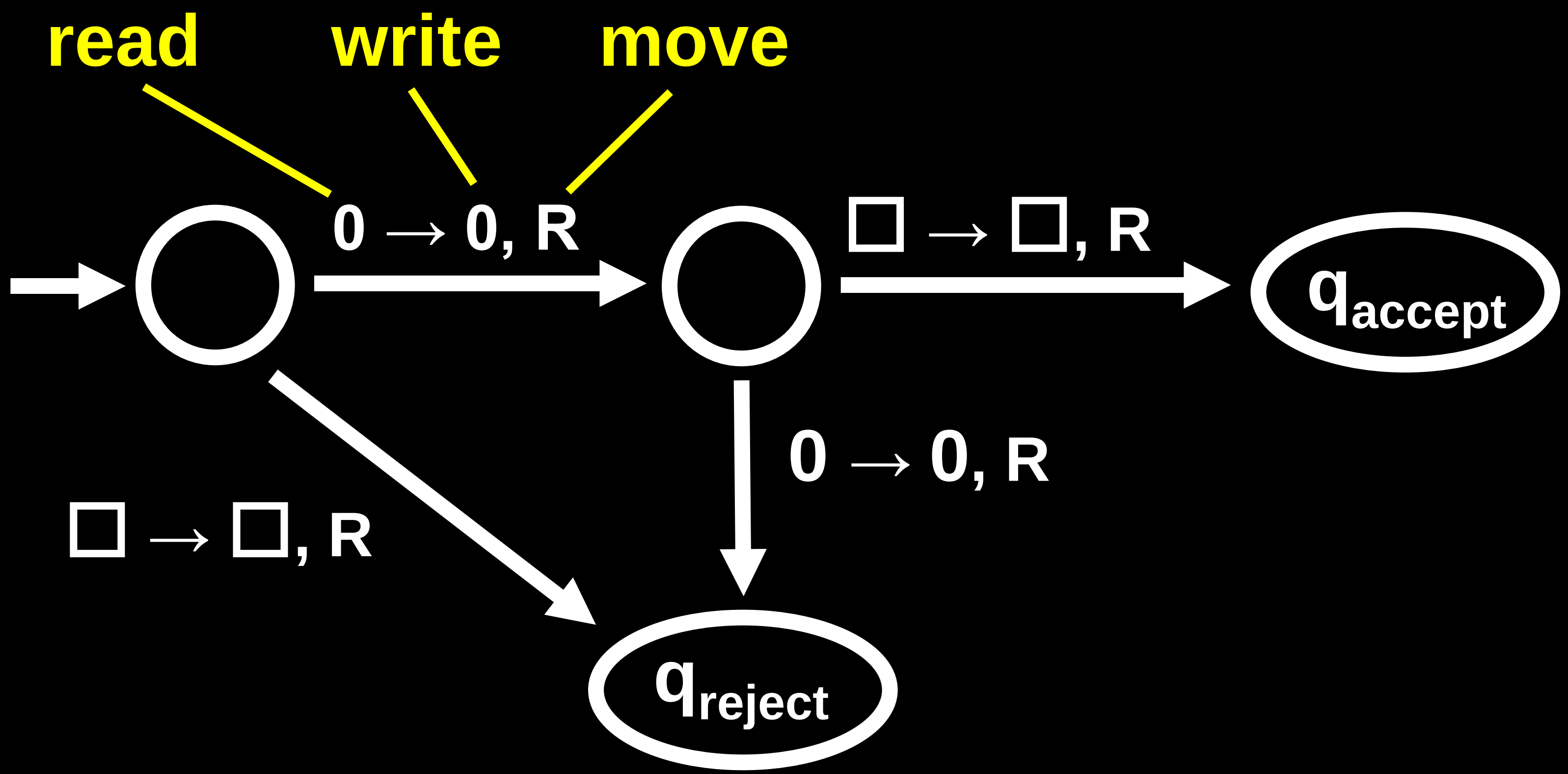
$B \to 00$

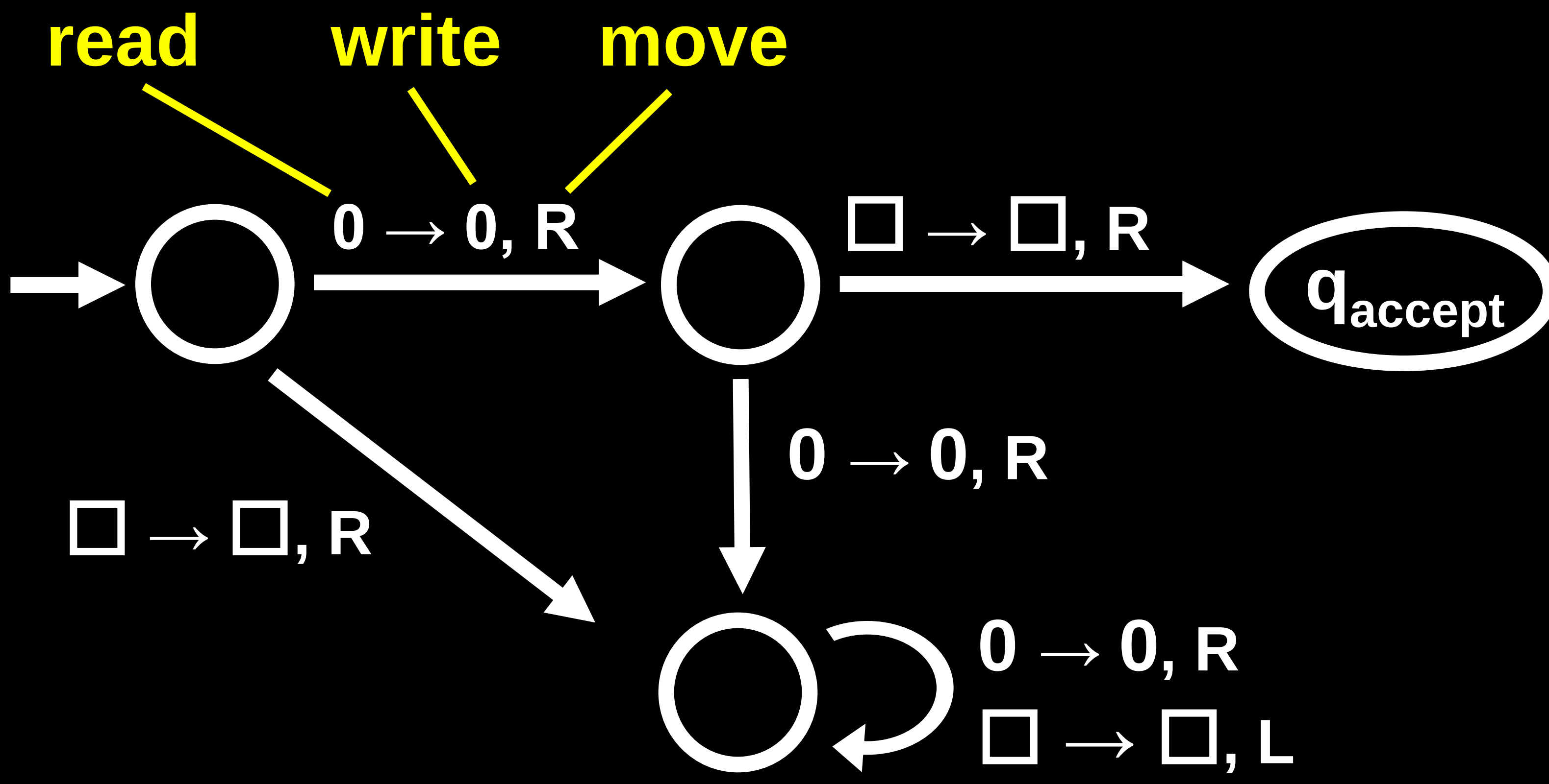
$S_0 \to BC \mid DD \mid BB \mid AB \mid BA \mid \varepsilon$, $C \to AB$,

$A \to BC \mid DD \mid BB \mid AB \mid BA$, $B \to DD$, $D \to 0$

TURING MACHINE




read
write
move
0 → 0, R
□ → □, R
q_accept
0 → 0, R
□ → □, R
q_reject


read
write
move
0 → 0, R
□ → □, R
q_{accept}
0 → 0, R
□ → □, R
0 → 0, R
□ → □, L

TMs VERSUS FINITE AUTOMATA

TM can both *write* to and read from the tape

The head can move *left and right*

The input doesn't have to be read entirely, and the computation can continue after all the input has been read

Accept and Reject take immediate effect

Definition: A Turing Machine is a 7-tuple
$T = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where:

Q is a finite set of states

Σ is the input alphabet, where $\square \notin \Sigma$

Γ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$

$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$

$q_0 \in Q$ is the start state

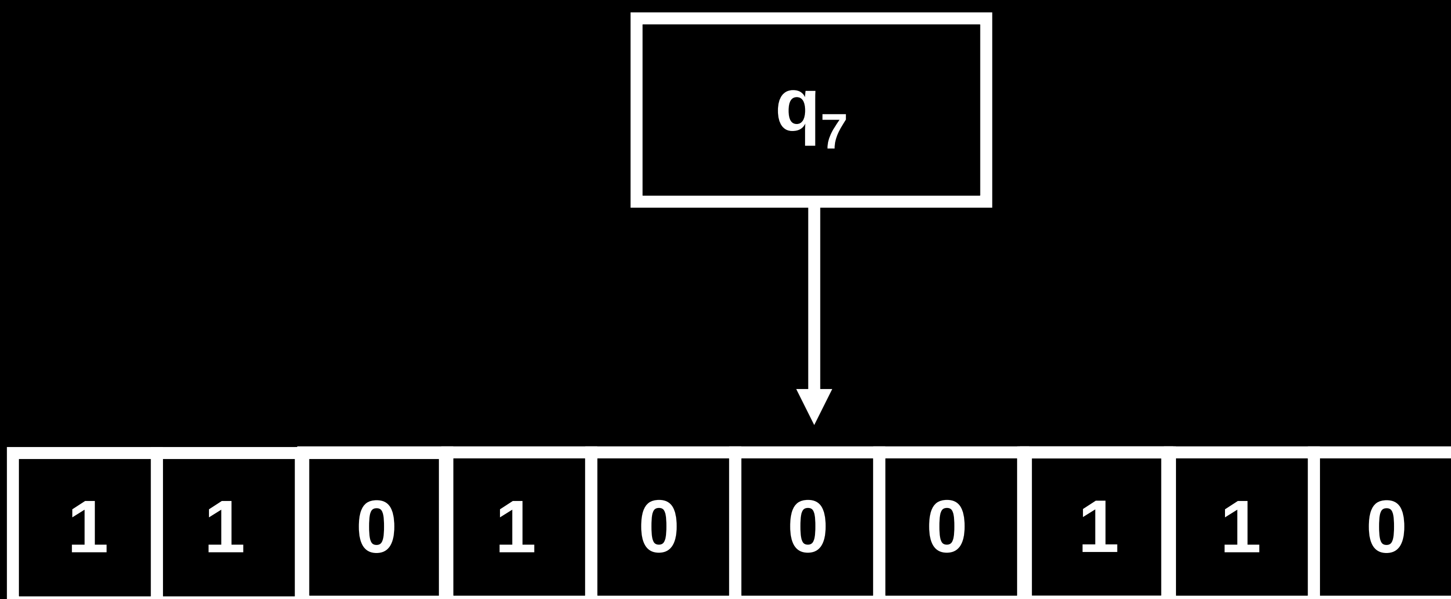
$q_{accept} \in Q$ is the accept state

$q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$

CONFIGURATIONS

$$11010q_7 00110$$

corresponds to:



A Turing Machine M accepts input w if there is a sequence of configurations $C_1, \dots, C_k$ such that

1. C_1 is a *start* configuration of M on input w, ie C_1 is q_0w
2. each C_i *yields* C_{i+1}, ie M can legally go from C_i to C_{i+1} in a single step

ua q_i bv *yields* u q_j acv if $\delta(q_i, b) = (q_j, c, L)$

ua qi bv *yields* uac q_j v if $\delta(q_i, b) = (q_j, c, R)$

A Turing Machine M accepts input w if there is a sequence of configurations $C_1, \ldots, C_k$ such that

1. **C_1 is a *start* configuration of M on input w, ie C_1 is $q_0 w$**
2. **each C_i *yields* C_{i+1}, ie M can legally go from C_i to C_{i+1} in a single step**
3. **C_k is an *accepting* configuration, ie the state of the configuration is q_{accept}**

A TM *recognizes* a language iff it accepts all and only those strings in the language

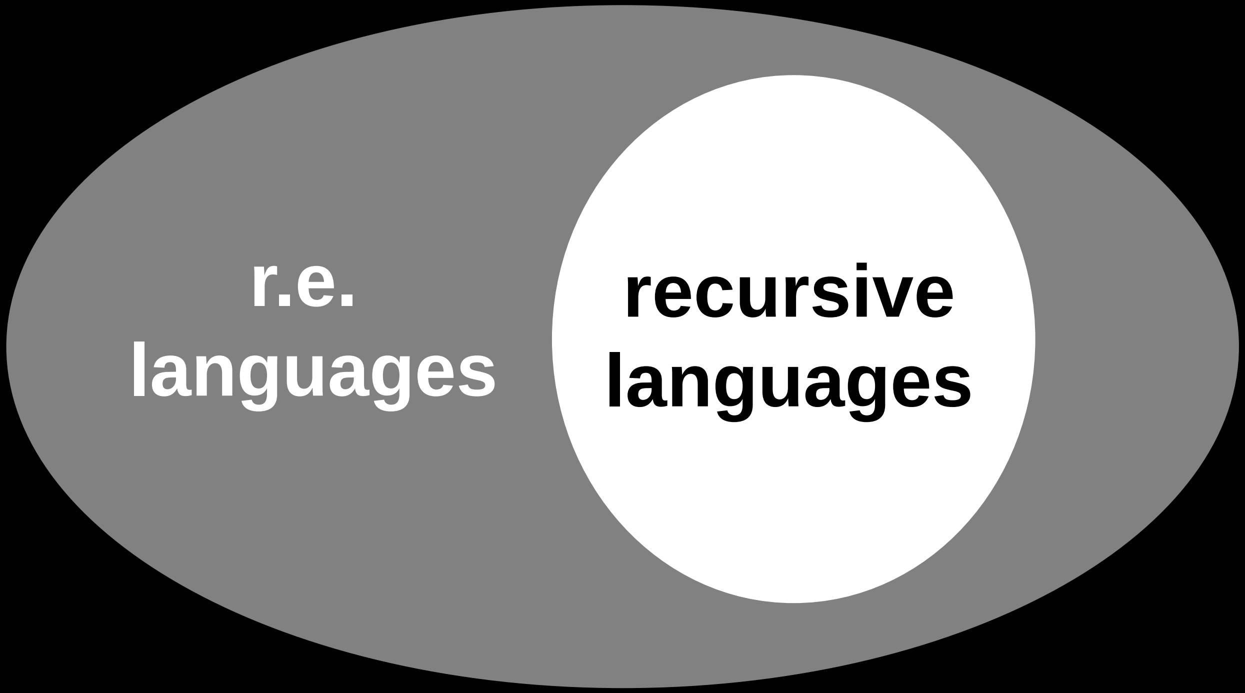
A language L is called Turing-recognizable or recursively enumerable or semi-decidable iff some TM recognizes L

A TM *decides* a language L iff it accepts all strings in L and rejects all strings not in L

A language L is called decidable or recursive iff some TM decides L

A language is called **Turing-recognizable** or **recursively enumerable (r.e.)** or **semi-decidable** if some TM **recognizes** it

A language is called **decidable** or **recursive** if some TM **decides** it



Theorem: If A and ¬A are r.e. then A is recursive

Theorem: If A and ¬A are r.e. then A is recursive

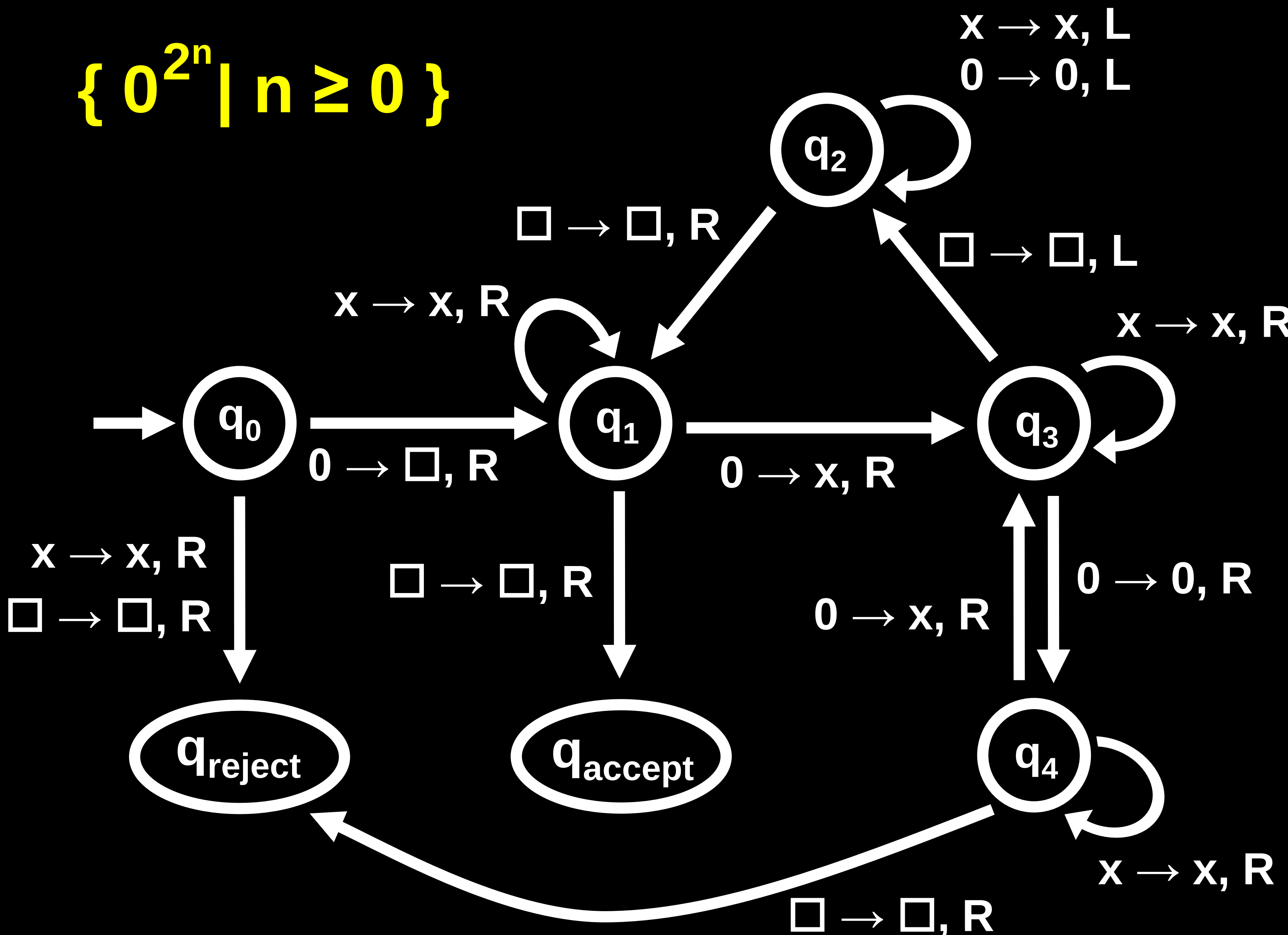
Given:
a TM that recognizes A and
a TM that recognizes ¬A,
we can build a new machine that *decides* A

$\{ 0^{2^n} \mid n \geq 0 \}$ Is decidable

PSEUDOCODE:

1. Sweep from left to right, cross out every other **0**
2. If in stage 1, the tape had only one **0**, *accept*
3. If in stage 1, the tape had an odd number of **0**’s, *reject*
4. Move the head back to the first input symbol.
5. Go to stage 1.

$\{ 0^{2^n} \mid n \geq 0 \}$



$\{ 0^{2^n} \mid n \geq 0 \}$ Is decidable

$q_0 0000$

$\square q_1 000$

$\square x q_3 00$

$\square x 0 q_4 0$

$\square x 0 x q_3$

$\square x 0 q_2 x$

$\square x q_2 0 x$

$\square q_2 x 0 x$

$q_2 \square x 0 x$

C = {$a^i b^j c^k$ | k = ij, and i, j, k ≥ 1}

PSEUDOCODE:

1. If the input doesn't match **a*b*c***, *reject.*
2. Move the head back to the leftmost symbol.
3. Cross off an **a**, scan to the right until **b**.
   Sweep between **b**'s and **c**'s, crossing off one of each until all **b**'s are crossed off.
4. Uncross all the **b**'s.
   If there's another **a** left, then repeat stage 3.
   If all **a**'s are crossed out,
      Check if all **c**'s are crossed off.
      If yes, then *accept*, else *reject.*

$C = \{a^i b^j c^k \mid k = ij, \text{ and } i, j, k \geq 1\}$

aabbbcccccc

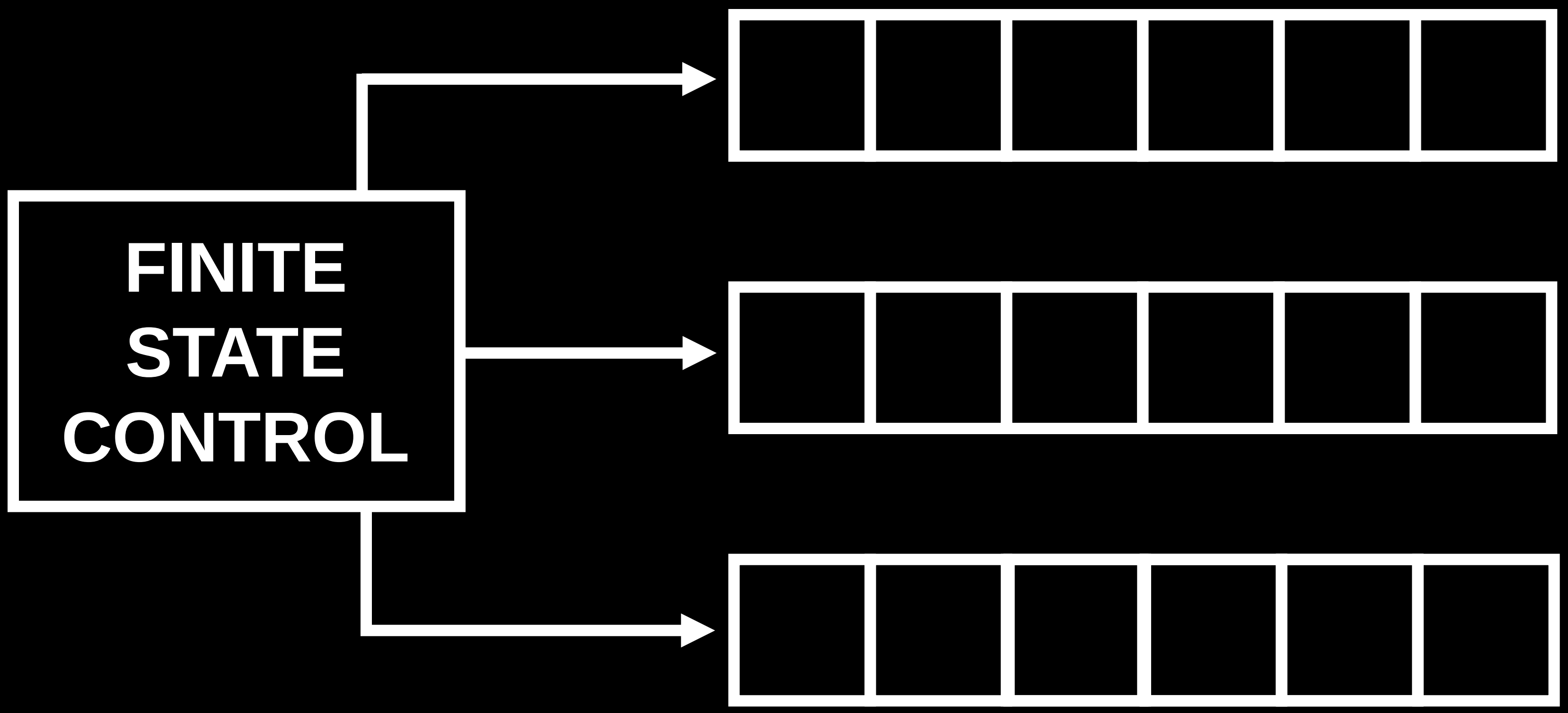
xabbbcccccc

xayyyzzzccc

xabbbzzzccc

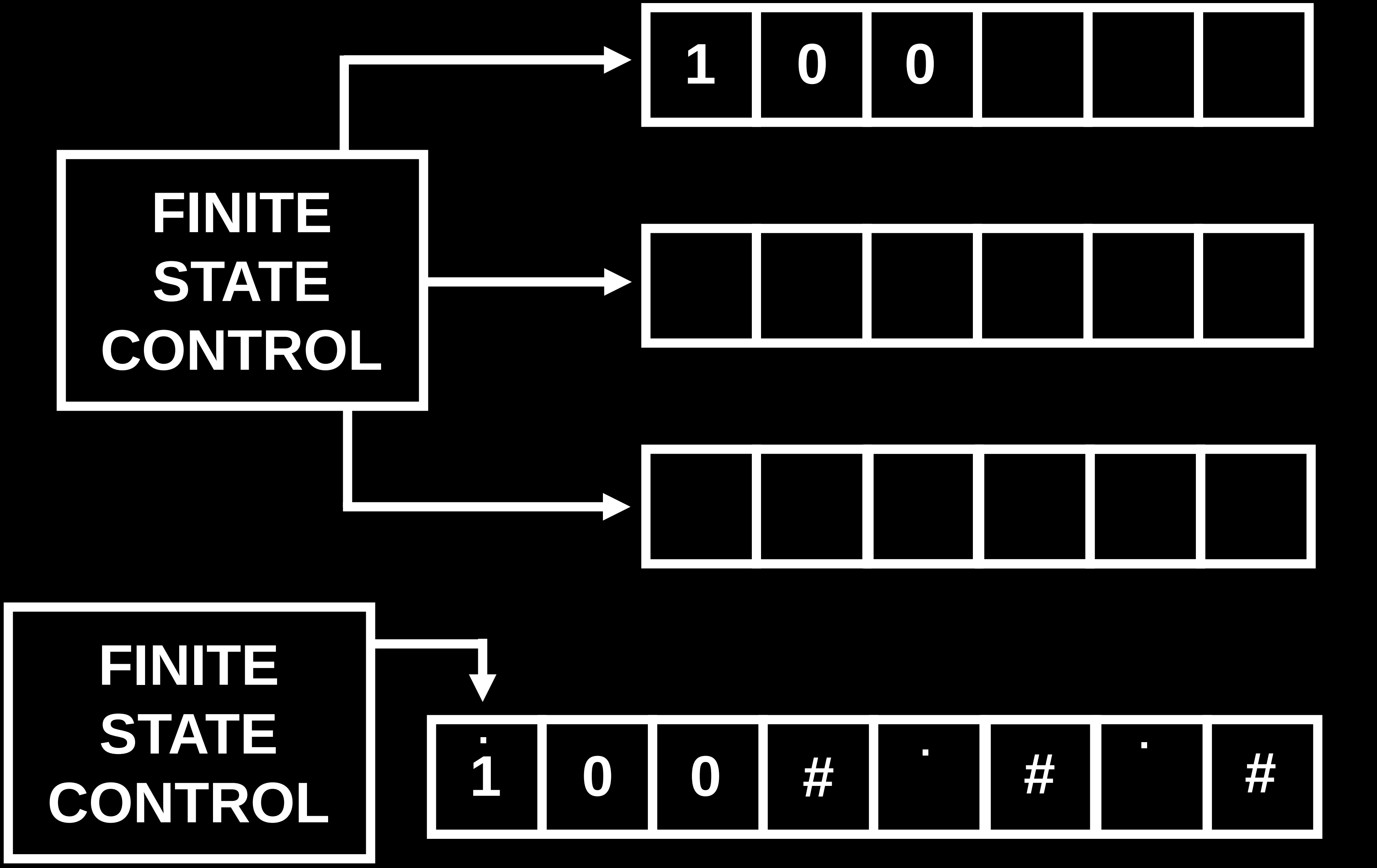
xxyyyzzzzzz

MULTITAPE TURING MACHINES



$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L,R\}^k$$

Theorem: Every Multitape Turing Machine can be transformed into a single tape Turing Machine



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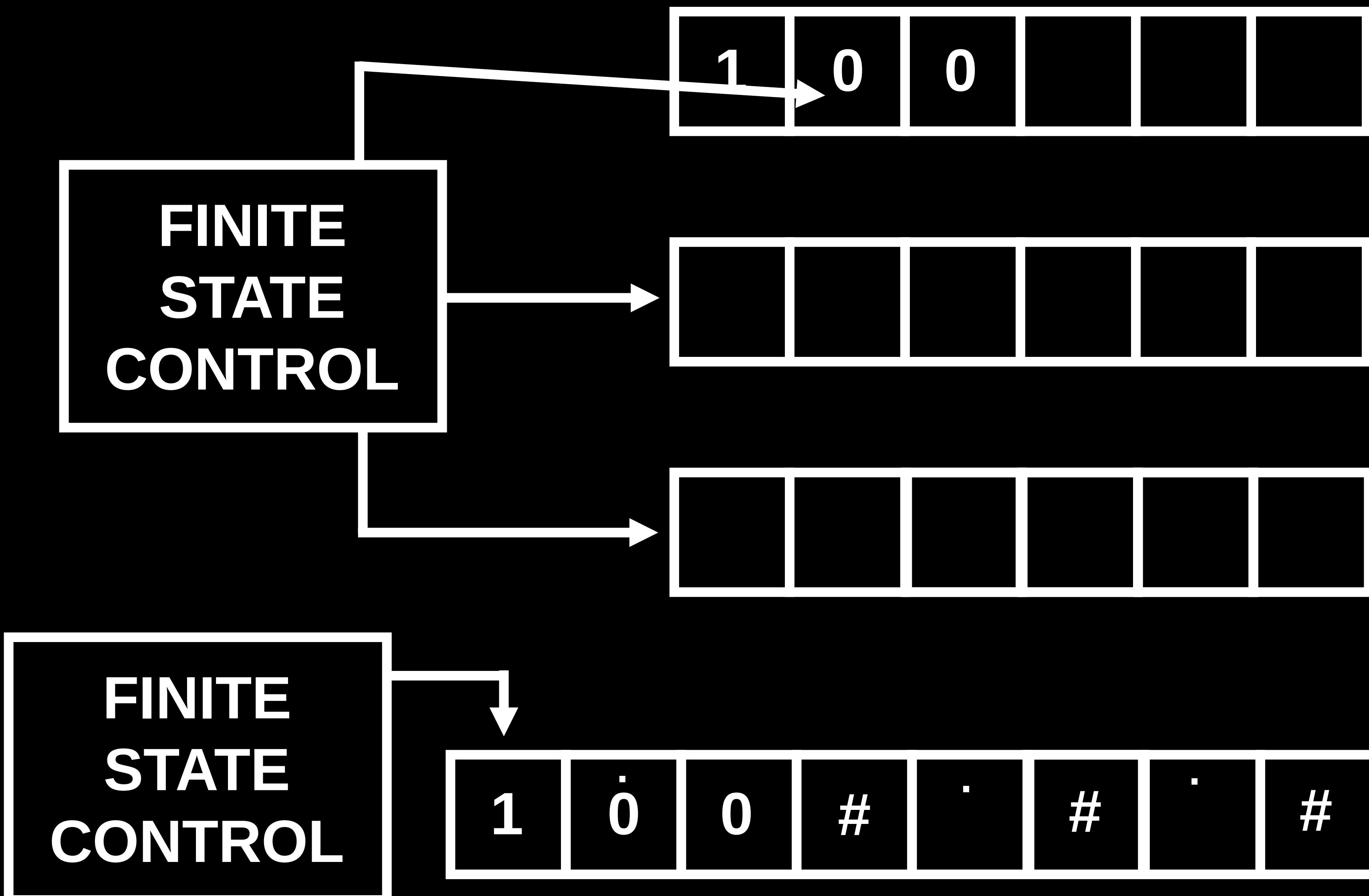

THE CHURCH-TURING THESIS

Intuitive Notion of Algorithms
EQUALS
Turing Machines

We can encode a TM as a string of 0s and 1s

$$0^n 1 0^m 1 0^k 1 0^s 1 0^t 1 0^r 1 0^u 1 \ldots$$

- 0^n: n states
- 0^m: m tape symbols (first k are input symbols)
- 0^k: (first k are input symbols)
- 0^s: start state
- 0^t: accept state
- 0^r: reject state
- 0^u: blank symbol

$$((p, a), (q, b, L)) = 0^p 1 0^a 1 0^q 1 0^b 1 0$$

$$((p, a), (q, b, R)) = 0^p 1 0^a 1 0^q 1 0^b 1 1$$

Similarly, we can encode DFAs, NFAs, CFGs, *etc.* into strings of 0s and 1s

So we can define the following languages:

A_{DFA} = { (B, w) | B is a DFA that accepts string w }

A_{NFA} = { (B, w) | B is an NFA that accepts string w }

A_{CFG} = { (G, w) | G is a CFG that generates string w }

Similarly, we can encode DFAs, NFAs, CFGs, *etc.* into strings of 0s and 1s

So we can define the following languages:

A_{DFA} = { (B, w) | B is a DFA that accepts string w }

Theorem: A_{DFA} is decidable

Proof Idea: Simulate B on w

A_{NFA} = { (B, w) | B is an NFA that accepts string w }

Theorem: A_{NFA} is decidable

A_{CFG} = { (G, w) | G is a CFG that generates string w }

Theorem: A_{CFG} is decidable

Proof Idea: Transform G into Chomsky Normal Form. Try all derivations of length up to 2|w|-1

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Read Chapter 3 of the book for next time