15-453

FORMAL LANGUAGES,
AUTOMATA AND
COMPUTABILITY
FINITE STATE CONTROL

INPUT

INFINITE TAPE

Computational Theology?

ORACLE TMs

Oracle

GANDALF
Is \((M, w)\) in \(A_{TM}\)?
FINITE STATE CONTROL

INFINITE TAPE

$A_{TM}$

Is $(M, w)$ in $A_{TM}$?

$q_{YES}$

Oracle for $A_{TM}$

Input

Put

Infinite tape
An **ORACLE** is a set $B$ to which the TM may pose membership questions "Is $w$ in $B$?" (formally: TM enters state $q_?$) and the TM always receives a correct answer in one step (formally: if the string on the oracle tape is in $B$, state $q_?$ is changed to $q_{\text{YES}}$, otherwise $q_{\text{NO}}$)

This makes sense even if $B$ is not decidable! (We do not assume that the oracle $B$ is a computable set!)
We say A is semi-decidable in B if there is an oracle TM M with oracle B that semi-decides A.

We say A is decidable in B if there is an oracle TM M with oracle B that decides A.
HALT\textsubscript{TM} is DECIDABLE in A\textsubscript{TM}

On input \((M, w)\), decide if \(M\) halts on \(w\) as follows:

1. Ask the oracle for A\textsubscript{TM} whether \(M\) accepts \(w\). If yes, then ACCEPT

2. Switch the accept and reject states of \(M\) to get \(M'\). Ask the oracle for A\textsubscript{TM} whether \(M'\) accepts \(w\). If yes, then ACCEPT

3. REJECT
\( A_{TM} \) is DECIDABLE in \( HALT_{TM} \)

On input \((M,w)\), decide if \( M \) accepts \( w \) as follows:

Ask the oracle for \( HALT_{TM} \) whether \( M \) halts on \( w \). If yes, then run \( M(w) \) and output its answer. If no, then REJECT.
Language A “Turing Reduces” to Language B

if A is decidable in B, ie if there is an oracle TM $M$ with oracle $B$ that decides $A$

$A \leq_T B$
\[ \leq_T \text{ VERSUS } \leq_m \]

**Theorem:** If \( A \leq_m B \) then \( A \leq_T B \)

**Proof:**

If \( A \leq_m B \) then there is a computable function \( f : \Sigma^* \rightarrow \Sigma^* \), where for every \( w \),

\[ w \in A \iff f(w) \in B \]

We can thus use an oracle for \( B \) to decide \( A \)

**Theorem:** \( \neg \text{HALT}_{TM} \leq_T \text{HALT}_{TM} \)

**Theorem:** \( \neg \text{HALT}_{TM} \not\leq_m \text{HALT}_{TM} \)
THE ARITHMETIC HIERARCHY

\[ \Delta^0_1 = \{ \text{decidable sets} \} \quad (\text{sets = languages}) \]

\[ \Sigma^0_1 = \{ \text{semi-decidable sets} \} \]

\[ \Sigma^0_{n+1} = \{ \text{sets semi-decidable in some } B \in \Sigma^0_n \} \]

\[ \Delta^0_{n+1} = \{ \text{sets decidable in some } B \in \Sigma^0_n \} \]

\[ \Pi^0_n = \{ \text{complements of sets in } \Sigma^0_n \} \]
Decidable Languages

Semi-decidable Languages

Co-semi-decidable Languages

\( \Delta^0_1 \cap \Sigma^0_1 \cap \Pi^0_1 = \Sigma^0_1 \cap \Pi^0_1 \)
Semidecidable Languages

Decidable Languages

Co-semidecidable Languages
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Read Chapter 6.4 for next time