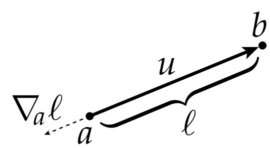
TRIANGLE MESH DERIVATIVES — CHEAT SHEET

[This "cheat sheet" is an excerpt from "Discrete Differential Geometry: An Applied Introduction" by Keenan Crane. There are sure to be errors! Please email `kmcrane@cs.cmu.edu` *if you find one.]*

A.1. List of Derivatives

We here give expressions for the derivatives of a variety of basic quantities often associated with triangle and tetrahedral meshes in $\mathbb{R}^3$. Unless otherwise noted, we assume quantities are associated with geometry in $\mathbb{R}^3$ (though many of the expressions easily generalize).

A.1.1. Edge Length.



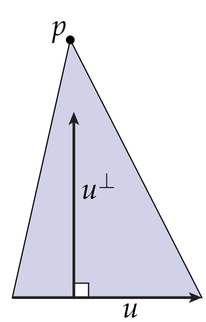
Let ℓ be the length of the vector $u := b - a$, where a and b are points in $\mathbb{R}^3$. Let $\hat{u} := u/\ell$ be the unit vector in the same direction as u. Then the gradient of ℓ with respect to the location of the point a is given by

$$\nabla_a \ell = -\hat{u}.$$

Similarly,

$$\nabla_b \ell = \hat{u}.$$

A.1.2. Triangle Area.

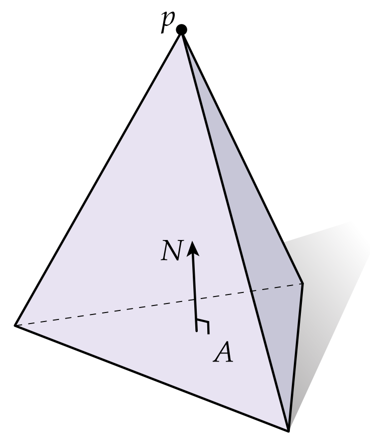


Consider any triangle in $\mathbb{R}^3$, and let u be the vector along the edge opposite a vertex p. Then the gradient of the triangle area A with respect to the location of p is

$$\nabla_p A = \tfrac{1}{2} N \times u,$$

where N is the unit normal of the triangle (oriented so that $N \times u$ points from u toward p, as in the figure above).

A.1.3. Tetrahedron Volume.

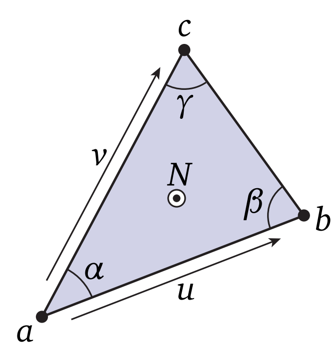


Consider any tetrahedron in $\mathbb{R}^3$, and let N be the unit normal of a triangle pointing toward the opposite vertex p (as in the figure above). The gradient of volume of the tetrahedron with respect to the location of p is

$$\nabla_p V = \frac{1}{3} A N,$$

where A is the area of the triangle with normal N.

A.1.4. Interior Angle.



Consider a triangle with vertices $a, b, c \in \mathbb{R}^3$, and let α be the signed angle between vectors $u := b - a$ and $v := c - a$. Then

$$\begin{aligned} \nabla_a \alpha &= -(\nabla_b \alpha + \nabla_c \alpha), \\ \nabla_b \alpha &= -(N \times u)/|u|^2, \\ \nabla_c \alpha &= (N \times v)/|v|^2. \end{aligned}$$

A.1.4.1. *Cosine.* Let θ be the angle between two vectors $u, v \in \mathbb{R}^3$. Then

$$\begin{aligned} \nabla_u \cos\theta &= (v - \langle v, \hat{u} \rangle \hat{u})/|u||v|, \\ \nabla_v \cos\theta &= (u - \langle u, \hat{v} \rangle \hat{v})/|u||v|, \end{aligned}$$

where $\hat{u} := u/|u|$ and $\hat{v} := v/|v|$. If u and v are edge vectors of a triangle with vertices $a, b, c \in \mathbb{R}^3$, namely $u := b - a$ and $v := c - a$, then

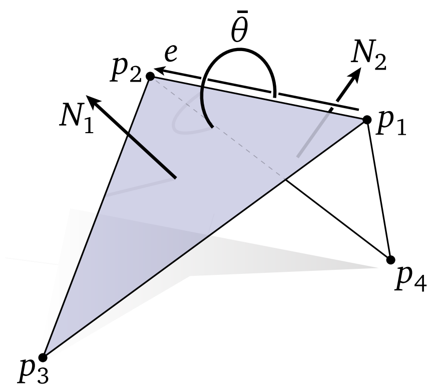
$$\nabla_a \cos\theta = -(\nabla_u \cos\theta + \nabla_v \cos\theta).$$

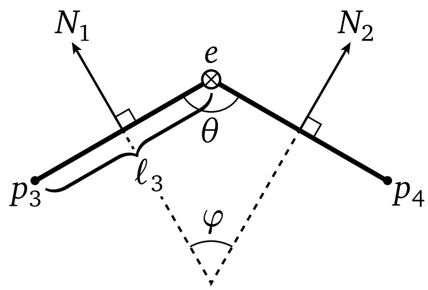
A.1.4.2. *Cotangent.* For any angle θ that depends on a vertex position p, we have

$$\nabla_p \cot\theta = -\frac{1}{\sin^2\theta} \nabla_p \theta.$$

The expression for the gradient of θ with respect to p can then be computed as above.

A.1.5. Dihedral Angle.





Consider a pair of triangles sharing an edge e, with vertices and normals labeled as in the figure above; let θ be the interior dihedral angle θ, complementary to the angle φ between normals. Explicitly, we can write θ as

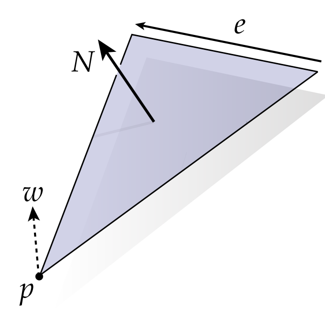
$$\theta = \text{atan2}(e \cdot (N_1 \times N_2), N_1 \cdot N_2),$$

where the use of the two-argument arc tangent function ensures we obtain the proper sign. Then

$$\begin{aligned} \nabla_{p_3}\theta &= |e|N_1/(2A_1), \\ \nabla_{p_4}\theta &= |e|N_2/(2A_2), \end{aligned}$$

where A_1, A_2 are the areas of the triangles with normals N_1, N_2, respectively. Gradients with respect to p_1 and p_2 can be found in the appendix to Wardetzky et al, *"Discrete Quadratic Curvature Energies"* (CAGD 2007).

A.1.6. Triangle Normal.



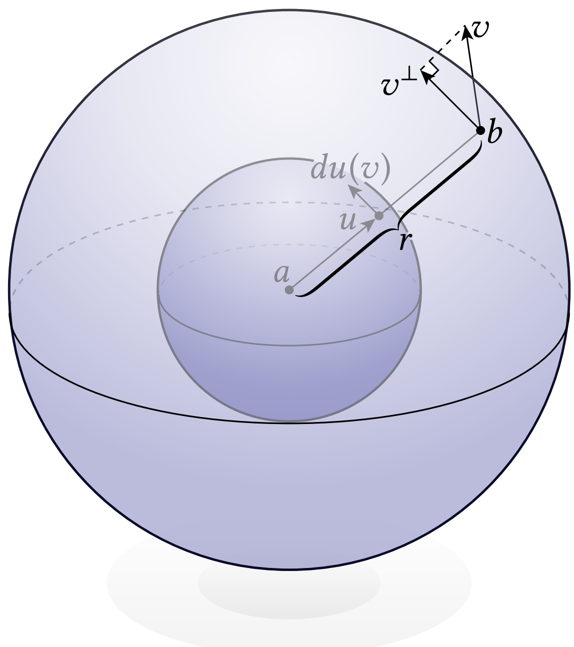
Consider a triangle in $\mathbb{R}^3$, and let e be the vector along an edge opposite a vertex p. If we move p in the direction w, the resulting change in the unit normal N (with orientation as depicted above) can be expressed as

$$dN(w) = \frac{\langle N, w \rangle}{2A} e \times N,$$

where A is the triangle area. The corresponding Jacobian matrix is given by

$$\frac{1}{2A}(e \times N)N^T.$$

A.1.7. Unit Vector.



Consider the unit vector $u := (b-a)/|b-a|$ associated with two points $a, b \in \mathbb{R}^3$. The change in this vector with respect to a motion of the endpoint b in the direction v can be expressed as

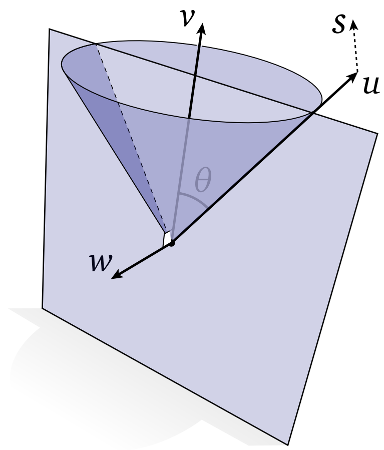
$$du(v) = \frac{v - \langle v, u \rangle u}{r},$$

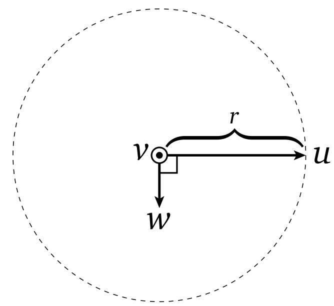
where $r := |b-a|$ is the distance between a and b. The corresponding Jacobian matrix is

$$\frac{1}{r}(I - uu^T),$$

where I denotes the 3×3 identity matrix.

A.1.8. Cross Product.





For any two vectors $u, v \in \mathbb{R}^3$, consider the vector

$$w := \frac{u \times v}{|u \times v|}.$$

If we move u in the direction s, then the resulting change to w is given by

$$dw(s) = \frac{\langle w, s \rangle}{|u \times v|} w \times v.$$

The corresponding Jacobian matrix is

$$\frac{1}{|u \times v|}(w \times v)w^T.$$