15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

TIME COMPLEXITY AND POLYNOMIAL TIME; NON DETERMINISTIC TURING MACHINES AND NP

THURSDAY Mar 20

COMPLEXITY THEORY

Studies what can and can't be computed under limited resources such as time, space, etc

Today: Time complexity

MEASURING TIME COMPLEXITY

We measure time complexity by counting the elementary steps required for a machine to halt

Consider the language $A = \{ 0^k1^k \mid k \geq 0 \}$

On input of length **n**:

1. Scan across the tape and reject if the string is not of the form 0^i1^j

2. Repeat the following if both 0s and 1s remain on the tape:

Scan across the tape, crossing off a single 0 and a single 1

3. If 0s remain after all 1s have been crossed off, or vice-versa, reject. Otherwise accept.

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~n^2 **2. Repeat the following if both 0s and 1s remain on the tape:**

Scan across the tape, crossing off a single 0 and a single 1

~n **3. If 0s remain after all 1s have been crossed off, or vice-versa, reject. Otherwise accept.**

Definition: Let M be a TM that halts on all inputs. The **running time or time-complexity** of M is the function $f : N \rightarrow N$, where f(n) is the maximum number of steps that M uses ***on any input of length n.***

ASYMPTOTIC ANALYSIS

$$5n^3 + 2n^2 + 22n + 6 = O(n^3)$$

BIG-O

Let f and g be two functions f, g : $N \to R^+$. We say that f(n) = O(g(n)) if there exist positive integers c and n_0 so that for every integer $n \geq n_0$

$$f(n) \leq cg(n)$$

When f(n) = O(g(n)), we say that g(n) is an asymptotic upper bound for f(n)

f asymptotically NO MORE THAN g

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If $c = 6$ and $n_0 = 10$, then $5n^3 + 2n^2 + 22n + 6 \leq cn^3$

$$2n^{4.1} + 200283n^4 + 2 = O(n^{4.1})$$

$$3n\log_2 n + 5n \log_2\log_2 n = O(n\log_2 n)$$

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$$\log_{10} n = \log_2 n / \log_2 10$$

$$O(n\log_2 n) = O(n\log_{10} n) = O(n\log n)$$

Definition: TIME(t(n)) = { L | L is a language decided by a O(t(n)) time Turing Machine }

$A = \{ 0^k 1^k \mid k \geq 0 \} \in \mathrm{TIME}(n^2)$

$A = \{ 0^k1^k \mid k \geq 0 \} \in \text{TIME}(n\log n)$

Cross off every other 0 and every other 1. If the # of 0s and 1s left on the tape is odd, reject

00000000000001111111111111

x0x0x0x0x0x0xx1x1x1x1x1x1x

xxx0xxx0xxx0xxxx1xxx1xxx1x

xxxxxxx0xxxxxxxxxxxx1xxxxx

xxxxxxxxxxxxxxxxxxxxxxxxxx

We can prove that a TM cannot decide A in less time than O(nlog n)

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***7.49 Extra Credit. Let f(n) = *o*(nlogn). Then Time(f(n)) contains only regular languages.**

where f(n) = *o*(g(n)) iff $\lim_{n \to \infty}$ f(n)/g(n) = 0

ie, for all c >0, ∃ n_0 such that f(n) < cg(n) for all n $\geq n_0$

f asymptotically LESS THAN g

Can A = { 0^k1^k | $k \geq 0$ } be decided in time O(n) with a two-tape TM?

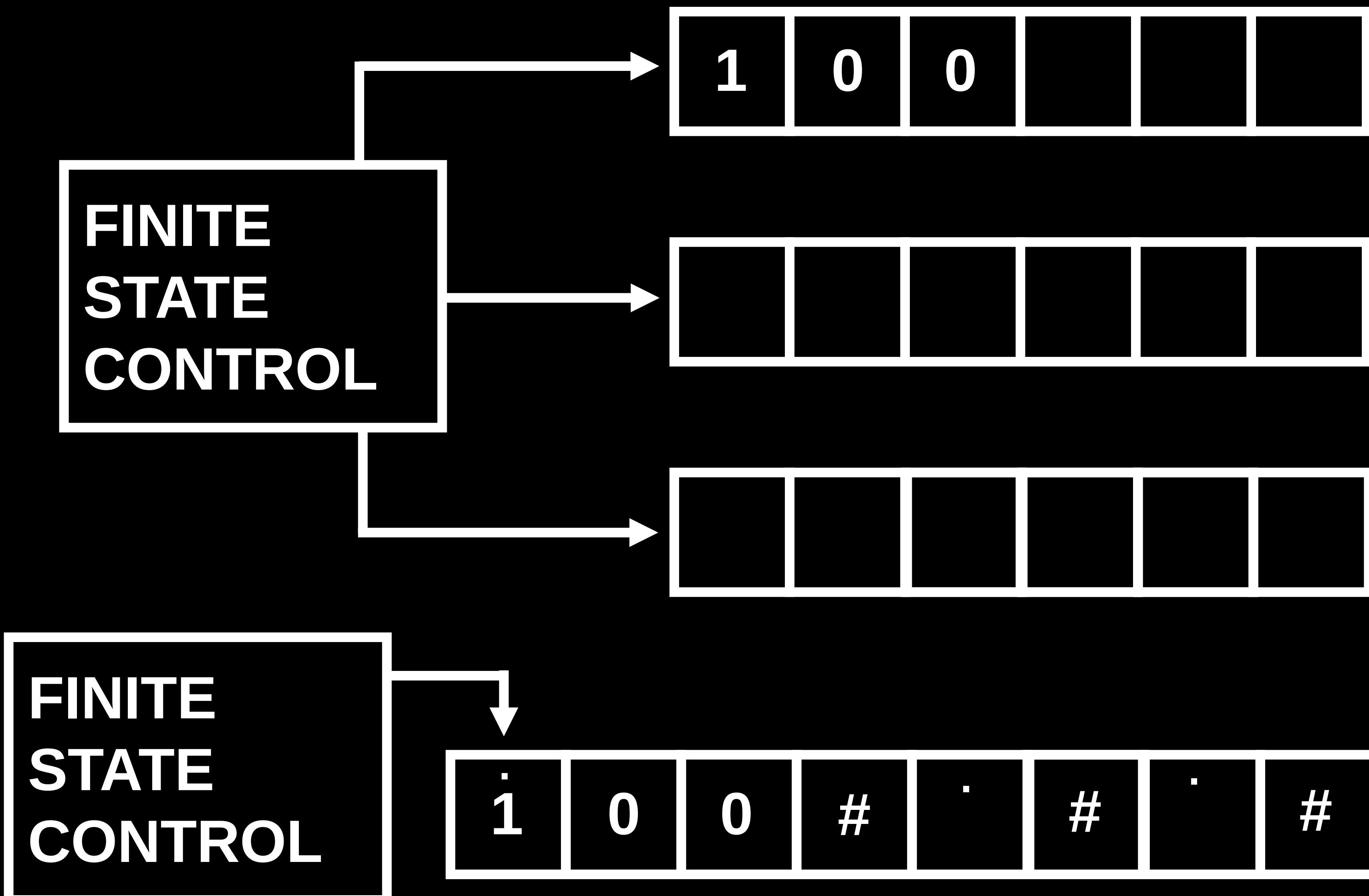
Scan all 0s and copy them to the second tape. Scan all 1s, crossing off a 0 from the second tape for each 1.

Different models of computation yield different running times for the same language!

Theorem: Let t(n) be a function such that t(n) $\geq$ n. Then every t(n)-time multi-tape TM has an equivalent $O(t(n)^2)$ single tape TM

Claim: Simulating each step in the multi-tape machine uses at most O(t(n)) steps on a single-tape machine.
Hence total time of simulation is $O(t(n)^2)$.

Theorem: Every Multitape Turing Machine can be transformed into a single tape Turing Machine



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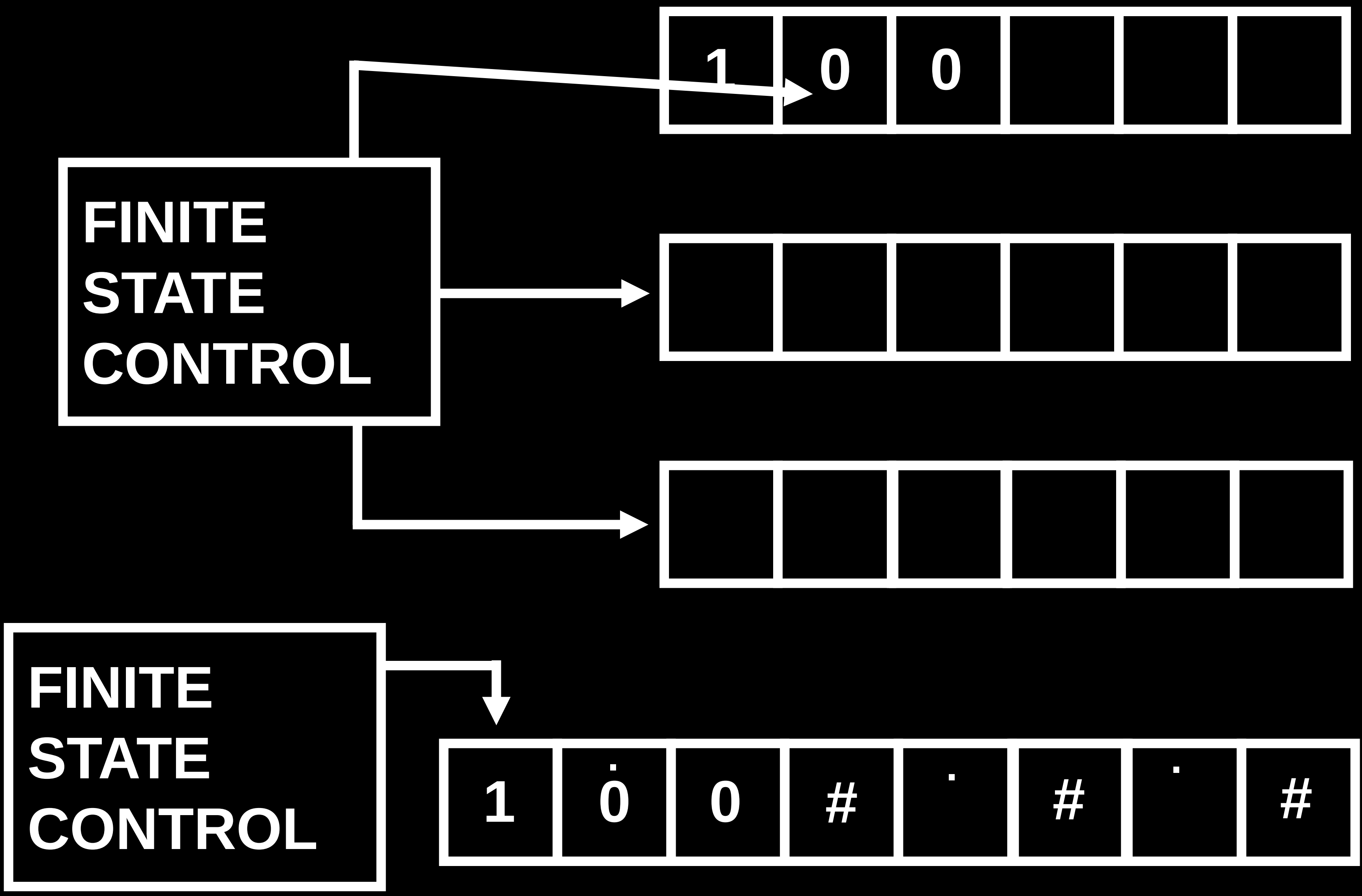

Theorem: **Let t(n) be a function such that t(n) ≥ n. Then every t(n)-time multi-tape TM has an equivalent $O(t(n)^2)$ single tape TM**

Analysis: (Note, k, the # of tapes, is fixed.)

Let S be simulator

- Put S's tape in proper format: O(n) steps
- Two scans to simulate one step,
  1. to optain info for next move O(t(n)) steps, why?
  2. to simulate it (may need to shift everything over to right possibly k times): O(t(n)) steps, why?

Theorem: **Let t(n) be a function such that $t(n) \geq n$. Then every t(n)-time multi-tape TM has an equivalent $O(t(n)^2)$ single tape TM**

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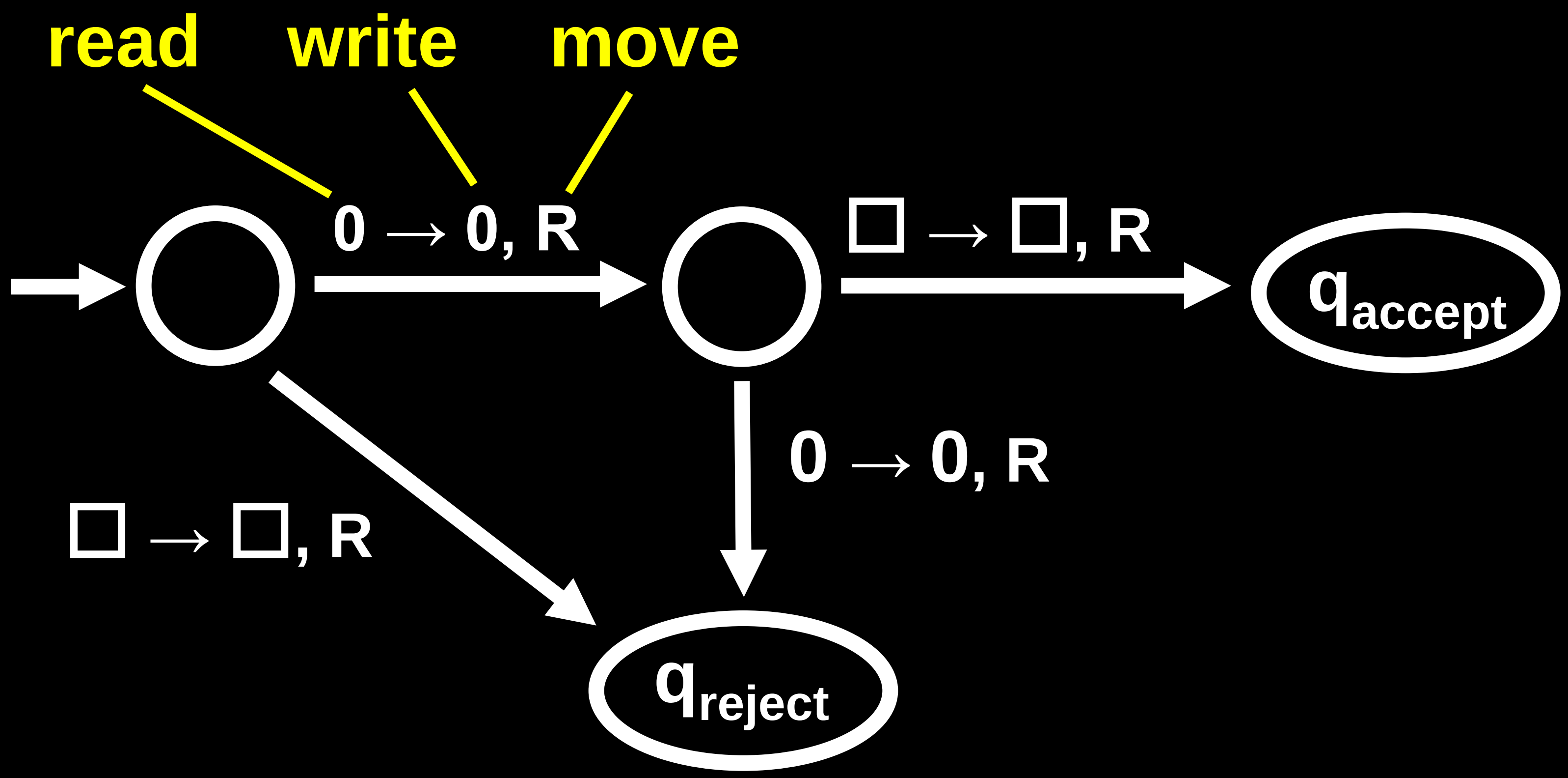
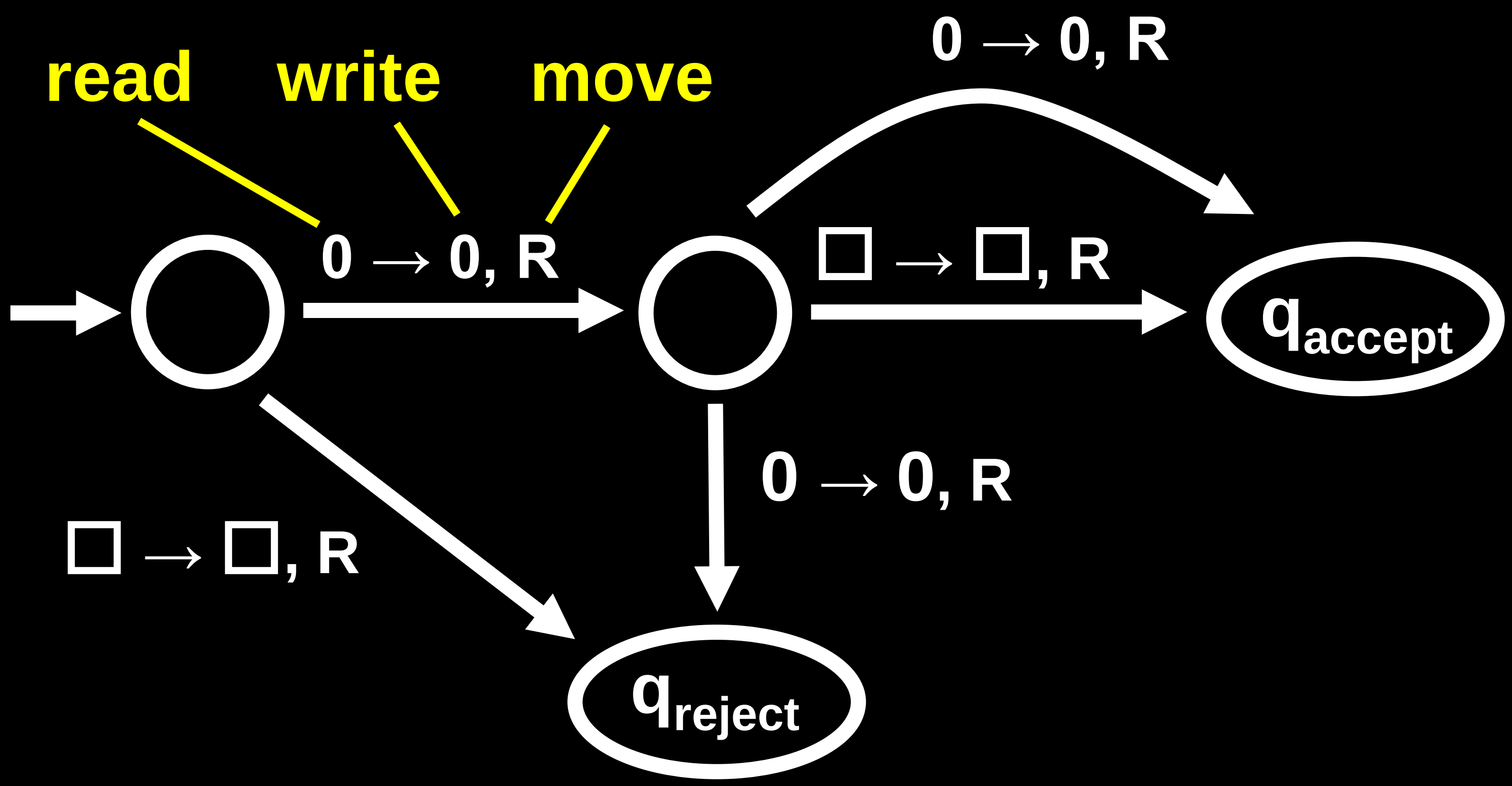
Therefore, $O(n) + t(n)\, O(t(n)) = O(t(n)^2)$ steps in simulation.

$$P = \bigcup_{k \in N} \text{TIME}(n^k)$$

NON-DETERMINISTIC TURING MACHINES AND NP


read
write
move
0 → 0, R
□ → □, R
q_{accept}
□ → □, R
0 → 0, R
q_{reject}


read
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move
0 → 0, R
0 → 0, R
□ → □, R
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Definition: A Non-Deterministic TM is a 7-tuple $T = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where:

Q is a finite set of states

Σ is the input alphabet, where $\Box \notin \Sigma$

Γ is the tape alphabet, where $\Box \in \Gamma$ and $\Sigma \subseteq \Gamma$

$\delta : Q \times \Gamma \rightarrow 2^{(Q \times \Gamma \times \{L,R\})}$

$q_0 \in Q$ is the start state

$q_{accept} \in Q$ is the accept state

$q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$

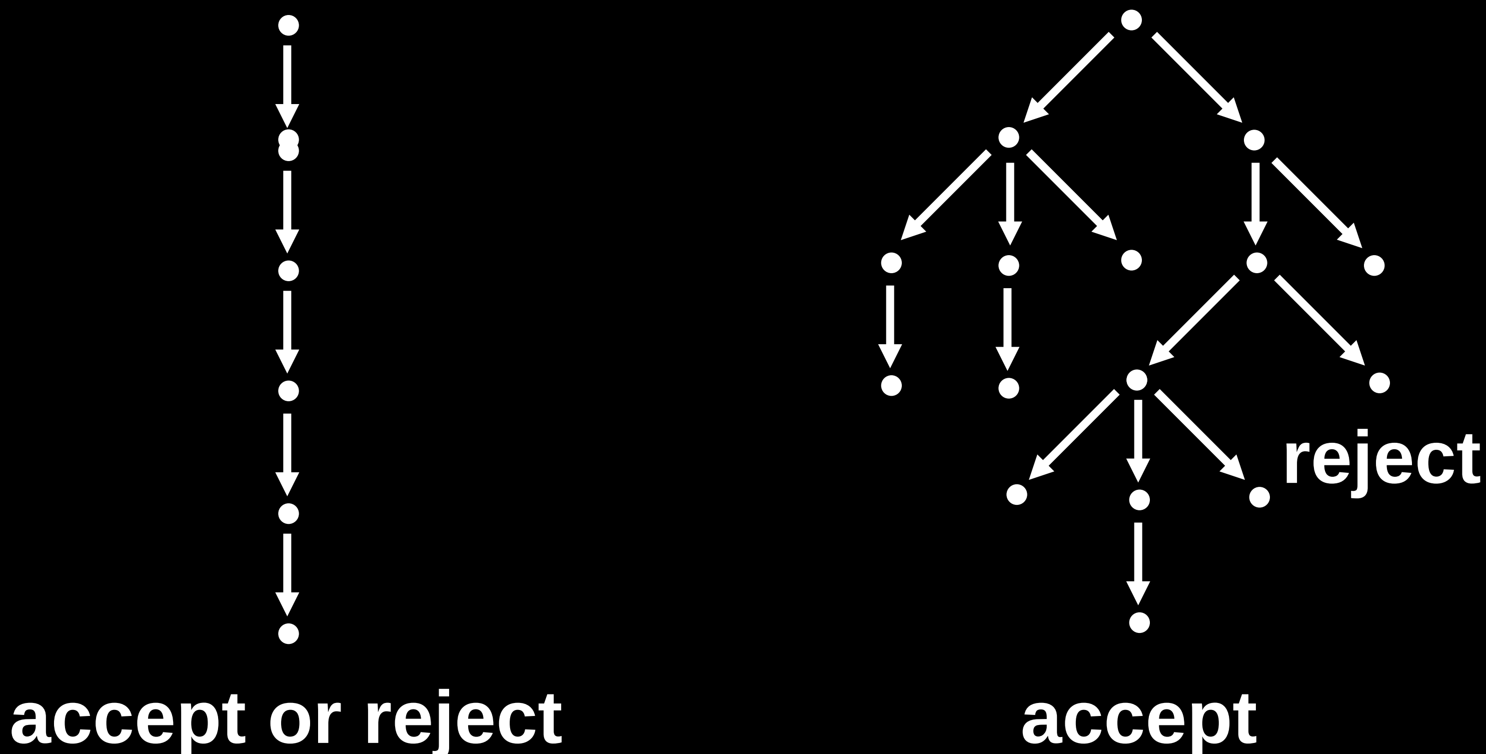
NON-DETERMINISTIC TMs

…are just like standard TMs, except:

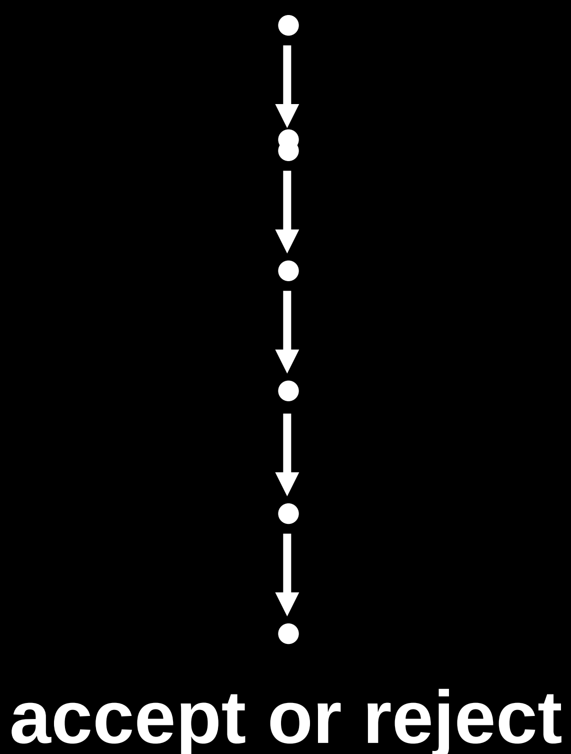
1. The machine may proceed according to several possibilities

2. The machine accepts a string if there exists a path from start configuration to an accepting configuration

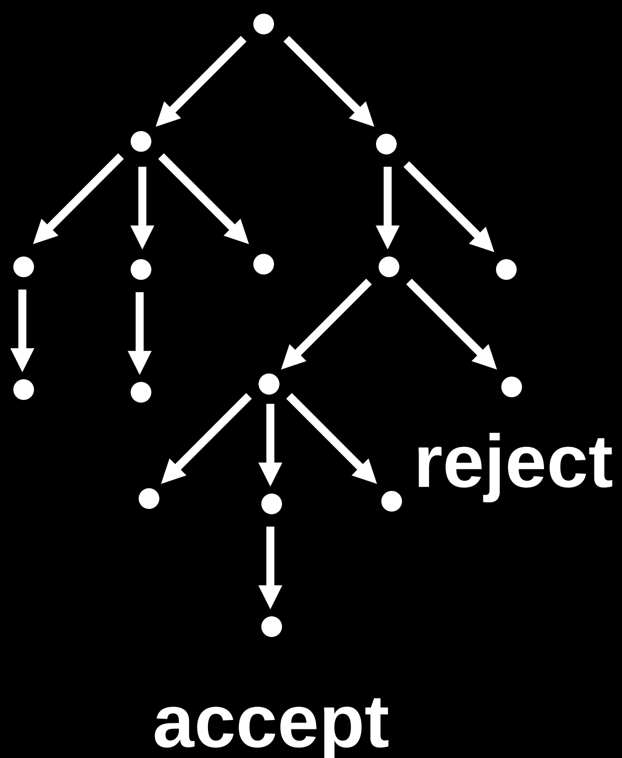
Deterministic Computation | **Non-Deterministic Computation**



Deterministic Computation



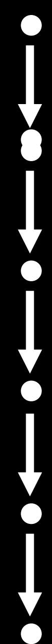
Non-Deterministic Computation

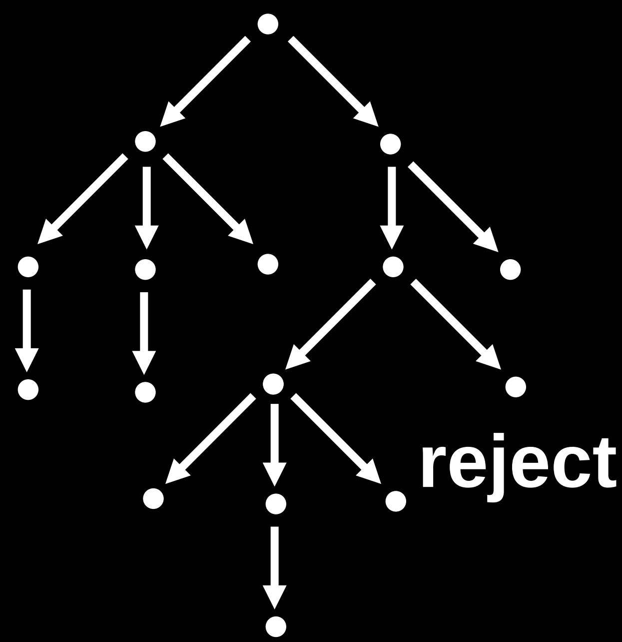


Definition*:* Let M be a NTM that is a decider, Ie on all inputes, all branches halt (with accept or reject). The running time or time-complexity of M is the function $f : N \rightarrow N$, where f(n) is the maximum number of steps that M uses *on any branch of its computation on any input of length n.*

Deterministic Computation

Non-Deterministic Computation





accept or reject | **accept**

Theorem: Let t(n) be a function such that t(n) ≥ n. Then every t(n)-time nondeterministic single-tape TM has an equivalent $2^{O(t(n))}$ deterministic single tape TM

Definition: NTIME(t(n)) = { L | L is decided by a O(t(n))-time non-deterministic Turing machine }

$$\text{TIME}(t(n)) \subseteq \text{NTIME}(t(n))$$

BOOLEAN FORMULAS

logical operations

parentheses

$$\phi = (\neg x \land y) \lor z$$

variables

A **satisfying assignment** is a setting of the variables that makes the formula true

$x = 1, y = 1, z = 1$ is a satisfying assignment for ϕ

A Boolean formula is satisfiable if there exists a satisfying assignment for it

YES $a \wedge b \wedge c \wedge \neg d$

NO $\neg(x \vee y) \wedge x$

SAT = { ϕ | ϕ is a satisfiable Boolean formula }

A **3cnf-formula** is of the form:

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_4 \vee x_2 \vee x_5) \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$$

literals

clauses

$$(x_1 \vee \neg x_2 \vee x_1)$$

$$(x_3 \vee x_1) \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$$

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_4 \vee x_2 \vee x_1) \vee (x_3 \vee x_1 \vee \neg x_1)$$

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NO $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_3 \wedge \neg x_2 \wedge \neg x_1)$

3SAT = { ϕ | ϕ is a satisfiable 3cnf-formula }

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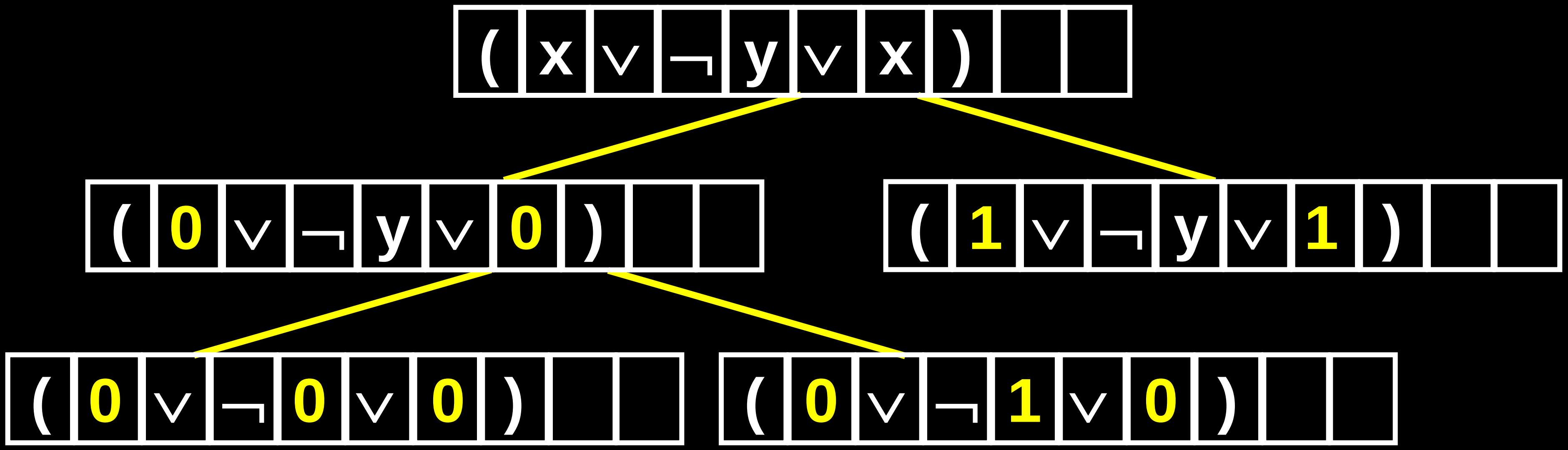
Theorem: 3SAT $\in$ NTIME(n^2)

On input ϕ:

1. Check if the formula is in 3cnf
2. For each variable, non-deterministically substitute it with 0 or 1



3. Test if the assignment satisfies ϕ

$$NP = \bigcup_{k \in N} NTIME(n^k)$$

Theorem: $L \in NP \Leftrightarrow$ if there exists a poly-time Turing machine V(erifier) with

$L = \{ x \mid \exists y$(witness) $|y| = poly(|x|)$ and $V(x,y)$ accepts $\}$

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Proof:

(1) If $L = \{ x \mid \exists y\ |y| = poly(|x|) \text{ and } V(x,y) \text{ accepts} \}$
then $L \in NP$

(2) If $L \in NP$ then
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Because we can guess y and then run V

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Theorem: $L \in NP \Leftrightarrow$ if there exists a poly-time Turing machine V(erifier) with

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Let N be a non-deterministic poly-time TM that decides L and define V(x,y) to accept if y is an accepting computation history of N on x

3SAT = $\{ \phi \mid \exists y$ **such that y is a satisfying assignment to** ϕ **and** ϕ **is in 3cnf** $\}$

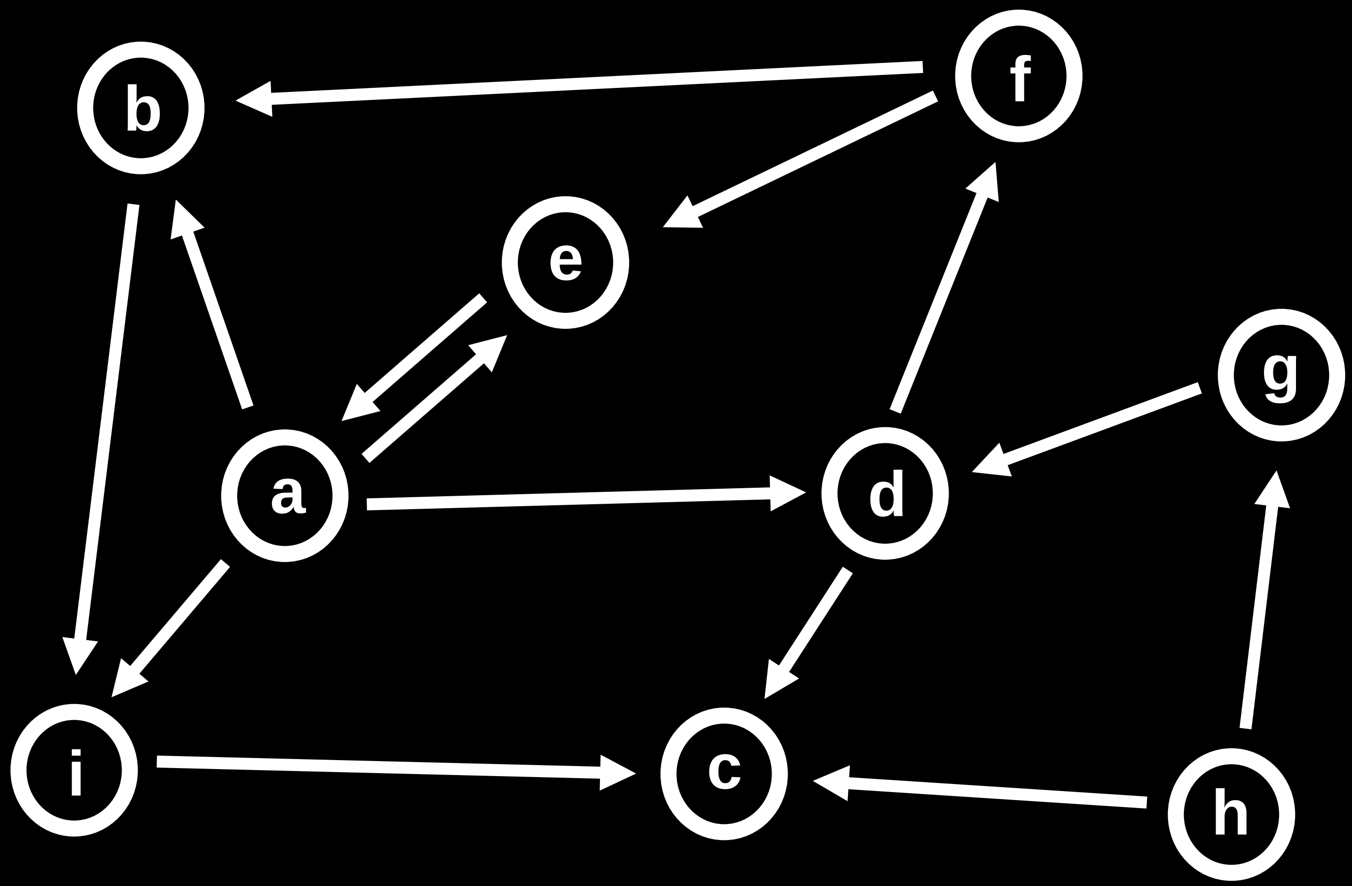
SAT = $\{ \phi \mid \exists y$ **such that y is a satisfying assignment to** ϕ $\}$

A language is in NP if and only if there exist polynomial-length certificates* for membership to the language

SAT is in NP because a satisfying assignment is a polynomial-length certificate that a formula is satisfiable

* that can be verified in poly-time

HAMILTONIAN PATHS



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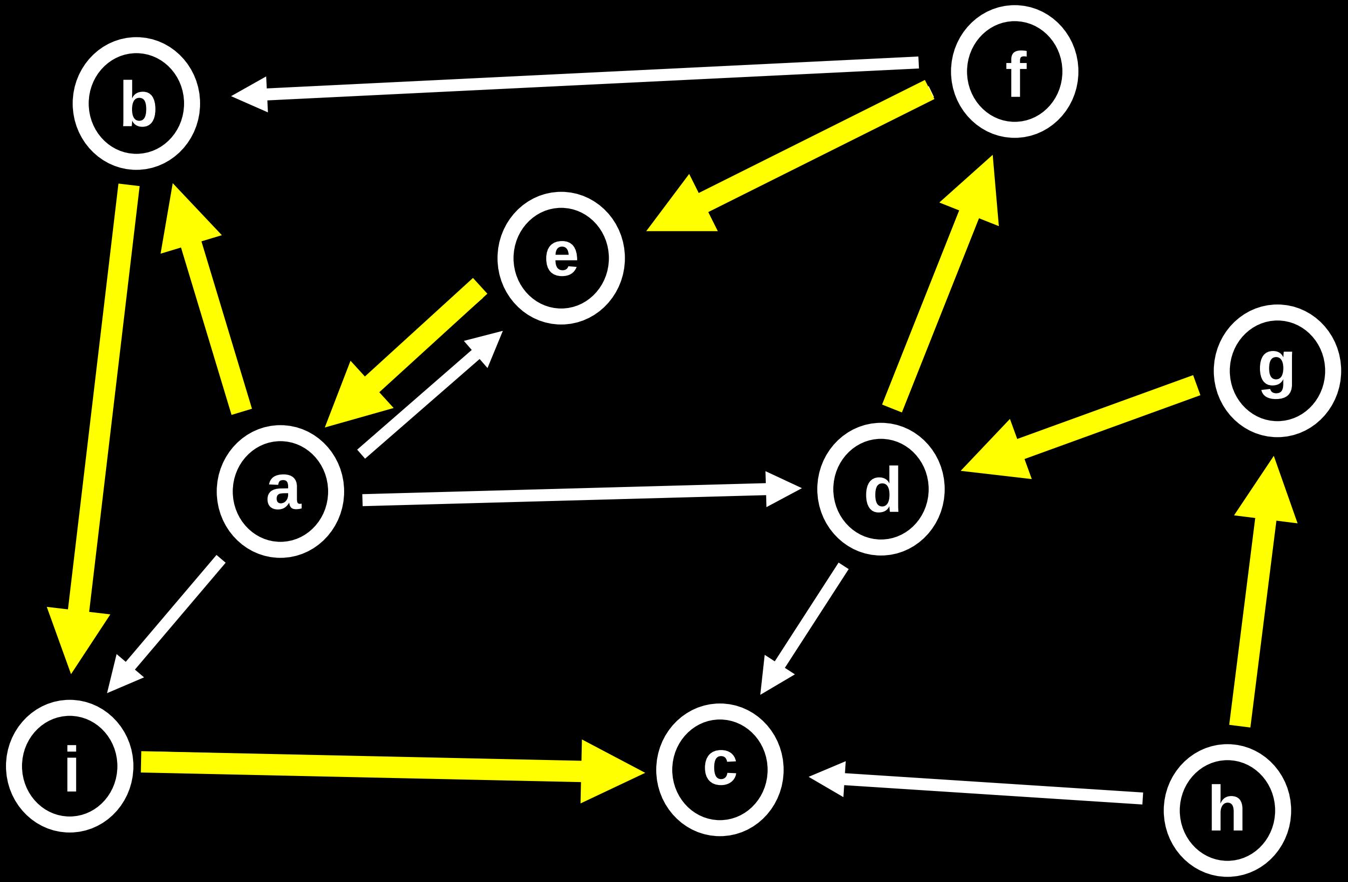
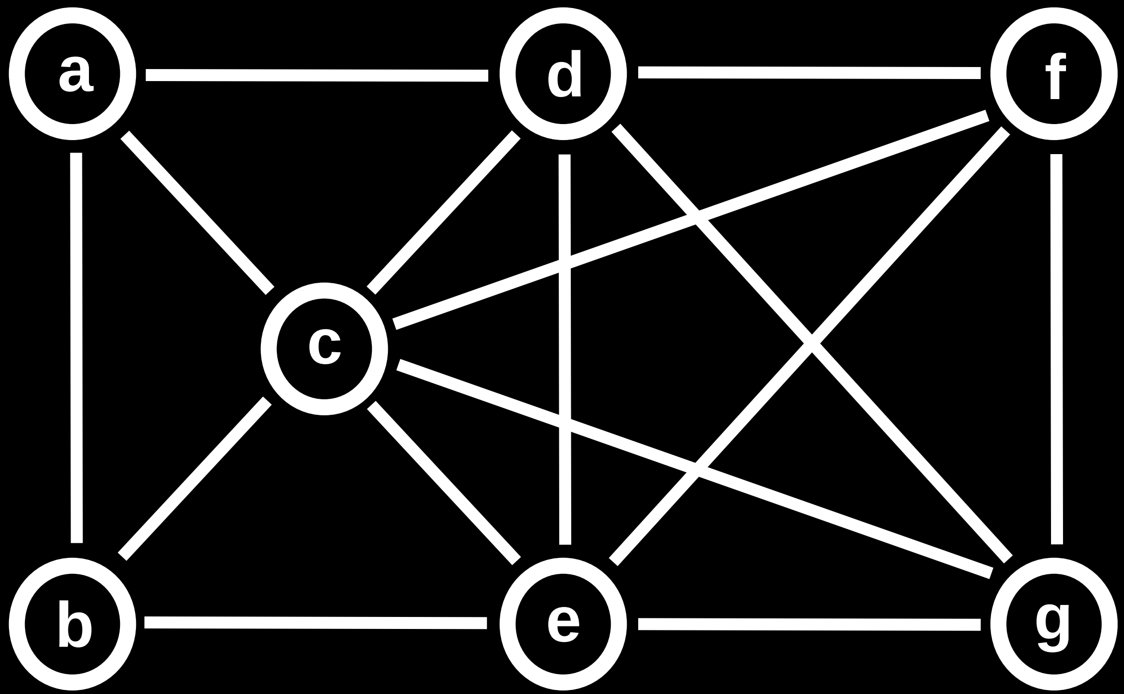


HAMPATH = { (G,s,t) | G is a directed graph with a Hamiltonian path from s to t }

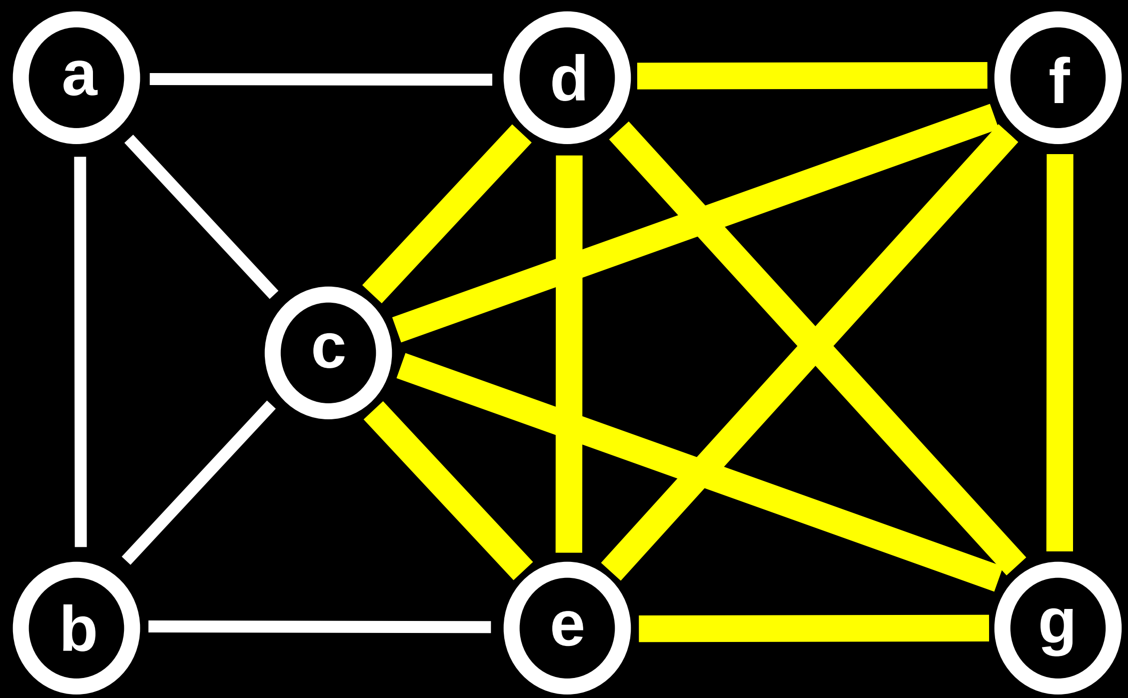
Theorem: HAMPATH $\in$ NP

The Hamilton path itself is a certificate

K-CLIQUES



K-CLIQUES



CLIQUE = { (G,k) | G is an undirected graph with a k-clique }

Theorem: CLIQUE $\in$ NP

The k-clique itself is a certificate

NP = all the problems for which once you have the answer it is easy (i.e. efficient) to verify

P = NP?

P = NP?

$$$

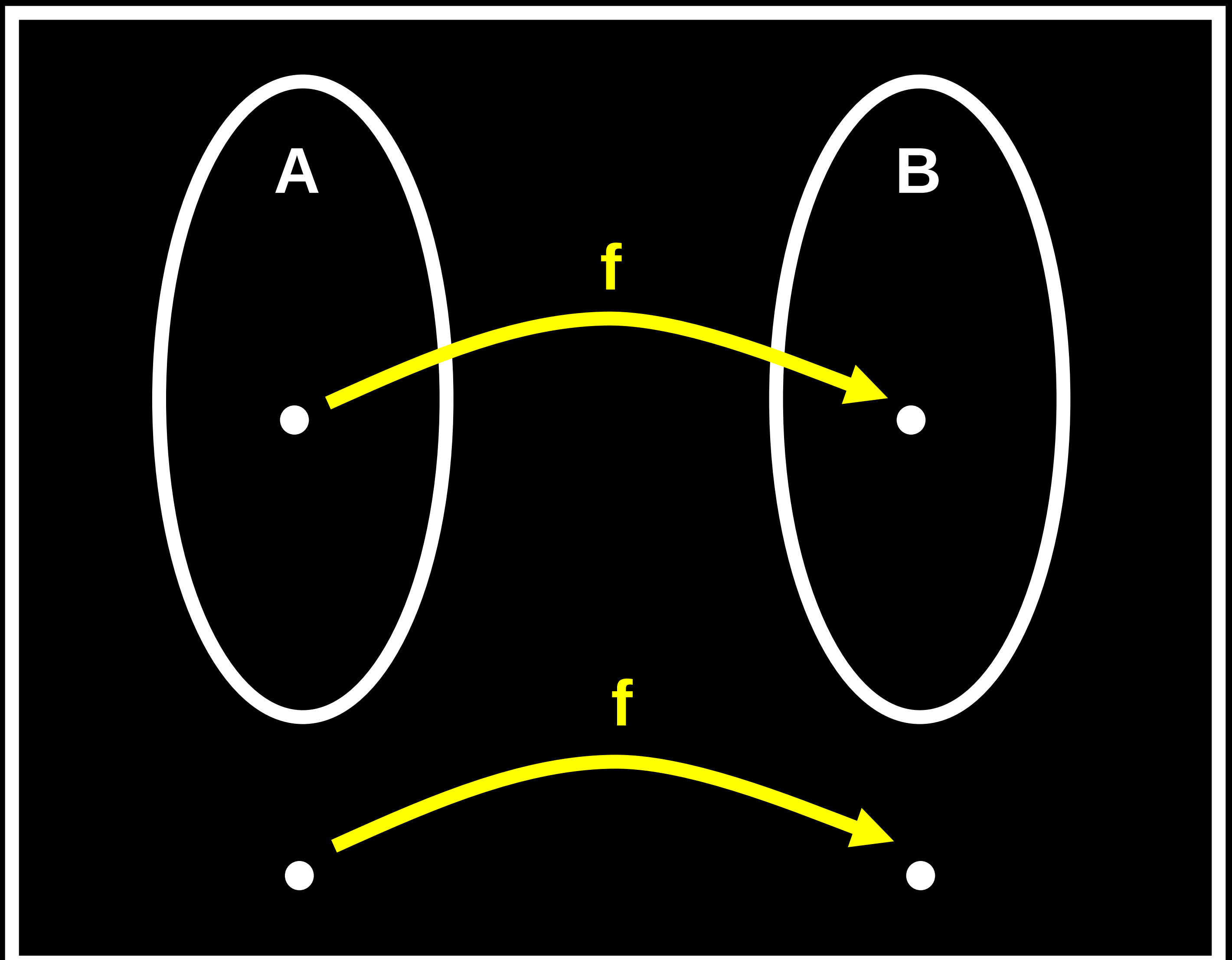
POLY-TIME REDUCIBILITY

$f : \Sigma^* \rightarrow \Sigma^*$ is a polynomial time computable function if some poly-time Turing machine M, on every input w, halts with just f(w) on its tape

Language A is polynomial time reducible to language B, written $A \leq_P B$, if there is a poly-time computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that:

$$w \in A \Leftrightarrow f(w) \in B$$

f is called a polynomial time reduction of A to B


A
B
f
f

Theorem: If $A \leq_P B$ and $B \in P$, then $A \in P$

Proof: Let M_B be a poly-time (deterministic) TM that decides B and let f be a poly-time reduction from A to B

We build a machine M_A that decides A as follows:

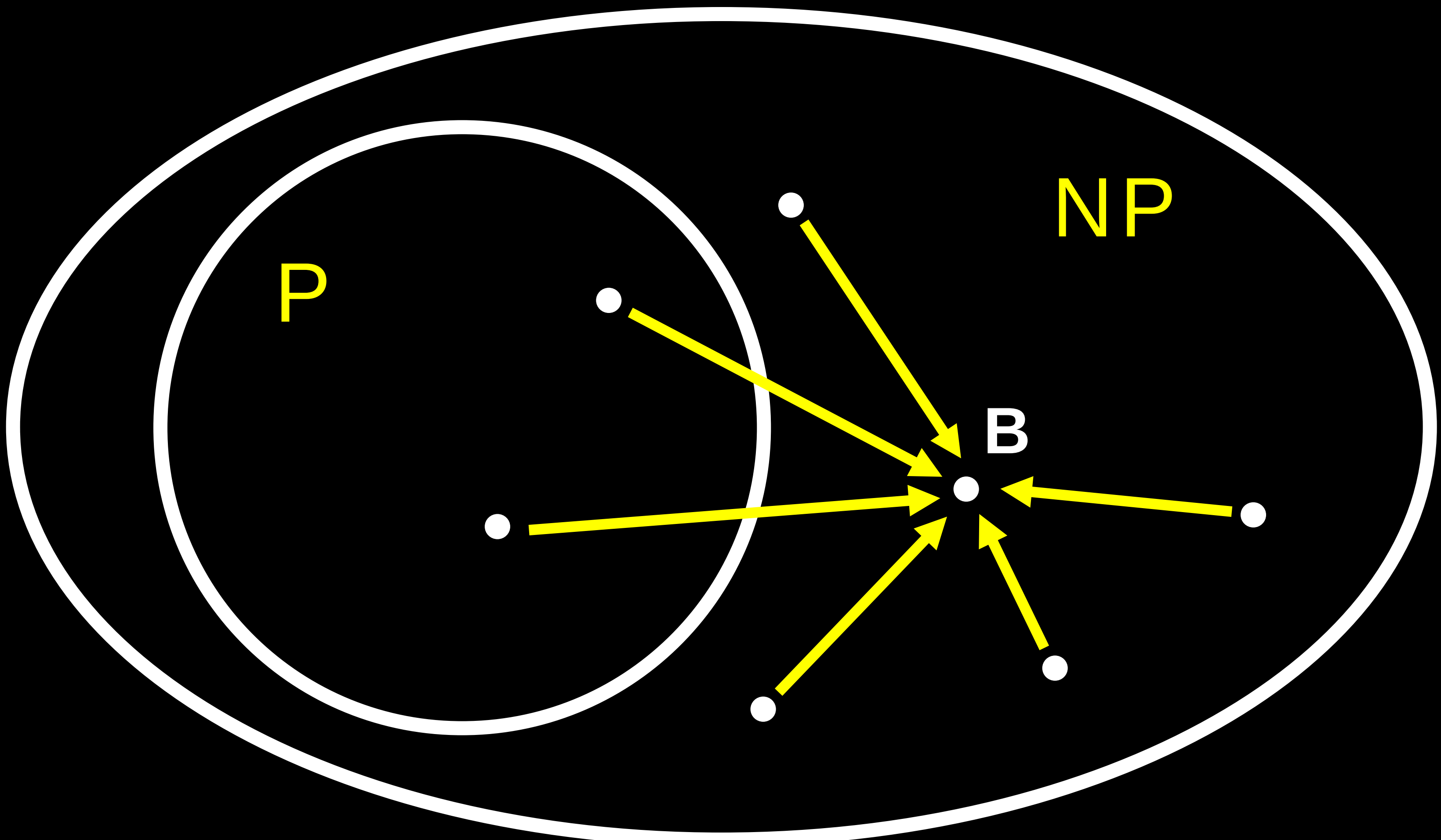
On input w:

1. Compute $f(w)$
2. Run M_B on $f(w)$

Definition: A language B is NP-complete if:

1. B $\in$ NP
2. Every A in NP is poly-time reducible to B (i.e. B is NP-hard)

Suppose B is NP-Complete



So, if B is NP-Complete and B $\in$ P then NP = P. Why?

Theorem (Cook-Levin): SAT is NP-complete

Corollary: SAT $\in$ P if and only if P = NP

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Read Chapter 7.3 of the book for next time