Trivial Connections on Discrete Surfaces Revisited: A Simplified Algorithm for Simply-Connected Surfaces

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1 Overview

A recent paper [2] describes an algorithm for computing trivial connections with prescribed singularities on general discrete surfaces. This note presents a simplification of that algorithm in the special case of simply-connected surfaces (e.g., topological spheres and disks) with no directional constraints. The main advantage of the simplified algorithm is that the constraint system $Ax = -b$ becomes a positive-semidefinite Poisson equation, which can be solved using a wide variety of existing numerical packages.

2 Algorithm

On a simply-connected mesh, a trivial connection can be computed in two steps:

I. Solve $\nabla^2 u = -\tilde{K}$ for $u \in \mathbb{R}^{|V|}$.

II. Compute $x = \star_1 d_0 u$.

Here $\tilde{K} = K - 2\pi k$ is the vector K of discrete Gaussian curvatures (i.e., 2π minus the sum of tip angles at each vertex) minus 2π times the vector k of singularity indices. The matrix $\star_1$ is diagonal with entries $(\star_1)_{kk} = \frac{1}{2}(\cot\varphi_i + \cot\varphi_j)$, and $\nabla^2 = d_0^T \star_1 d_0$ is the usual cotan-Laplace matrix. This matrix has constant functions in its kernel, but many solvers naturally produce solutions of minimum norm (otherwise, the constant component can simply be projected out). The solution x can be used to construct direction fields on surfaces as described in [2].

3 Derivation

Note: We adopt notation from discrete exterior calculus [3], dropping subscripts on operators (e.g., d_0, $\star_1$, etc.) for brevity.

The general algorithm for computing a trivial connection solves the optimization problem

$$\begin{aligned} \min_x \quad & \|x\|_2 \\ \text{s.t.} \quad & Ax = -b \end{aligned}$$

where A is the constraint matrix, b is the vector of modified angle defects, and x is the vector of adjustment angles. On a simply-connected surface we have $A = \mathrm{d}$ and $b = \tilde{K}$, and the Hodge decomposition theorem tells us that x can be expressed as

$$x = \mathrm{d}\alpha + \delta\beta$$

for some 0-form α and 2-form β.

Since $\mathrm{d} \circ \mathrm{d} = 0$, our constraint equation simplifies to

$$\mathrm{d}\delta\beta = -\tilde{K},$$

and since this puts no constraint on α our optimization problem becomes

$$\begin{array}{ll} \min\limits_{\beta} & \|\delta\beta\|_2 \\ \text{s.t.} & \mathrm{d}\delta\beta = -\tilde{K}. \end{array}$$

We can then make a change of variables $u = \star\beta$ to get

$$\begin{array}{ll} \min\limits_{\beta} & \| \star du\|_2 \\ \text{s.t.} & \nabla^2 u = -\tilde{K}. \end{array}$$

However, since ∇^2 has only the constant functions in its kernel, this problem is equivalent to computing *any* solution to $\nabla^2 u = -\tilde{K}$ and projecting out the constant part. The final solution is recovered via

$$x = \delta\beta = \star\mathrm{d}u.$$

4 Acknowledgements

Thanks to Mirela Ben-Chen for pointing out the relationship with [1].

References

[1] Mirela Ben-chen, Craig Gotsman, and Guy Bunin. Conformal flattening by curvature prescription and metric scaling. In *Computer Graphics Forum*, 2008.

[2] Keenan Crane, Mathieu Desbrun, and Peter Schröder. Trivial connections on discrete surfaces. In *Computer Graphics Forum (SGP)*, 2010.

[3] Mathieu Desbrun, Eva Kanso, and Yiying Tong. Discrete differential forms for computational modeling. In *SIGGRAPH '06: ACM SIGGRAPH 2006 Courses*, 2006.