15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

RICE'S THEOREM, THE RECURSION THEOREM, AND THE FIXED-POINT THEOREM

THURSDAY FEB 27

FIN_{TM} = { M | M is a TM and L(M) is finite}

Is **FIN_{TM}** Decidable?

Note Properties of this language:

- **FIN_{TM} is a language of Turing Machines**
- **If $M_1 \equiv M_2$ (ie $L(M_1) = L(M_2)$), then either both M_1 and M_2 are in FIN_{TM} or both are not.**
- **There are TMs M_1 and M_2, such that $M_1 \in FIN_{TM}$ and $M_2 \notin FIN_{TM}$**

RICE'S THEOREM

Let L be a language over Turing machines. Assume that L satisfies the following properties:

1. For TMs M_1 and M_2, if $M_1 \equiv M_2$ then $M_1 \in L \Leftrightarrow M_2 \in L$

2. There are TMs M_1 and M_2, such that $M_1 \in L$ and $M_2 \notin L$

Then L is undecidable

EXTREMELY POWERFUL!

RICE'S THEOREM

Let L be a language over Turing machines. Assume that L satisfies the following properties:

1. For TMs M_1 and M_2, if $M_1 \equiv M_2$ then $M_1 \in L \Leftrightarrow M_2 \in L$

2. There are TMs M_1 and M_2, such that $M_1 \in L$ and $M_2 \notin L$

Then L is undecidable

FIN_{TM} = { M | M is a TM and L(M) is finite}

RICE'S THEOREM

Let L be a language over Turing machines. Assume that L satisfies the following properties:

1. For TMs M_1 and M_2, if $M_1 \equiv M_2$ then $M_1 \in L \Leftrightarrow M_2 \in L$

2. There are TMs M_1 and M_2, such that $M_1 \in L$ and $M_2 \notin L$

Then L is undecidable

E_{TM} = { M | M is a TM and L(M) = $\emptyset$ }

REG_{TM} = { M | M is a TM and L(M) is regular}

RICE'S THEOREM

Let L be a language over Turing machines. Assume that L satisfies the following properties:

1. For TMs M_1 and M_2, if $M_1 \equiv M_2$ then $M_1 \in L \Leftrightarrow M_2 \in L$

2. There are TMs M_1 and M_2, such that $M_1 \in L$ and $M_2 \notin L$

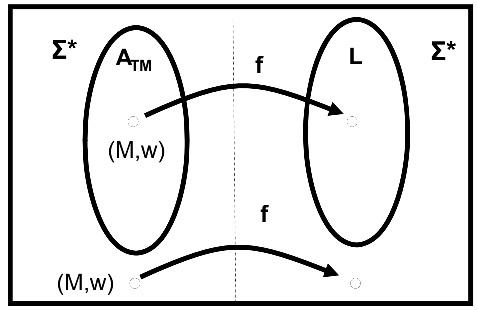
Then L is undecidable

Proof: Will show:

A_{TM} is mapping reducible to L

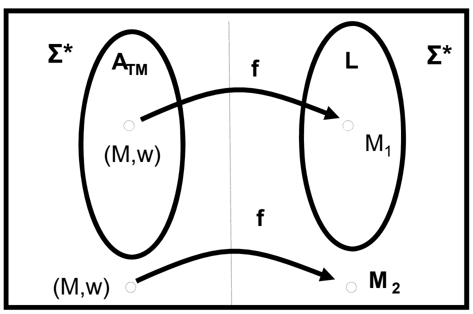
Proof: Show L is undecidable

Show: A_{TM} is mapping reducible to L



Proof: Show L is undecidable

Show: A_{TM} is mapping reducible to L



RICE'S THEOREM

Proof:

Define $M_\emptyset$ to be a TM that never halts

Assume, WLOG, that $M_\emptyset \notin L$ Why?

Let $M_1 \in L$ (such M_1 exists, by assumption)

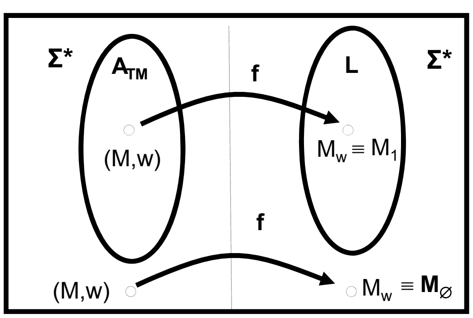
Show A_{TM} is mapping reducible to L :

Map $(M, w) \rightarrow M_w$ where

$M_w(s)$ = accepts if both $M(w)$ and $M_1(s)$ accept
loops otherwise

What is the language of M_w ?

A_{TM} is mapping reducible to L



QED

Problem

Let S = { M | M is a TM with the property: for all w, M(w) accepts implies $M(w^R)$ accepts}.

S is undecidable.

A_{TM} = { (M,w) | M is a TM that accepts string w }

$HALT_{TM}$ = { (M,w) | M is a TM that halts on string w }

E_{TM} = { M | M is a TM and L(M) = ∅ }

REG_{TM} = { M | M is a TM and L(M) is regular}

EQ_{TM} = {(M, N) | M, N are TMs and L(M) =L(N)}

ALL_{PDA} = { P | P is a PDA and L(P) = Σ* }

ALL UNDECIDABLE

Where is Rice's Theorm Applicable?

Which are SEMI-DECIDABLE?

The rest of the content of today's lecture has been a major source of headaches and misunderstandings

"The recursion theorem is just like tennis. Unless you're exposed to it at age five, you'll never become world class."

-Juris Hartmanis (Turing Award 1993)

(Note: Juris didn't see the recursion theorem until he was in his 20's….)

THE RECURSION THEOREM

Theorem: Let T be a Turing machine that computes a function t : Σ* × Σ* → Σ*.

Then there is a Turing machine R that computes a function r : Σ* → Σ*, where for every string w,

r(w) = t(<R>, w)

(a,b) → T → t(a,b)

w → R → t(<R>,w)

Recursion Theorem says: A Turing machine can obtain its own description, and compute with it

. We can use the operation:
"Obtain your own description"
in pseudocode!

Given a computable t, we can get a computable r such that r(w) = t(<R>,w) where <R> is a description of r

INSIGHT: T (or t) is really R (or r)

Theorem: A_{TM} is undecidable

Proof (using the Recursion Theorem)**:**

Assume H decides A_{TM}

Construct machine R such that on input w:

1. **Obtains its own description < R>**
2. **Runs H on (<R>, w) and flips the output**

Running R on input w always does the opposite of what H says it should!

Theorem: A_{TM} is undecidable

Proof (using the Recursion Ttheorem)**:**

Assume H decides A_{TM}

Let T_H(x, w) = **Reject if H (x, w) accepts**
Accept if H (x, w) rejects

(Here x is viewed as a code for a TM)

By the *Recursion Theorem*, there is a TM R such that:

R(w) = T_H(<R>, w) = **Reject if H (<R>, w) accepts**
Accept if H (<R>, w) rejects

Contradiction!

MIN_{TM} = {<M>| M is a minimal TM, wrt |<M>|}

Theorem: MIN_{TM} is not RE.

Proof (using the Recursion Theorem)**:**

Assume E enumerates MIN_{TM}
Construct machine R such that on input w:

1. **Obtains its own description <R>**
2. **Runs E until a machine D appears with a longer description than of R**
3. **Simulate D on w**

Contradiction. Why?

MIN_{TM} = {<M>| M is a minimal TM, wrt |<M>|}

Theorem: MIN_{TM} is not RE.

Proof (using the Recursion Theorem)**:**

Assume E enumerates MIN_{TM}

Let T_E(x, w) = D(w) where <D> is first in E's enumeration s.t. |<D>| > |x|

By the *Recursion Theorem*, there is a TM R such that:

R(w) = T_E(<R>, w) = D(w)

where <D> is first in E's enumeration s.t. |<D>| > |<R>|

Contradiction. Why?

THE FIXED-POINT THEOREM

Theorem: Let f : $\Sigma^* \to \Sigma^*$ be a computable function. There is a TM R such that f(<R>) describes a TM that is *equivalent* to R.

Proof: Pseudocode for the TM R:

On input w:

1. **Obtain the description <R>**
2. **Let g = f(<R>) and interpret g as a code for a TM G**
3. **Accept w iff G(w) accepts**

THE FIXED-POINT THEOREM

Theorem: Let f : $\Sigma^* \to \Sigma^*$ be a computable function. There is a TM R such that f(<R>) describes a TM that is *equivalent* to R.

Proof: Let T_f(x, w) = G(w) where <G> = f (x)
(Here f(x) is viewed as a code for a TM)

By the *Recursion Theorem*, there is a TM R such that

R(w) = T_f(<R>, w) = G(w) where <G> = f (<R>)

Hence R $\equiv$ G **where <G> = f (<R>), ie <R> "$\equiv$" f (<R>)**

So R is a fixed point of f !

THE FIXED-POINT THEOREM

Theorem: Let f : $\Sigma^* \to \Sigma^*$ be a computable function. There is a TM R such that f(<R>) describes a TM that is *equivalent* to R.

Proof: Let T_f(x, w) = G(w) where <G> = f (x)
(Here f(x) is viewed as a code for a TM)

By the *Recursion Theorem*, there is a TM R such that

R(w) = T_f(<R>, w) = G(w) where <G> = f (<R>)

Hence R $\equiv$ G **where <G> = f (<R>), ie <R> "$\equiv$" f (<R>)**

So R is a fixed point of f !

THE FIXED-POINT THEOREM

Theorem: Let $f : \Sigma^* \rightarrow \Sigma^*$ be a computable function. There is a TM R such that f(<R>) describes a TM that is *equivalent* to R.

Examples:

1. For any 1-1 computable enumeration of Σ^* (or Gödel numbering of TMs) , there will always be a TM R that is equivalent to its successor in the enumeration

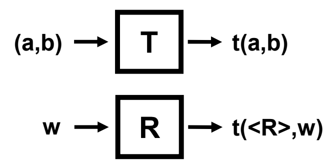
2. Let a virus flip the first bit of each word w in Σ^* (or in each TM). Then there is a TM R that "remains uninfected".

THE RECURSION THEOREM

Theorem: Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$.

Then there is a Turing machine R that computes a function $r : \Sigma^* \rightarrow \Sigma^*$, where for every string w,

$$r(w) = t(<R>, w)$$



THE RECURSION THEOREM

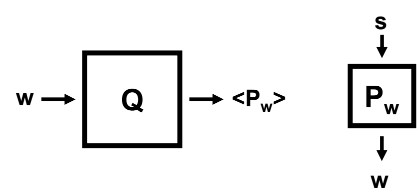
Theorem: Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$.

Then there is a Turing machine R that computes a function $r : \Sigma^* \rightarrow \Sigma^*$, where for every string w,

$$r(w) = t(<R>, w)$$

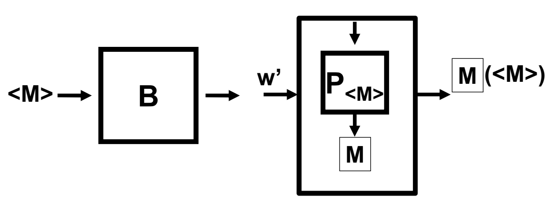
To Start: Need to show how to construct a TM that computes its own description.

Lemma: There is a computable function $q : \Sigma^* \rightarrow \Sigma^*$, where for any string w, q(w) is the *description* of a TM P_w that on any input, prints out w and then accepts



TM Q computes q

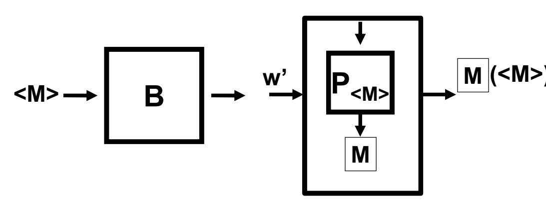
A TM SELF THAT PRINTS <SELF>



TM B, on any input <M>, prints the code for a TM that on any input outputs the result of M with input <M>

What about B on input <B>?

A TM SELF THAT PRINTS <SELF>



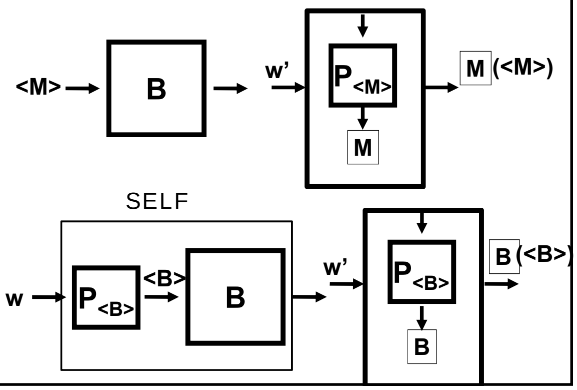
B (<M>) = < $P_{<M>}$ **M**> **where** $P_{<M>}$ **M** (w) = M (<M>)

So, B (<B>) = < $P_{<B>}$**B** > where $P_{<B>}$**B** (w) = B (<B>)

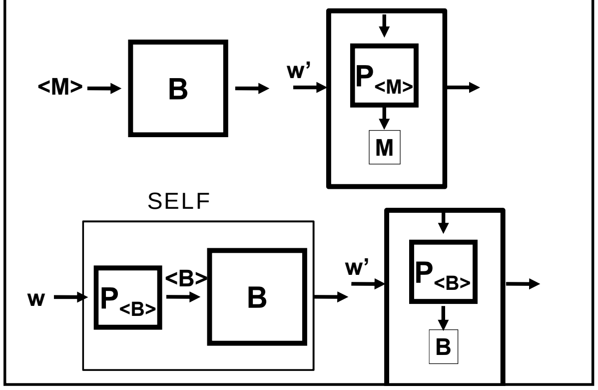
Now, $P_{<B>}$**B** (w) = B(<B>) = <$P_{<B>}$**B** >

So, let **SELF** = $P_{<B>}$**B**

A TM SELF THAT PRINTS <SELF>



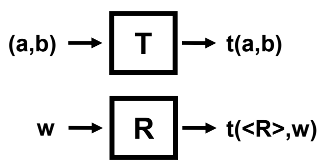
A TM SELF THAT PRINTS <SELF>



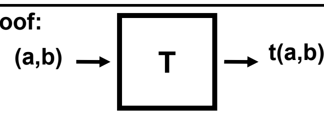
THE RECURSION THEOREM

Theorem: Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \to \Sigma^*$. There is a Turing machine R that computes a function $r : \Sigma^* \to \Sigma^*$, where for every string w,

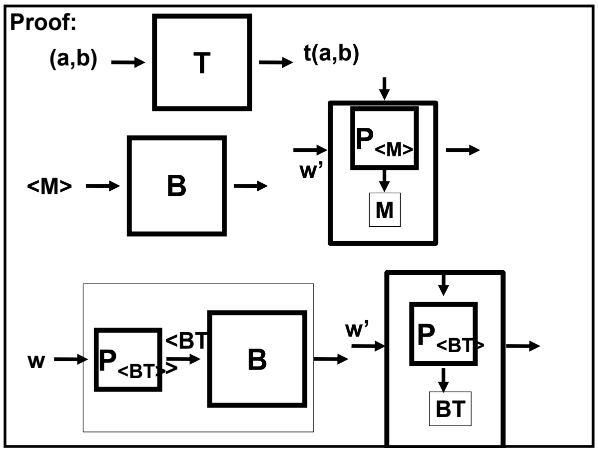
$$r(w) = t(<R>, w)$$



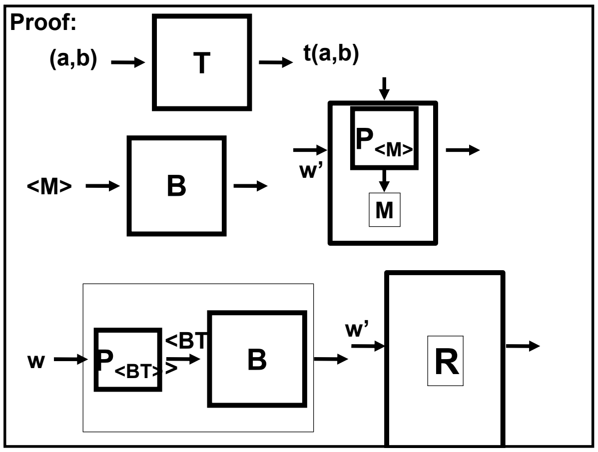
Proof:

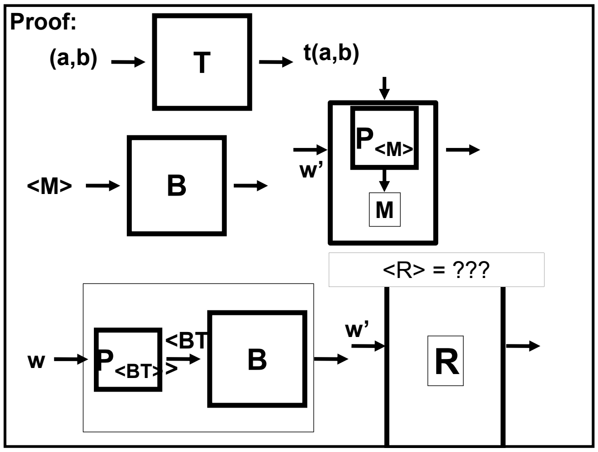


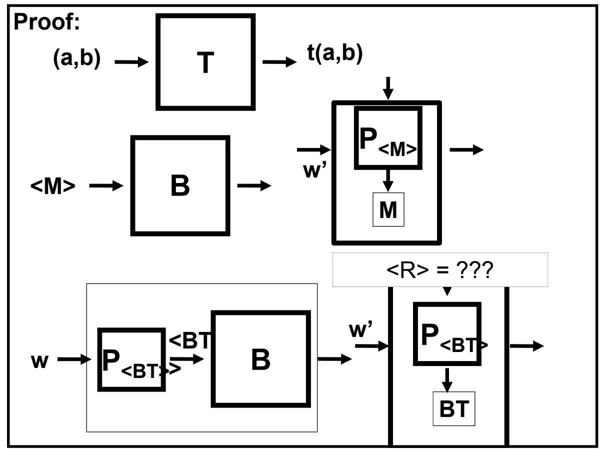
Proof:

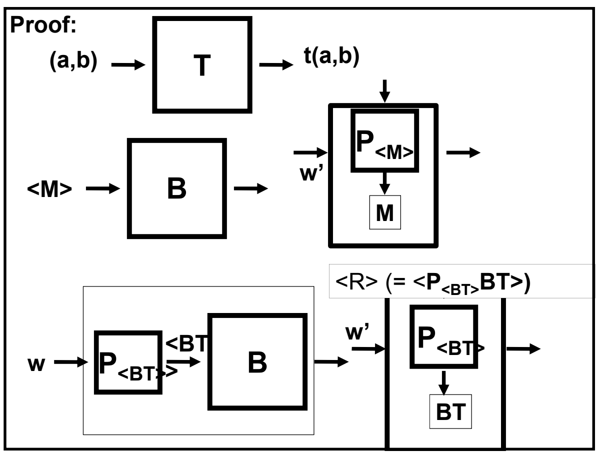


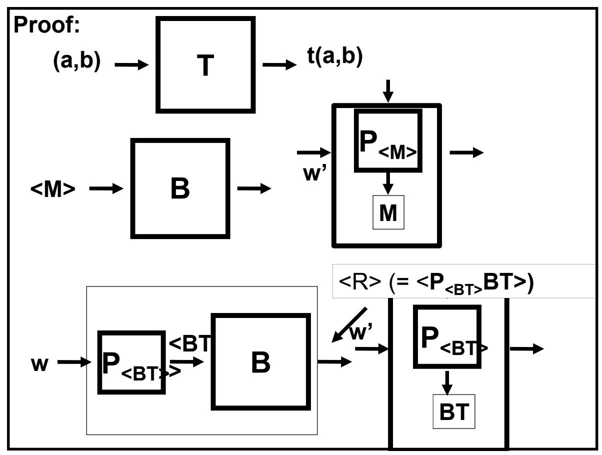
Proof:

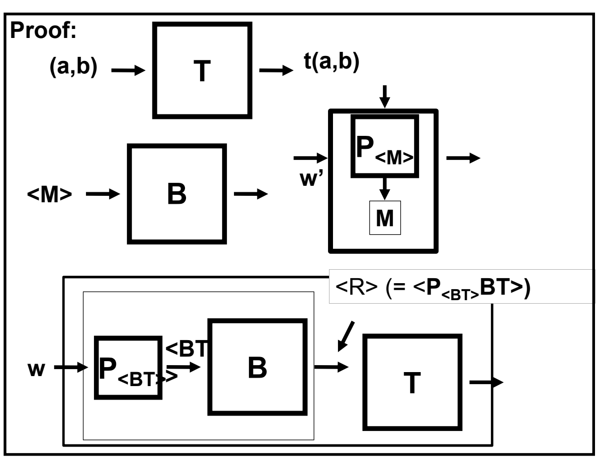


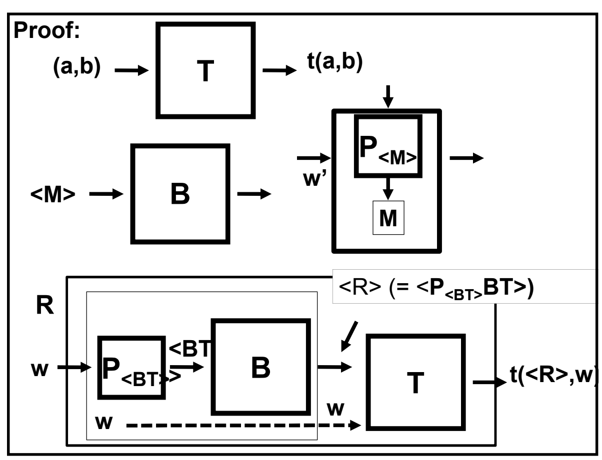

Proof:
(a,b)
T
t(a,b)
<M>
B
w'
P<M>
M
<R> = ???
w
P<BT>
<BT>
B
w'
R


Proof:
(a,b)
T
t(a,b)
<M>
B
w'
P<M>
M
<R> = ???
w
P<BT>
<BT>
B
w'
P<BT>
BT


Proof:
(a,b)
T
t(a,b)
<M>
B
w'
P<M>
M
<R> (= <P<BT>BT>)
w
P<BT>
<BT>
B
w'
P<BT>
BT


Proof:
(a,b)
T
t(a,b)
<M>
B
w'
P<M>
M
<R> (= <P<BT>BT>)
w
P<BT>
<BT>
B
w'
P<BT>
BT


Proof:
(a,b)
T
t(a,b)
<M>
B
w'
P<M>
M
<R> (= <P<BT>BT>)
w
P<BT>
<BT>
B
T


Proof:
(a,b)
T
t(a,b)
<M>
B
w'
P<M>
M
<R> (= <P<BT>BT>)
R
w
P<BT>
<BT>
B
w
w
T
t(<R>,w)

THE RECURSION THEOREM

Theorem: Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \to \Sigma^*$. There is a Turing machine R that computes a function $r : \Sigma^* \to \Sigma^*$, where for every string w,

$$r(w) = t(<R>, w)$$

$$(a,b) \longrightarrow \boxed{T} \longrightarrow t(a,b)$$

$$w \longrightarrow \boxed{R} \longrightarrow t(<R>,w)$$

WWW.FLAC.WS

Read Chapter 6.1 and 6.3 for next time