15-453
FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY
THE ARITHMETIC HIERARCHY

THURSDAY, MAR 6
THE ARITHMETIC HIERARCHY

\[ \Delta^0_{n+1} = \{ \text{decidable sets} \} \quad (\text{sets} = \text{languages}) \]

\[ \Sigma^0_{n+1} = \{ \text{semi-decidable sets} \} \]

\[ \Sigma^0_n \quad = \{ \text{sets semi-decidable in some } B \in \Sigma^0_n \} \]

\[ \Delta^0_{n+1} \quad = \{ \text{sets decidable in some } B \in \Sigma^0_n \} \]

\[ \Pi^0_n \quad = \{ \text{complements of sets in } \Sigma^0_n \} \]
Decidable Languages

Semi-decidable Languages

Co-semi-decidable Languages

\[ \sum_1^0 \cap \Pi_1^0 = \Delta_1^0 \]

Decidable Languages
Semi-decidable Languages

Decidable Languages

Co-semi-decidable Languages

\[ \Sigma_1^0 \cap \Pi_1^0 = \Delta_1^0 \]

\[ \Sigma_3^0 \cap \Pi_3^0 = \Delta_3^0 \]

\[ \Sigma_2^0 \cap \Pi_2^0 = \Delta_2^0 \]
Definition: A decidable predicate \( R(x,y) \) is some proposition about \( x \) and \( y \), where there is a TM \( M \) such that

for all \( x, y \), \( R(x,y) \) is TRUE \( \Rightarrow \) \( M(x,y) \) accepts
\( R(x,y) \) is FALSE \( \Rightarrow \) \( M(x,y) \) rejects

We say \( M \) “decides” the predicate \( R \).

1. \( x, y \) are positive integers or elements of \( \Sigma^* \)
Definition: A decidable predicate \( R(x,y) \) is some proposition about \( x \) and \( y \), where there is a TM \( M \) such that

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We say \( M \) “decides” the predicate \( R \).

EXAMPLES:
\( R(x,y) = \) “\( x + y \) is less than 100”
\( R(<N>,y) = \) “\( N \) halts on \( y \) in at most 100 steps”

Kleene’s T predicate, \( T(<M>, x, y) \): \( M \) accepts \( x \) in \( y \) steps.

1. \( x, y \) are positive integers or elements of \( \sum^* \)
Definition: A decidable predicate $R(x,y)$ is some proposition about $x$ and $y^1$, where there is a TM $M$ such that

for all $x, y$, $R(x,y)$ is TRUE $\Rightarrow$ $M(x,y)$ accepts

$R(x,y)$ is FALSE $\Rightarrow$ $M(x,y)$ rejects

We say $M$ “decides” the predicate $R$.

EXAMPLES:
$R(x,y) =$ “$x + y$ is less than 100”
$R(<N>,y) =$ “$N$ halts on $y$ in at most 100 steps”
Kleene’s $T$ predicate, $T(<M>, x, y): M$ accepts $x$ in $y$ steps.

Note: $A$ is decidable $\iff A = \{x \mid R(x,\varepsilon)\}$,
for some decidable predicate $R$. 
Theorem: A language $A$ is semi-decidable if and only if there is a decidable predicate $R(x, y)$ such that: 

$$A = \{ x \mid \exists y \ R(x,y) \}$$

Proof:
Theorem: A language $A$ is semi-decidable if and only if there is a **decidable** predicate $R(x, y)$ such that:

$$A = \{ x \mid \exists y \ R(x,y) \}$$

**Proof:**

(1) If $A = \{ x \mid \exists y \ R(x,y) \}$ then $A$ is semi-decidable

(2) If $A$ is semi-decidable, then $A = \{ x \mid \exists y \ R(x,y) \}$
Theorem: A language $A$ is semi-decidable if and only if there is a decidable predicate $R(x, y)$ such that: $A = \{ x \mid \exists y \ R(x,y) \}$

Proof:
(1) If $A = \{ x \mid \exists y \ R(x,y) \}$ then $A$ is semi-decidable
Because we can enumerate over all $y$'s

(2) If $A$ is semi-decidable, then $A = \{ x \mid \exists y \ R(x,y) \}$
Theorem: A language $A$ is semi-decidable if and only if there is a decidable predicate $R(x,y)$ such that: $A = \{ x | \exists y \ R(x,y) \}$

Proof:

1. If $A = \{ x | \exists y \ R(x,y) \}$ then $A$ is semi-decidable
   Because we can enumerate over all $y$'s

2. If $A$ is semi-decidable, then $A = \{ x | \exists y \ R(x,y) \}$
Let $M$ semi-decide $A$
Then, $A = \{ x | \exists y \ T(<M>, x, y) \}$ (Here $M$ is fixed.)
where
Kleene’s $T$ predicate, $T(<M>, x, y)$: $M$ accepts $x$ in $y$ steps.
Theorem

$\Sigma^0_1 = \{ \text{semi-decidable sets} \}$

$= \{ x | \exists y \ R(x,y) \}$

$\Pi^0_1 = \{ \text{complements of semi-decidable sets} \}$

$= \{ x | \forall y \ R(x,y) \}$

$\Delta^0_1 = \{ \text{decidable sets} \}$

$= \Sigma^0_1 \cap \Pi^0_1$

Where $R$ is a decidable predicate
Theorem

$\Sigma_2^0 = \{ \text{sets semi-decidable in some semi-dec. B} \}$

= languages of the form $\{ x | \exists y_1 \forall y_2 R(x,y_1,y_2) \}$

$\Pi_2^0 = \{ \text{complements of } \Sigma_2^0 \text{ sets} \}$

= languages of the form $\{ x | \forall y_1 \exists y_2 R(x,y_1,y_2) \}$

$\Delta_2^0 = \Sigma_2^0 \cap \Pi_2^0$

Where $R$ is a decidable predicate
Theorem

\[ \sum_0^n = \text{languages } \{ x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q y_n \ R(x, y_1, \ldots, y_n) \} \]

\[ \Pi_0^n = \text{languages } \{ x \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n \ R(x, y_1, \ldots, y_n) \} \]

\[ \Delta_0^n = \sum_0^n \cap \Pi_0^n \]

Where \( R \) is a decidable predicate
Example

$$\sum_1^0 = \text{languages of the form } \{ x \mid \exists y \ R(x,y) \}$$

We know that $$A_{TM}$$ is in $$\sum_1^0$$ Why?

Show it can be described in this form:
$\Sigma^0_1 = \text{languages of the form } \{ x | \exists y \ R(x,y) \}$

We know that $A_{TM}$ is in $\Sigma^0_1$

Show it can be described in this form:

$A_{TM} = \{ \langle M, w \rangle | \exists t \ [M \text{ accepts } w \text{ in } t \text{ steps}] \}$

Decidable predicate
Example

\[ \Sigma_1^0 = \text{languages of the form } \{ x \mid \exists y \ R(x,y) \} \]

We know that \( A_{TM} \) is in \( \Sigma_1^0 \) Why?

Show it can be described in this form:

\[ A_{TM} = \{ \langle M, w \rangle \mid \exists t \ [M \text{ accepts } w \text{ in } t \text{ steps}] \} \]

Decidable predicate

\[ A_{TM} = \{ \langle M, w \rangle \mid \exists t \ T (<M>, w, t) \} \]
Example

$\Sigma^0_1$ = languages of the form \{ $x$ | \exists $y$ $R(x,y)$ \}

We know that $A_{TM}$ is in $\Sigma^0_1$ Why?

Show it can be described in this form:

$A_{TM} = \{ <(M,w)> | \exists t [M \text{ accepts } w \text{ in } t \text{ steps}] \}$

decidable predicate

$A_{TM} = \{ <(M,w)> | \exists t T(<M>, w, t) \}$

$A_{TM} = \{ <(M,w)> | \exists v (v \text{ is an accepting computation history of } M \text{ on } w) \}$
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

$\sum_1^0 \cap \pi_2^0 = \Delta_2^0$

$\Delta_3^0$

$\Delta_1^0$

$\sum_3^0$

$\pi_3^0$

$\pi_1^0$

$\sum_2^0$

$\pi_2^0$
$$\Pi^0_1 = \text{languages of the form } \{ \ x \mid \forall y \ R(x,y) \ \}$$

Show that \text{EMPTY} (ie, \text{ETM}) = \{ \ M \mid L(M) = \emptyset \ \} is in $$\Pi^0_1$$
$\Pi^0_1 = \text{languages of the form } \{ x \mid \forall y \ R(x,y) \} \}

\text{Show that } \text{EMPTY (ie, } E_{TM}) = \{ M \mid L(M) = \emptyset \} \text{ is in } \Pi^0_1

\text{EMPTY} = \{ M \mid \forall w \forall t [M \text{ doesn’t accept } w \text{ in } t \text{ steps}] \}
\[ \Pi^0_1 = \text{languages of the form } \{ x \mid \forall y \ R(x,y) \} \]

Show that \text{EMPTY (ie, } E_{TM}) = \{ M \mid L(M) = \emptyset \} \text{ is in } \Pi^0_1

\text{EMPTY} = \{ M \mid \forall w \forall t \ [ \neg T(<M>, w, t) ] \}

\text{decidable predicate}
$\Pi^0_1$ = languages of the form \{ $x \mid \forall y \ R(x,y)$ \}

Show that $\text{EMPTY}$ (ie, $E_{TM}$) = \{ $M \mid L(M) = \emptyset$ \} is in $\Pi^0_1$

$\text{EMPTY} = \{ M \mid \forall w \forall t \ [ \neg T(<M>, w, t) ] \}$

two quantifiers?? decidable predicate
THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function \(<, \,> : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*\) and computable functions \(\pi_1\) and \(\pi_2 : \Sigma^* \rightarrow \Sigma^*\) such that

\[ z = <w, t> \implies \pi_1 (z) = w \text{ and } \pi_2(z) = t \]
THE PAIRING FUNCTION

**Theorem.** There is a 1-1 and onto computable function \(<, >: \Sigma^* \times \Sigma^* \rightarrow \Sigma^*\) and computable functions \(\pi_1\) and \(\pi_2: \Sigma^* \rightarrow \Sigma^*\) such that

\[ z = \langle w, t \rangle \implies \pi_1(z) = w \text{ and } \pi_2(z) = t \]

\[ \text{EMPTY} = \{ M \mid \forall w \forall t \ [M \text{ doesn't accept } w \text{ in } t \text{ steps}] \} \]

\[ \text{EMPTY} = \{ M \mid \forall z [M \text{ doesn't accept } \pi_1(z) \text{ in } \pi_2(z) \text{ steps}] \} \]

\[ \text{EMPTY} = \{ M \mid \forall z [ \neg T(\langle M \rangle, \pi_1(z), \pi_2(z)) ] \} \]
Theorem. There is a 1-1 and onto computable function $<, >: \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ and computable functions $\pi_1$ and $\pi_2 : \Sigma^* \rightarrow \Sigma^*$ such that

$z = <w, t> \Rightarrow \pi_1(z) = w$ and $\pi_2(z) = t$

Proof: Let $w = w_1...w_n \in \Sigma^*$, $t \in \Sigma^*$. Let $a, b \in \Sigma$, $a \neq b$.

$<w, t> := a w_1... a w_n b t$

$\pi_1(z) := \text{“if } z \text{ has the form } a w_1... a w_n b t, \text{ then output } w_1... w_n, \text{ else output } \varepsilon”$

$\pi_2(z) := \text{“if } z \text{ has the form } a w_1... a w_n b t, \text{ then output } t, \text{ else output } \varepsilon”$
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \Delta_3^0 \]

\[ \Sigma_3^0 \]

\[ \Pi_3^0 \]

\[ \Sigma_2^0 \]

\[ \Pi_2^0 \]

\[ \Sigma_1^0 \]

\[ \Pi_1^0 \]

\[ \Delta_2^0 = \Sigma_2^0 \cap \Pi_2^0 \]

\[ \Delta_1^0 \]

\[ \Delta_2^0 \]

\[ \Delta_3^0 \]

\[ A_{TM} \]

EMPTY
\( \Pi_2^0 \) = languages of the form \( \{ x \mid \forall y \exists z \ R(x,y,z) \} \)

Show that TOTAL = \( \{ M \mid M \text{ halts on all inputs} \} \) is in \( \Pi_2^0 \)
$\Pi^0_2 = \text{languages of the form } \{ x | \forall y \exists z \ R(x,y,z) \}$

Show that $\text{TOTAL} = \{ M | M \text{ halts on all inputs} \}$ is in $\Pi^0_2$

$\text{TOTAL} = \{ M | \forall w \exists t \ [M \text{ halts on } w \text{ in } t \text{ steps}] \}$

decidable predicate
$$\Pi^0_2 = \text{languages of the form } \{ x \mid \forall y \exists z \ R(x,y,z) \}$$

Show that $\text{TOTAL} = \{ M \mid M \text{ halts on all inputs} \}$ is in $\Pi^0_2$

$$\text{TOTAL} = \{ M \mid \forall w \ \exists t \ [ T(<M>, w, t) ] \}$$

decidable predicate
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \sum_3^0 \]

\[ \sum_2^0 \]

\[ \sum_1^0 \]

\[ \Delta_3^0 \]

\[ \Delta_2^0 \]

\[ \Delta_1^0 \]

\[ \sum_2^0 \cap \Pi_2^0 \]

\[ \pi_3^0 \]

\[ \pi_2^0 \]

\[ \pi_1^0 \]

TOTAL

EMPTY

\[ A_{TM} \]
\[ \Sigma^0_2 = \text{languages of the form } \{ x \mid \exists y \forall z \ R(x,y,z) \} \]

Show that \( \text{FIN} = \{ M \mid L(M) \text{ is finite} \} \) is in \( \Sigma^0_2 \)
$\sum^0_2 = \text{languages of the form } \{ x \mid \exists y \forall z \ R(x,y,z) \}$

Show that $\text{FIN} = \{ M \mid \text{L}(M) \text{ is finite} \}$ is in $\sum^0_2$

$\text{FIN} = \{ M \mid \exists n \forall w \forall t \ [\text{Either } |w| < n, \text{ or } M \text{ doesn’t accept } w \text{ in } t \text{ steps}] \}$

$\text{FIN} = \{ M \mid \exists n \forall w \forall t \ ( |w| < n \lor \neg T(<M>,w,t) ) \}$

decidable predicate
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

$\sum_3^0$

$\sum_2^0$

$\sum_1^0$

$\Delta_3^0$

$\Delta_2^0$

$\Delta_1^0$

$\Pi_3^0$

$\Pi_2^0$

$\Pi_1^0$

$\sum_2^0 \cap \Pi_2^0$

$\text{FIN}

\text{TOTAL}$

$A_{TM}$

\text{EMPTY}
$\Sigma^0_3 = \text{languages of the form } \{ x \mid \exists y \forall z \exists u \; R(x,y,z,u) \}$

Show that $\text{COF} = \{ M \mid L(M) \text{ is cofinite} \}$ is in $\Sigma^0_2$
\( \sum_3^0 \) = languages of the form \( \{ x \mid \exists y \forall z \exists u \ R(x,y,z,u) \} \)

Show that \( \text{COF} = \{ M \mid \text{L}(M) \text{ is cofinite } \} \) is in \( \sum_2^0 \)

\( \text{COF} = \{ M \mid \exists n \forall w \exists t \ [ |w| > n \Rightarrow M \text{ accept } w \text{ in } t \text{ steps}] \} \)

\( \text{COF} = \{ M \mid \exists n \forall w \exists t \ ( |w| \leq n \forall T(<M>,w,t)) \} \)

decidable predicate
Decidable languages

Semi-decidable languages

\[ \sum_0^0 \]

\[ \sum_0^3 \]

\[ \Pi_3^0 \]

\[ \Pi_1^0 \]

\[ \Delta_0^0 \]

\[ \Delta_2^0 \]

\[ \Delta_3^0 \]

\[ \Pi_2^0 \]

\[ \Pi_0^0 \]

\[ \sum_2^0 \cap \Pi_2^0 \]

\[ \sum_2^0 \]

\[ \sum_1^0 \]

Co-semi-decidable languages

\[ \text{COF} \]

\[ \text{FIN} \]

\[ \text{TOTAL} \]

\[ \text{EMPTY} \]

\[ \text{집계} \]

\[ \text{TM} \]
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \Sigma^0_0 \]

\[ \Delta^0_1 \]

\[ \Delta^0_2 \]

\[ \Sigma^0_2 \]

\[ \Pi^0_2 \]

\[ \Pi^0_3 \]

\[ \sum^0_2 \cap \Pi^0_2 \]

\[ \Delta^0_3 \]

\[ \Pi^0_1 \]

\[ \text{ATM} \]

\[ \text{TOTAL} \]

\[ \text{EMPTY} \]

\[ \text{REG} \]
Decidable languages

Semi-decidable languages

Co-semi-decidable languages

\[ \Sigma^0_3 \]
\[ \sum^0_2 \]
\[ \sum^0_1 \]

\[ \Delta^0_3 \]
\[ \Delta^0_2 \]

\[ \Delta^0_1 \]

\[ \Pi^0_3 \]
\[ \Pi^0_2 \]
\[ \Pi^0_1 \]

\[ \Sigma^0_2 \cap \Pi^0_2 \]

\[ \text{DEc} \]
\[ \text{FIN} \]
\[ \text{TOTAL} \]

\[ \text{ATM} \]
\[ \text{TOTAL} \]

\[ \text{EMPTY} \]
Decidable languages

Semi-decidable languages

CFL

\[ \sum_3^0 \]

\[ \sum_2^0 \]

FIN

\[ \sum_1^0 \]

\[ \Delta_3^0 \]

\[ \Delta_2^0 \]

\[ \Delta_1^0 \]

\[ \Pi_3^0 \]

\[ \Pi_2^0 \]

\[ \Pi_1^0 \]

\[ \text{TOTAL} \]

\[ \text{EMPTY} \]

\[ \text{ATM} \]

\[ \text{FIN} \]

\[ \text{CFL} \]

\[ \sum_2^0 \cap \Pi_2^0 \]
Each is m-complete for its level in hierarchy and cannot go lower (by the SuperHalting Theorem, which shows the hierarchy does not collapse).
Each is m-complete for its level in hierarchy and cannot go lower (by the SuperHalting Theorem, which shows the hierarchy does not collapse).

$L$ is m-complete for class $C$ if

i) $L \in C$ and

ii) $L$ is m-hard for $C$,

ie, for all $L' \in C$, $L' \leq_m L$
$A_{TM}$ is m-complete for class $C = \sum_{1}^{0}$

i) $A_{TM} \in C$

ii) $A_{TM}$ is m-hard for $C$,
\( A_{TM} \) is m-complete for class \( C = \Sigma_1^0 \)

i) \( A_{TM} \in C \)

ii) \( A_{TM} \) is m-hard for \( C \),

Suppose \( L \in C \). Show: \( L \leq_m A_{TM} \)

Let \( M \) semi-decide \( L \). Then Map

\( \Sigma^* \rightarrow \Sigma^* \)

where \( w \rightarrow (M, w) \).

Then, \( w \in L \iff (M, w) \in A_{TM} \)

QED
FIN is $m$-complete for class $C = \sum_2^0$

i) $FIN \in C$

ii) $FIN$ is $m$-hard for $C$,

Suppose $L \in C$. Show: $L \leq_m FIN$
Supose $L \in \Sigma_2^0$ ie $L = \{ w | \exists y \forall z \ R(w,y,z) \}$ where $R$ is decided by some TM $D$

Show: $L \leq_m \text{FIN}$
Suppose \( L \in \Sigma^0_2 \) ie \( L = \{ w | \exists y \forall z \ R(w, y, z) \} \) where \( R \) is decided by some TM \( D \).

Show: \( L \leq_m \text{FIN} \)

Map \( \Sigma^* \rightarrow \Sigma^* \)

where \( w \rightarrow N_{D,w} \)
Suppose \( L \in \Sigma^0_2 \) i.e. \( L = \{ w \mid \exists y \forall z \ R(w, y, z) \} \) where \( R \) is decided by some TM \( D \)

Show: \( L \leq_m \text{FIN} \)

Map \( \Sigma^* \to \Sigma^* \)

where \( w \to N_{D, w} \)

Define \( N_{D, w} \) on input \( s \):

1. Write down all strings \( y \) of length \(|s|\)
2. For each \( y \), try to find a \( z \) such that
   \( \neg R(w, y, z) \) and accept if all are successful
   (here use \( D \) and \( w \))

So, \( w \in L \iff N_{D, w} \in \text{FIN} \)
ORACLES not all powerful

The following problem cannot be decided, even by a TM with an oracle for the Halting Problem:

SUPERHALT = \{ (M,x) \mid M, \text{ with an oracle for the Halting Problem, halts on } x \}
ORACLES not all powerful

The following problem cannot be decided, even by a TM with an oracle for the Halting Problem:

SUPERHALT = \{ (M,x) \mid M, with an oracle for the Halting Problem, halts on x \}

Can use diagonalization here!

Suppose H decides SUPERHALT (with oracle)
Define \( D(X) = \text{"if } H(X,X) \text{ accepts (with oracle) then LOOP, else ACCEPT."} \)

\( D(D) \) halts \( \iff \) \( H(D,D) \) accepts \( \iff \) \( D(D) \) loops…
ORACLES not all powerful

**Theorem:** The arithmetic hierarchy is strict. That is, the nth level contains a language that isn’t in any of the levels below n.

**Proof IDEA:** Same idea as the previous slide.
**Theorem:** The arithmetic hierarchy is strict. That is, the $n$th level contains a language that isn’t in any of the levels below $n$.

**Proof IDEA:** Same idea as the previous slide.

$	ext{SUPERHALT}^0 = \text{HALT} = \{ (M,x) \mid M \text{ halts on } x \}$.

$	ext{SUPERHALT}^1 = \{ (M,x) \mid M, \text{ with an oracle for the Halting Problem, halts on } x \}$.

$	ext{SUPERHALT}^n = \{ (M,x) \mid M, \text{ with an oracle for } \text{SUPERHALT}^{n-1}, \text{ halts on } x \}$.
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Read Chapter 6.4 for next time