# 15-453

# FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

# Space Complexity II

**THURSDAY April 17**

# PSPACE = NPSPACE

**Definition:** Language B is PSPACE-complete if:

1. B $\in$ PSPACE
2. Every A in PSPACE is poly-time reducible to B
(i.e. B is PSPACE-hard)

# QUANTIFIED BOOLEAN FORMULAS

(in prenex normal form)

$$\exists x \exists y\, [\, x \lor \neg y\, ]$$

$$\forall x\, [\, x \lor \neg x\, ]$$

$$\forall x\, [\, x\, ]$$

$$\forall x \exists y\, [\, (x \lor y) \land (\neg x \lor \neg y)\, ]$$

**Allow constants, 0 and 1, eg.** $\forall x\, [\, 0 \lor \neg x\, ]$

**Definition:**

**A fully quantified Boolean formula is a Boolean formula** (in prenex normal form) **where every variable is quantified**

$$\exists x \exists y\, [\, x \vee \neg y \,]$$

$$\forall x\, [\, x \vee \neg x \,]$$

$$\forall x\, [\, x \,]$$

$$\forall x \exists y\, [\, (x \vee y) \wedge (\neg x \vee \neg y) \,]$$

$$\forall x \exists y\, [\, (x \vee 0) \wedge (\neg x \vee \neg y) \,]$$

TQBF = { $\phi$ | $\phi$ is a **true** fully quantified Boolean formula}

**Theorem:** TQBF is PSPACE-complete

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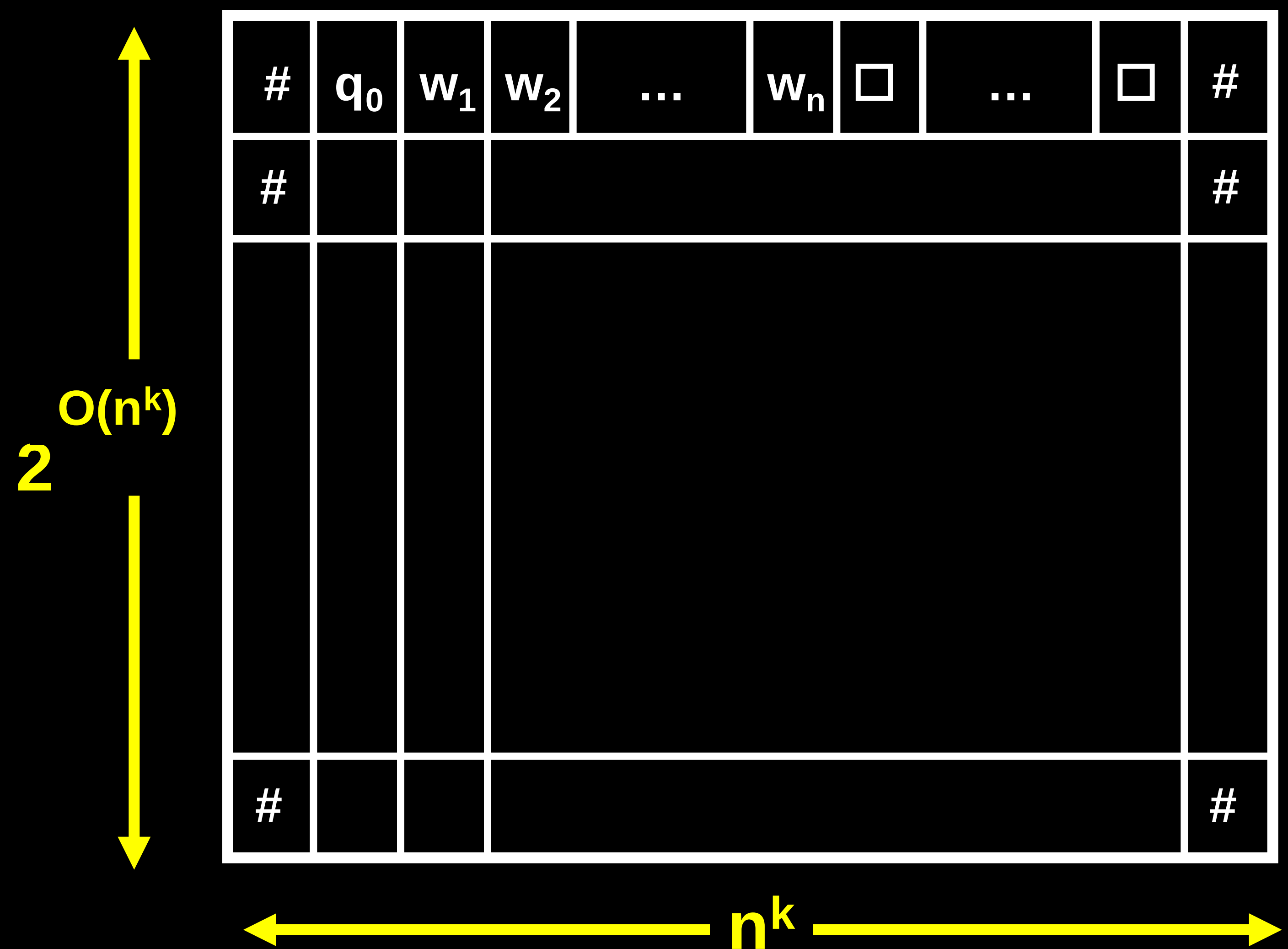
The reduction turns a string w into a fully quantified Boolean formula $\phi$ that simulates the PSPACE machine for A on w

Let M be a deterministic TM that decides A in space $n^k$ How do we know M exists?

**A tableau for M on w is an table whose rows are the configurations of the computation of M on input w**



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M can go from c to d in t steps

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We let $\phi = \phi_{c_{start},\ c_{accept},\ h}$, where $h = 2^{e\ s(n)}$ for a constant $e$ chosen so that $M$ has less than $2^{e\ s(n)}$ possible configurations on an input of length $n$

Here $s(n) = n^k$

# HIGH-LEVEL IDEA:

**Encode the Algorithm of Savitch's Theorem with a Quantified Boolean Formula**

**If M uses $n^k$ space,**

**then the QBF $\phi$ will have size $O(n^{2k})$**

**If we assign c and d to actual configurations,**

**$\phi_{c,d,t}$ will say:**

**"there exists a configuration m such that**

**$\phi_{c,m,t/2}$ is true and $\phi_{m,d,t/2}$ is true"**

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Each config has $n^k$ cells so and is encoded by $O(n^k)$ variables.

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"d follows from c in a single step of M"?

Use 2 x 3 windows as in the Cook-Levin theorem, and write a CNF formula

If $t > 1$, we construct $\phi_{c,d,t}$ *recursively*:

$$\phi_{c,d,t} = \exists m\, [\phi_{c,m,t/2} \wedge \phi_{m,d,t/2}\, ]$$

$$\exists x_1 \exists x_2 \ldots \exists x_L \quad L = O(n^k)$$

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Every level of the recursion cuts $t$ in half but roughly doubles the size of the formula (so back to length $O(t)$)
So, we modify the formula to be:

$$\phi_{c,d,t} = \exists m \forall a,b[\ [(a,b)=(c,m) \vee (a,b)=(m,d)] \Rightarrow [\ \phi_{a,b,t/2}\ ]\ ]$$

This folds the 2 recursive sub-formulas into 1

$$\phi_{c,d,t} = \exists m \forall a,b [\ [(a,b)=(c,m) \vee (a,b)=(m,d)] \Rightarrow [\ \phi_{a,b,t/2}\ ]\ ]$$

Set $\phi = \phi_{c_{start},\, c_{accept},\, h}$ where $h = 2^{d\, s(n)}$

- Each recursive step adds a portion that is linear in the size of the configurations, so has size $O(s(n))$
- Number of levels of recursion is $\log h = O(s(n))$
- Hence, the size of $\phi$ is $O(s(n)^2)$

PSPACE is often called
the class of games

Formalizations of many popular
games are PSPACE-Complete

# THE FORMULA GAME (FG)

**…is played between two players, E and A**

**Given a fully quantified Boolean formula**

$$\exists y \forall x \, [\, (x \lor y) \land (\neg x \lor \neg y) \,]$$

**E chooses values for variables quantified by $\exists$**

**A chooses values for variables quantified by $\forall$**

**Start at the leftmost quantifier**

**E wins if the resulting formula is true**

**A wins otherwise**

$$\forall x \exists y\, [\, (x \vee y) \wedge (\neg x \vee \neg y)\, ]$$

$$\exists x \exists y\, [\, x \vee \neg y\, ]$$

FG = { $\phi$ | Player E has a winning strategy in $\phi$ }

**Theorem:** FG is PSPACE-Complete

**Proof:**

# FG = TQBF

# GEOGRAPHY

Two players take turns naming cities from anywhere in the world

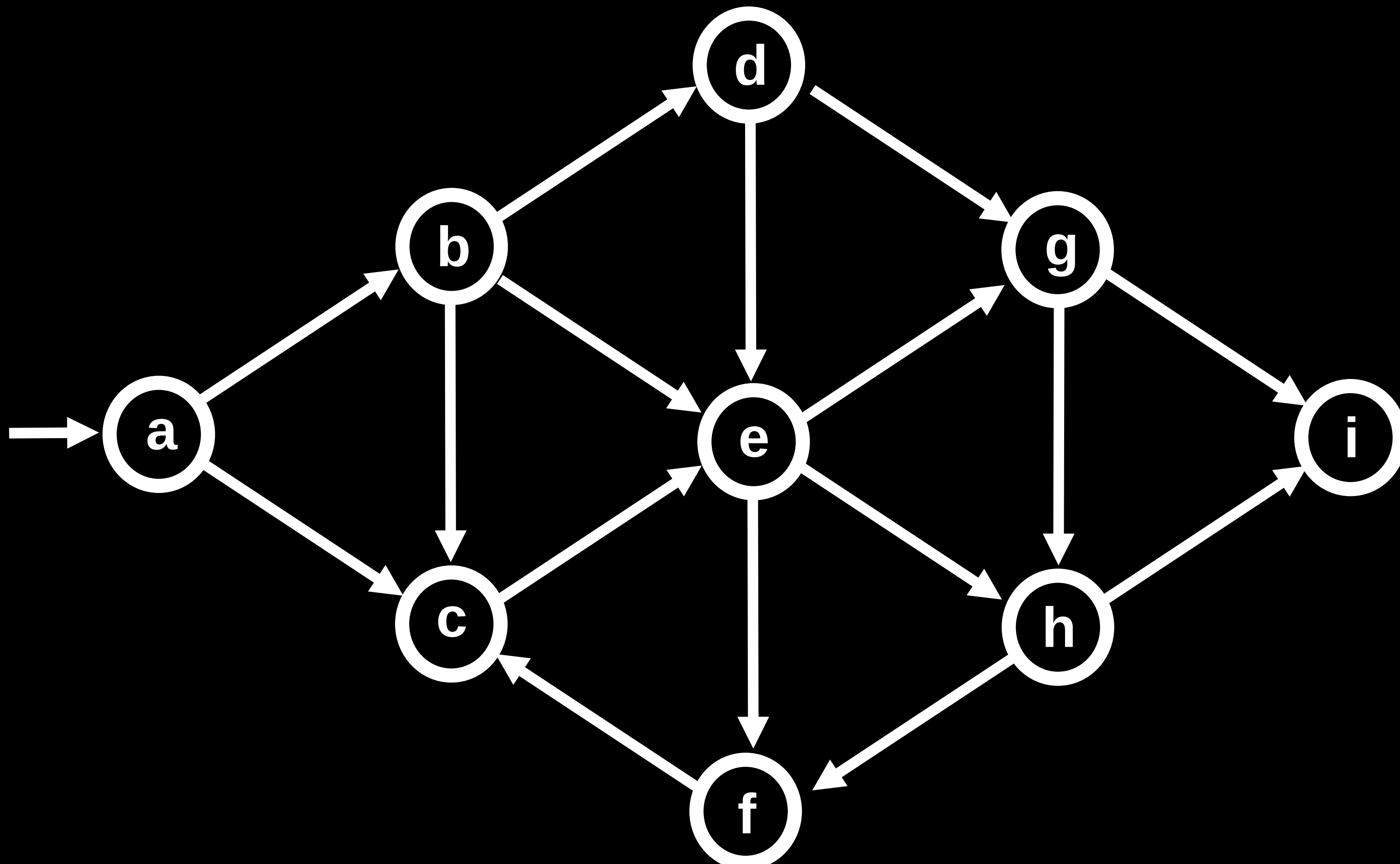
Each city chosen must begin with the same letter that the previous city ended with

Cities cannot be repeated

**Austin $\rightarrow$ Nashua $\rightarrow$ Albany $\rightarrow$ York**

Whoever cannot name any more cities loses

# GENERALIZED GEOGRAPHY



GG = { (G, b) | Player 1 has a winning strategy for generalized geography played on graph G starting at node b }

**Theorem:** GG is PSPACE-Complete

# GG $\in$ PSPACE

**WANT: Machine M that accepts (G,b)**

**$\Leftrightarrow$ Player 1 has a winning strategy on (G, b)**

**M(G, b): If b has no outgoing edges, *reject*.**

**1. Remove node b and all edges touching it to get to a new graph $G_1$**

**2. For each of the nodes $b_1$, $b_2$, …, $b_k$ that b originally pointed at, recursively call M($G_1$, $b_i$)**

**3. If all of these accept, Player 2 has a winning strategy, so *reject*.**
**Otherwise, *accept*.**

# GG IS PSPACE-HARD

**We show that FG $\leq_P$ GG**

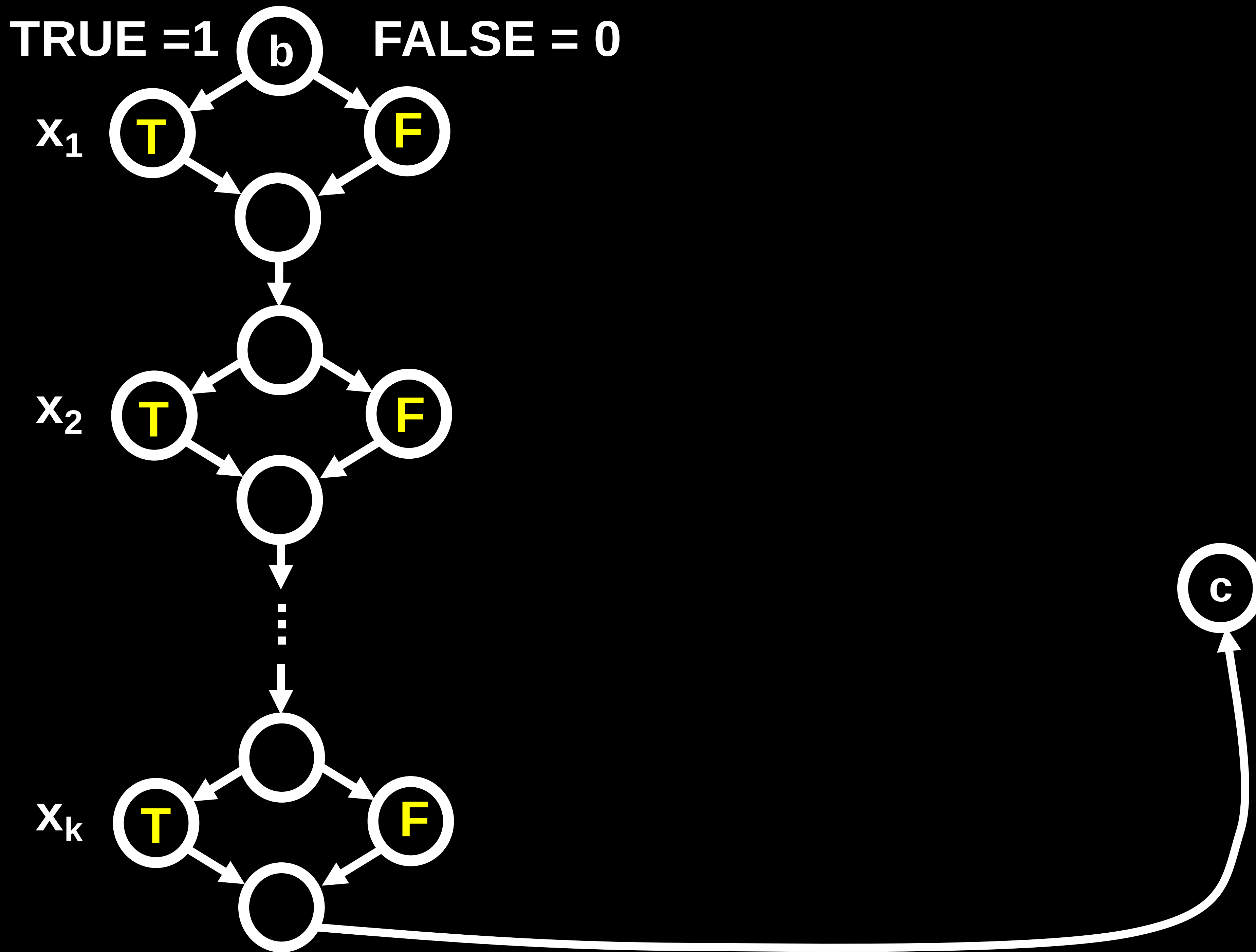
**We convert a formula $\phi$ into (G, b) such that:**

**Player E has winning strategy in $\phi$**
**if and only if**
**Player 1 has winning strategy in (G, b)**

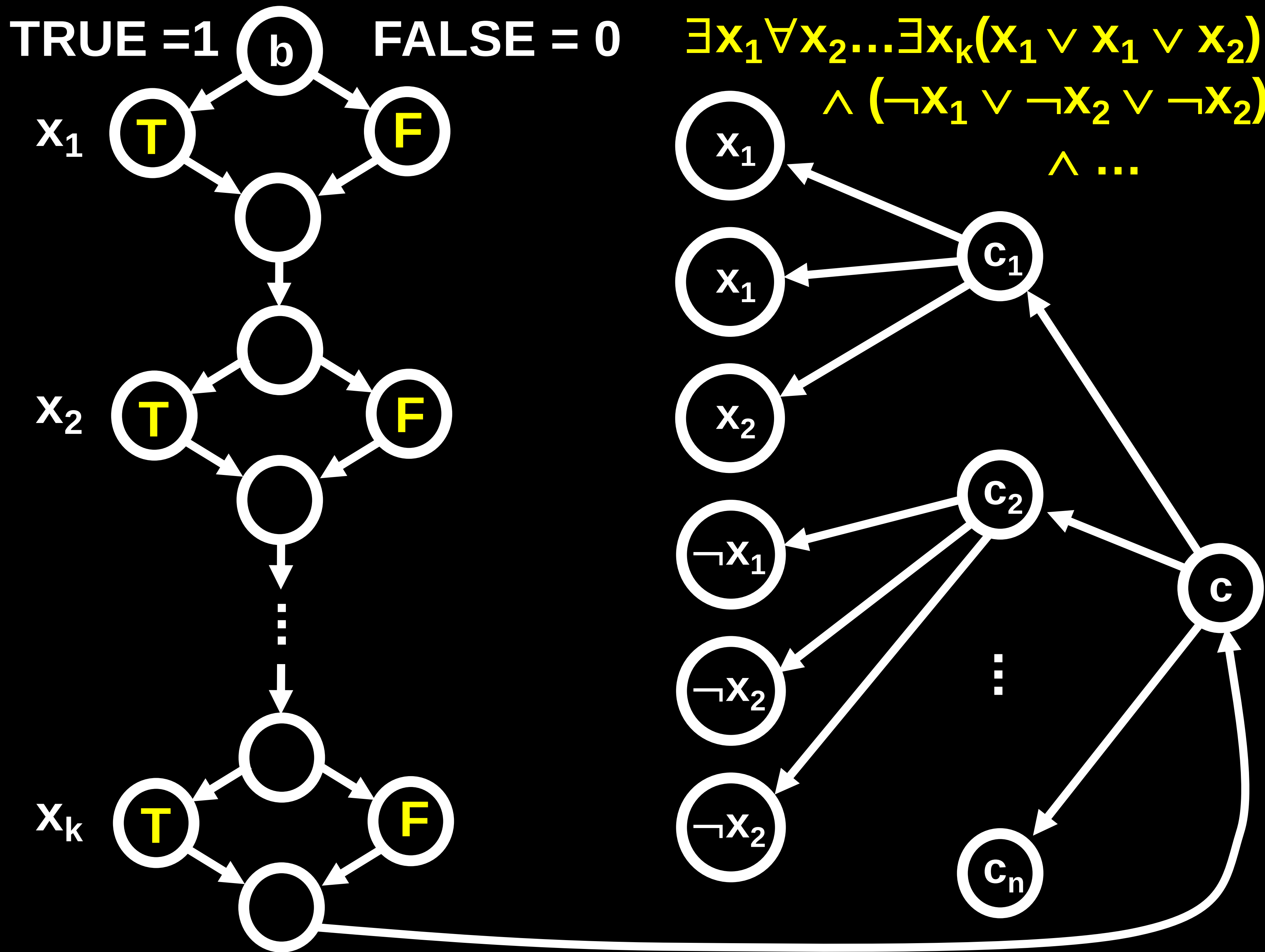
**For simplicity we assume $\phi$ is of the form:**

$$\phi = \exists x_1 \forall x_2 \exists x_3 \ldots \exists x_k \, [\psi]$$

**where $\psi$ is in cnf.**
**(Quantifiers alternate, and the last move is E's)**



TRUE =1
b
FALSE = 0
$x_1$
T
F
$x_2$
T
F
$x_k$
T
F
c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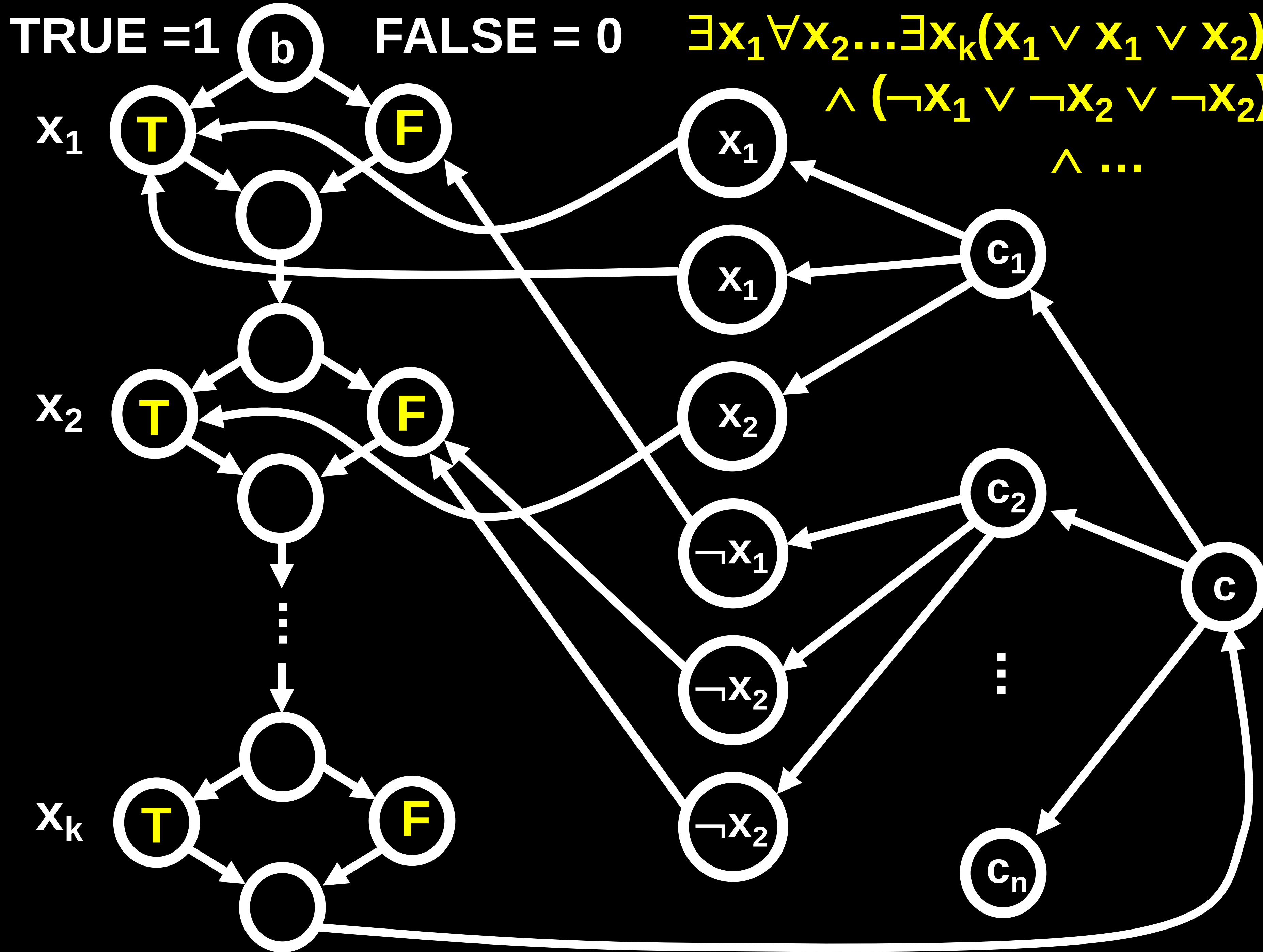
$\exists x_1 \forall x_2 \ldots \exists x_k (x_1 \vee x_1 \vee x_2)$
$\wedge (\neg x_1 \vee \neg x_2 \vee \neg x_2)$
$\wedge \ldots$
$x_1$
$x_1$
$x_2$
$\neg x_1$
$\neg x_2$
$\neg x_2$
$c_1$
$c_2$
c
$c_n$

$$\exists x_1 \forall x_2 \ldots \exists x_k (x_1 \vee x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_2) \wedge \ldots$$



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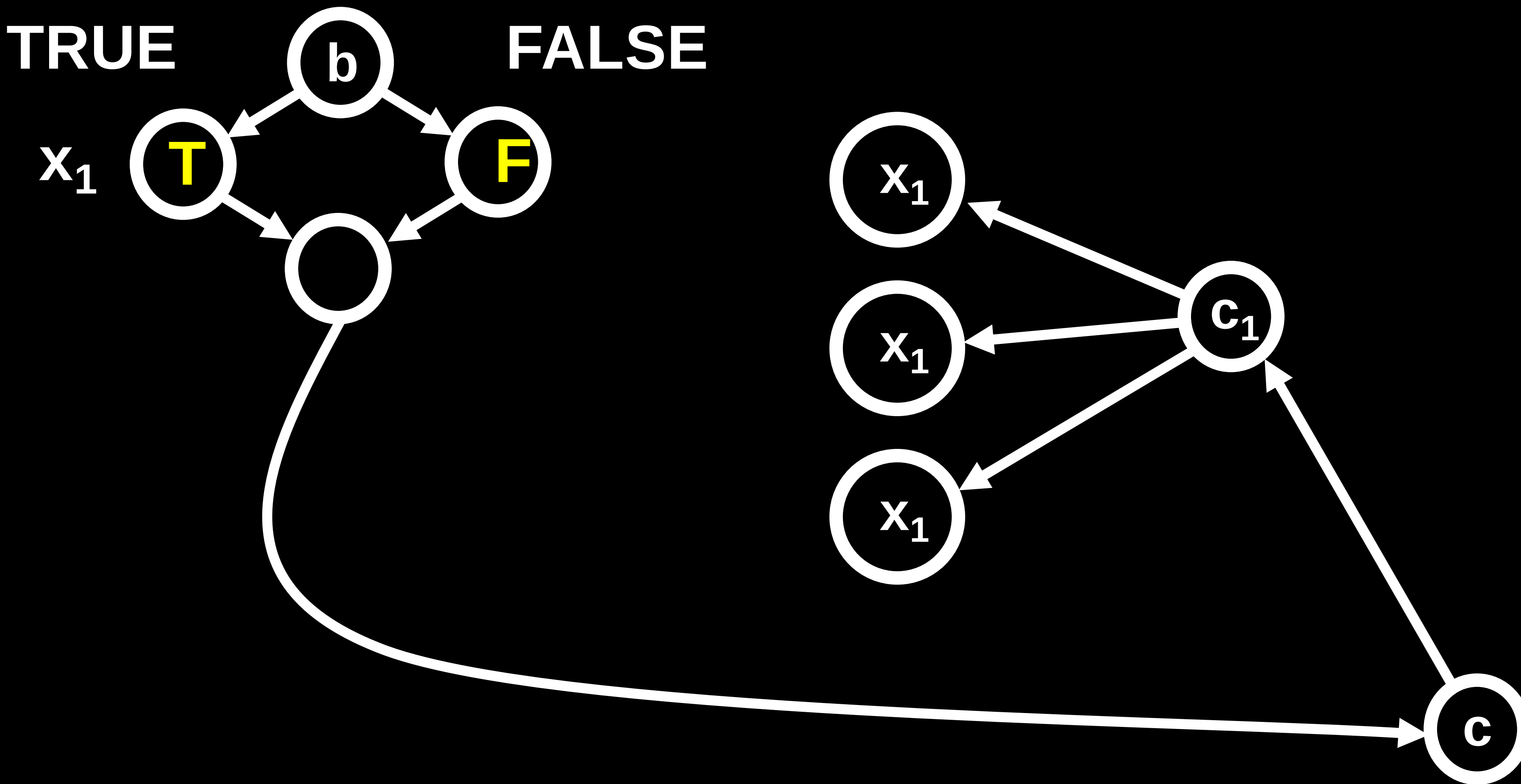

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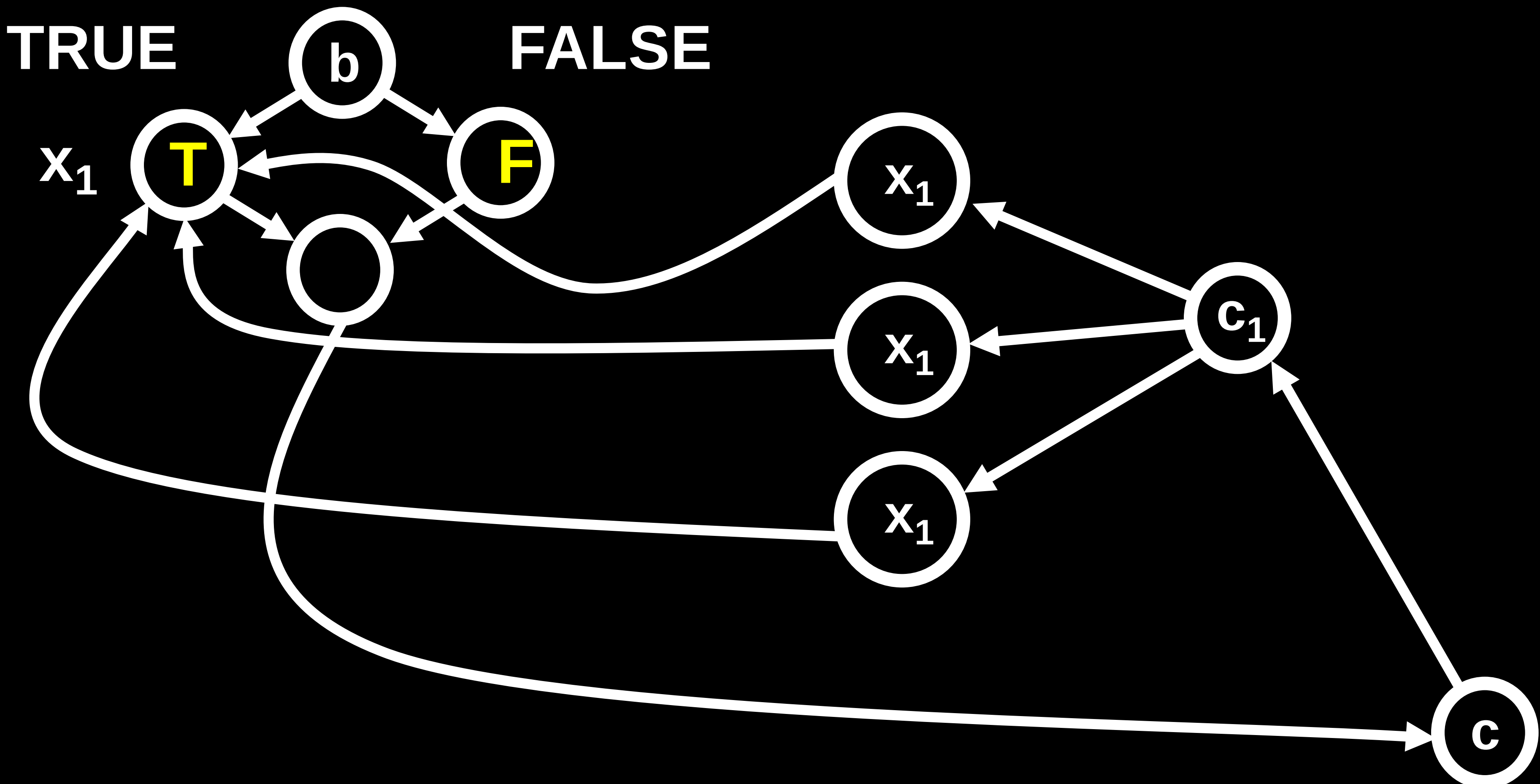

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**Question:**
**Is Chess a PSPACE complete problem?**

**No, because determining whether a player has a winning strategy takes CONSTANT time and space (OK, the constant is large…)**

## But n x n GO, Chess and Checkers can be shown to be PSPACE-hard

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**Read Chapter 10.2 of the book for next time**