15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY
\( A_{TM} = \{ (M, w) \mid M \text{ is a TM that accepts string } w \} \)

\( HALT_{TM} = \{ (M, w) \mid M \text{ is a TM that halts on string } w \} \)

\( E_{TM} = \{ M \mid M \text{ is a TM and } L(M) = \emptyset \} \)

\( REG_{TM} = \{ M \mid M \text{ is a TM and } L(M) \text{ is regular} \} \)

\( EQ_{TM} = \{ (M, N) \mid M, N \text{ are TMs and } L(M) = L(N) \} \)

\( ALL_{PDA} = \{ P \mid P \text{ is a PDA and } L(P) = \Sigma^* \} \)

**ALL UNDECIDABLE**

Use Reductions to Prove

Which are SEMI-DECIDABLE?
THE POST CORRESPONDENCE PROBLEM

TUESDAY   FEB 26
THE PCP GAME

ba
---
a

a
---
ab

b
---
bcb

b
---
a
THE PCP GAME

\[
\begin{array}{cccc}
ba & a \\
\hline
ab & a \\
\end{array} & 
\begin{array}{cccc}
a & b \\
\hline
ab & bcb \\
\end{array} & 
\begin{array}{cccc}
b & b \\
\hline
ba & a \\
\end{array}
\]
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GENERAL RULE #1

If every top string is longer than the corresponding bottom one, there can’t be a match
GENERAL RULE #2

If there is a domino with the same string on the top and on the bottom, there is a match.
POST CORRESPONDENCE PROBLEM

Given a collection of dominos, is there a match?
PCP = \{ P \mid P \text{ is a set of dominos with a match} \}

PCP is undecidable!
THE FPCP GAME

... is just like the PCP game except that a match has to start with the first domino
FPCP
Theorem: FPCP is undecidable

Proof: Assume machine C decides FPCP

We will show how to use C to decide $A_{TM}$
Given \((M,w)\)

we will construct a set of dominoes \(P_{M,w}\) where a match is an accepting computation history for \(M\) on \(w\)

\[ P_{M,w} = \begin{array}{c}
\text{caa} \\
\text{c} \\
\text{aba} \\
\text{bb} \\
\ldots \\
\text{a} \\
\text{d}
\end{array} \]
\$\left\{ 0^{2^n} \mid n \geq 0 \right\}$

Diagram:
- States: $q_0, q_1, q_2, q_3, q_4, q_{\text{accept}}, q_{\text{reject}}$
- Transitions:
  - $0 \rightarrow \square, R$
  - $x \rightarrow x, R$
  - $\square \rightarrow \square, R$
  - $x \rightarrow x, L$
  - $0 \rightarrow 0, L$
  - $0 \rightarrow x, R$
  - $0 \rightarrow 0, R$
  - $x \rightarrow x, R$
  - $\square \rightarrow \square, L$
  - $\square \rightarrow \square, R$
- Initial state: $q_0$
- Accepting states: $q_{\text{accept}}$
- Rejecting state: $q_{\text{reject}}$
\{0^{2^n} \mid n \geq 0\}
Given \((M,w)\), we will construct an instance \(P\) of FPCP in 7 steps.

Assume \(M\) on \(w\) never attempts to move off left hand edge of tape.
STEP 1

Put

\[ \#q_0w_1w_2\ldots w_n\# \]

into P

For start configuration

START
STEP 2

If $\delta(q,a) = (p,b,R)$ then add $qa bp$

STEP 3

If $\delta(q,a) = (p,b,L)$ then add $cqa pcb$ for all $c \in \Gamma$

RULES
\{ 0^{2^n} \mid n \geq 0 \}
\{ 0^{2^n} \mid n \geq 0 \}
STEP 4
add a for all $a \in \Gamma$

STEP 5
add #

For tape cells not adjacent to head

For configuration separator

To simulate the blanks on the right hand side of tape

CONTINUE
**STEP 4**

add 

for all \( a \in \Gamma \)

**STEP 5**

add 

Add pseudo-steps after TM halts (catch up)

**STEP 6**

add 

for all \( a \in \Gamma \)
STEP 7
add $q_{acc}##$

END
Given \((M, w)\), we can construct an instance of FPCP that has a match if and only if \(M\) accepts \(w\).
Can convert an instance of FPCP into one of PCP:

Let $u = u_1u_2...u_n$, define:

$$\star u = \star u_1 \star u_2 \star u_3 \star ... \star u_n$$

$$u\star = u_1 \star u_2 \star u_3 \star ... \star u_n \star$$

$$\star u\star = \star u_1 \star u_2 \star u_3 \star ... \star u_n \star$$
Given \((M,w)\), we can construct an instance of PCP that has a match if and only if \(M\) accepts \(w\).
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Read Chapters 5.2 and 5.3 of the book for next time