An LF representation of a variant of HOL Light Kevin Watkins / June 15, 2004

The basic idea of natural deduction in Martin-Löf's style is to let each logical connective unfold into its meta-level representation in terms of hypothetical and parametric judgments. The proof really takes place in the meta-language, using the properties of these judgments. Logical connectives are *internalizations* of pre-existing meta-level judgments.

The question arises: how concise can we be? What is the minimal number of connectives needed to internalize *all* the essential features of the meta-language (hyp. and para. judgments)?

The answer, perhaps surprisingly, is *one* connective, together with function types in the term language. The steps of the construction are:

1. Parametric judgments are converted to meta-functions.
2. Hypothetical judgments are converted to meta-equality.
3. Meta-functions are internalized by object functions.
4. Meta-equality is internalized by object equality.

First, we establish the syntactic categories of types, terms, and proofs.

```
tp  :  type.
tm  :  tp -> type.
o   :  tp.
pf  :  tm o -> type.
```

We define some preliminary concepts surrounding meta-equality (Leibniz equality).

```
eqv  = [P] [Q] (pf P -> pf Q) & (pf Q -> pf P).
eq   = [S] [T] {p:tm A -> tm o} eqv (p S) (p T).
feq  = [F] [G] {p:(tm A -> tm B) -> tm o} eqv (p F) (p G).
```

Now we introduce axioms for each step of the construction.

1. Parametric judgments are converted to meta-functions.

```
abs  :  ({x} eq (F x) (G x)) -> feq F G.
```

The converse of `abs` is already a consequence of the general properties of hypothetico-parametric judgments.

2. Hypothetical judgments are converted to meta-equality.

```
oext  :  eqv P Q -> eq P Q.
```

Again, the converse follows by general properties of hypothetico-parametric judgments.

3. Meta-functions are internalized by object functions.

```
-->   :  tp -> tp -> tp.
lam   :  (tm A -> tm B) -> tm (A --> B).
app   :  tm (A --> B) -> (tm A -> tm B).
beta  :  feq (app (lam F)) F.
eta   :  eq (lam (app T)) T.
```

4. Meta-equality is internalized by object equality.

```
==   :  tm A -> tm A -> tm o.
in   :  eq S T -> pf (S == T).
out  :  pf (S == T) -> eq S T.
```

The definition of all the usual connectives of intuitionistic higher-order logic in terms of == is left as an exercise for the reader. I conjecture that the provable closed formulae of this system are exactly those of HOL Light (without set types, infinity, or choice).