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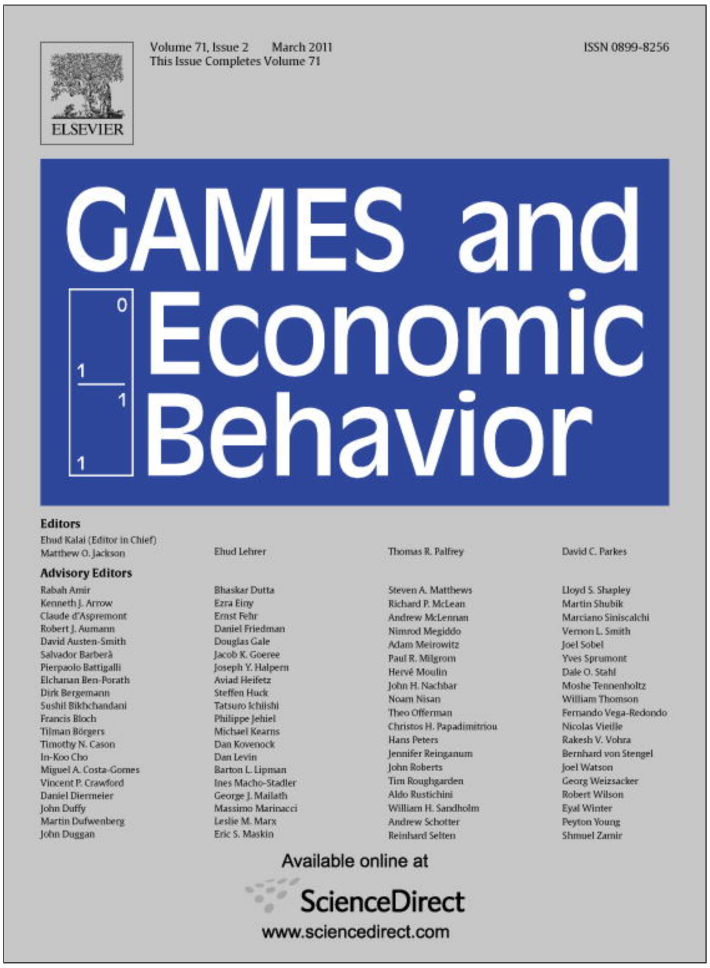

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Two equivalence results for two-person strict games

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A game is strict if for both players, different profiles have different payoffs. Two games are best response equivalent if their best response functions are the same. We prove that a two-person strict game has at most one pure Nash equilibrium if and only if it is best response equivalent to a strictly competitive game, and that it is best response equivalent to an ordinal potential game if and only if it is best response equivalent to a quasi-supermodular game.

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1. Introduction

In this paper we prove two results about pure Nash equilibria (PNEs) for 2-person *strict* games, which are those where payoff functions are one-to-one. The first result says that a 2-person strict game has at most one PNE iff it is *best-response equivalent* (Rosenthal, 1974) to a strictly competitive game (Moulin, 1976; Friedman, 1983). The second result says that a 2-person strict game is best-response equivalent to an ordinal potential game (Monderer and Shapley, 1996) iff it is best-response equivalent to a quasi-supermodular game (Topkis, 1998).

These results came out of our project on using computers to discover theorems in game theory (see, e.g. Tang and Lin, 2009). We were looking for conditions that imply the uniqueness and existence of the PNEs, and were using computers to run through a large set of conjectures. Given the number of all possible games, it made sense to test these conditions first on some special classes of games, and the class of strict games is a natural choice as in these games, the payoff functions are one-to-one, thus easier to generate. As it turned out, some interesting conditions hold only for this class of games. We have described some of them in Tang and Lin (2009). The two results that we report here are special in that their proofs are non-trivial and had to be done manually.

2. Basic definitions

A 2-person game is a tuple (A, B, u_1, u_2), where A and B are sets of pure strategies of players 1 and 2, respectively, and u_1 and u_2 are functions from $A \times B$ to reals called the *payoff functions* for players 1 and 2, respectively. In this paper we assume both A and B are finite.

A game is said to be *strict* if both players' payoff functions are one-to-one: for any (a_1, b_1) and (a_2, b_2) in $A \times B$, if $(a_1, b_1) \neq (a_2, b_2)$ then $u_i(a_1, b_1) \neq u_i(a_2, b_2)$, $i = 1, 2$. A related notion in the literature is *non-degenerate games* (see, e.g.

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Berger, 2007): a game is non-degenerate if for any $a_1, a_2 \in A$ and $b_1, b_2 \in B$, $u_1(a_1, b_1) \neq u_1(a_2, b_1)$ whenever $a_1 \neq a_2$, and $u_2(a_1, b_1) \neq u_2(a_1, b_2)$ whenever $b_1 \neq b_2$. Clearly, a strict game is also non-degenerate, but the converse is not true in general.

For each $b \in B$, let $B_1(b)$ be the set of best responses by player one to the action b by player two:

$$B_1(b) = \{a \mid a \in A, \text{ and for all } a' \in A, u_1(a', b) \leqslant u_1(a, b)\}.$$

Similarly, for each $a \in A$, let

$$B_2(a) = \{b \mid b \in B, \text{ and for all } b' \in N, u_2(a, b') \leqslant u_2(a, b)\}.$$

A profile $(a, b) \in A \times B$ is a pure Nash equilibrium (PNE) if $a \in B_1(b)$ and $b \in B_2(a)$.

In this paper we consider only PNEs. Our initial motivation for this work was to capture the class of games with at most one PNE. We started with strictly competitive games (see below). While every strictly competitive game can have at most one PNE, the converse is not true in general. This led us to consider notions of equivalence between games that will preserve PNEs. The one that suits our purpose here is that of *best-response equivalence* (Rosenthal, 1974): two 2-person games $G_1 = (A, B, u_1, u_2)$ and $G_2 = (A, B, u'_1, u'_2)$ are said to be best-response equivalent if for each $a \in A$, $B_2(a)$ in G_1 and G_2 are the same, and for each $b \in B$, $B_1(b)$ in G_1 and G_2 are the same.

It is easy to see that best-response equivalent games have the same sets of PNEs.

3. Strictly competitive games and at most one PNE

A game (A, B, u_1, u_2) is strictly competitive if for every pair of profiles s_1 and s_2 in $A \times B$, we have that $u_1(s_1) \leqslant u_1(s_2)$ iff $u_2(s_2) \leqslant u_2(s_1)$. We have the following result for strict games.

Theorem 1. *A strict 2-person game has at most one PNE if and only if it is best-response equivalent to a strictly competitive game.*

The "if" part of the theorem follows from the well-known fact that all PNEs in a strictly competitive game have the same payoff for each player (cf. p. 22, Osborne and Rubinstein, 1994), and that for strict games, profiles with same payoffs must be identical.

To prove the "only if" part of the theorem, for any given game $G = (A, B, u_1, u_2)$, we associate with it the following directed graph R: the vertices of R are profiles of the game and the set of arcs is:

$$\{((a, b), (a', b)) \mid a' \in B_1(b), a \notin B_1(b)\} \cup$$
$$\{((a, b), (a, b')) \mid b \in B_2(a), b' \notin B_2(a)\}.$$

Informally, this is a graph where player 1 moves to her best response, and player 2 moves away from her best response. Thus for a strictly competitive game, player 1's payoffs increase while player 2's payoffs decrease along every path of this graph.

The "only if" part of the theorem follows directly from the following two lemmas that relate strictly competitive games and acyclicity of the associated graphs.

Lemma 3.1. *A 2-person game is best-response equivalent to a strictly competitive game if and only if its directed graph has no cycle.*

Proof. Let $G = (A, B, u_1, u_2)$ be a 2-person game, and R its directed graph as defined above. If R has a cycle, then G cannot be equivalent to a strictly competitive game because if $G' = (A, B, u'_1, u'_2)$ is such a game, then for any profiles s_1 and s_2, if there is an arc from s_1 to s_2 in R, then $u'_1(s_1) < u'_1(s_2)$. So along the cycle, there must be a sequence $u'_1(t_1) < u'_1(t_2) < \cdots < u'_1(t_k) < u'_1(t_1)$, which is a contradiction.

Now if R has no cycle, construct a game $G' = (A, B, u'_1, u'_2)$ where u'_1 and u'_2 are defined as follows:

- $R_0 = \{(s, s') \mid \text{there is an arc from } s \text{ to } s' \text{ in } R\}$, $S_0 = \{s \mid s \in A \times B, \text{ there is no } s' \text{ such that } (s', s) \in R_0\}$.
- Suppose that R_k and S_k is defined, let

$$R_{k+1} = \{(s, s') \mid (s, s') \in R_k, \text{ and } s, s' \notin S_k\},$$
$$S_{k+1} = \{s \mid \text{for some } s', (s, s') \in R_k \text{ but there is no } s' \text{ such that } (s', s) \in R_{k+1}\}.$$

- Since R has no cycles, there is a finite number n such that for all $k > n$, $R_k = S_k = \emptyset$, and $A \times B = S_0 \cup \cdots \cup S_n$.
- Let u'_1 be a one-to-one function from $A \times B$ to the set of positive integers such that if $i < j$, then for any $s \in S_i$ and $s' \in S_j$, $u'_1(s) < u'_1(s')$.
- Let $u'_2 = -u'_1$.

Clearly, G' is strictly competitive, and best-response equivalent to G. □

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Lemma 3.2. *If the directed graph of a 2-person game has a cycle, then the game has at least two PNEs.*

Proof. Let $G = (A, B, u_1, u_2)$ be a 2-person game, and R its directed graph as defined above. Suppose $s_1, \ldots, s_k, s_{k+1}$ is a cycle in R. Suppose $s_1 = (a, b)$. Then either $s_2 = (a', b)$ for some $a' \neq a$ or $s_2 = (a, b')$ for some $b' \neq b$. Suppose it is the first case, $s_2 = (a', b)$. The proof for the second case is similar. Then by our construction of R, $s_3 = (a', b')$ for some $b' \neq b$, $s_4 = (a'', b')$ for some $a'' \neq a'$. We show that s_2 and s_4 are both PNEs of G. Because there is an arc from (a, b) to (a', b) in R, $a' \in B_1(b)$. Because there is an arc from (a', b) to (a', b') in R, $b \in B_2(a')$. Thus $s_2 = (a', b)$ is a PNE. Similarly, because there is arc from (a', b') to (a'', b') and an arc from (a'', b') to (a'', b'') in R for some b'' (it is possible that $a = a''$ and $b = b''$) $a'' \in B_1(b')$ and $b' \in B_2(a'')$. Thus $s_4 = (a'', b')$ is a PNE as well. □

One way to visualize this lemma[1] is to think of a path in this directed graph as a sequence of connected arrows that go either horizontally (player 1's move) or vertically (player 2's move). Thus a cycle must have an even number of arcs and vertices (profiles), and half of those profiles, the ones at the end of the horizontal arrows, must be PNEs.

3.1. Discussion

- The "if" part of Theorem 1 does not hold for general 2-person games – one can easily find a strictly competitive game that is not strict and has more than one PNE.
- The "only if" part of Theorem 1 holds for general 2-person games as well. This is because our proofs of the two lemmas above do not use the assumption that the games are strict, thus hold for general 2-person games as well. However, for general 2-person games, the "only if" part does not hold if we replace "at most one PNE" by "at most one payoff for the PNE" (meaning all the PNEs have the same payoff for each player). For instance, in the following game

1, 1	2, 2
2, 2	1, 1

  the two PNEs have the same payoff pair $(2, 2)$. But this game is not best-response equivalent to any strictly competitive game as the clockwise movement between the profiles is a cycle in its directed graph.

4. Ordinal potential and quasi-supermodular games

We first review *ordinal potential games* (Monderer and Shapley, 1996). Given a game $G = (A, B, u_1, u_2)$, a function P from $A \times B$ to reals is an ordinal potential for G, if for every $y \in B$ and every $x, z \in A$,

$$u_1(x, y) - u_1(z, y) > 0 \quad \text{iff} \quad p(x, y) - p(z, y) > 0,$$

and for every $x \in A$ and every $y, w \in B$,

$$u_2(x, y) - u_2(x, w) > 0 \quad \text{iff} \quad p(x, y) - p(x, w) > 0.$$

G is called an ordinal potential game if it admits an ordinal potential function.

Monderer and Shapley (1996) proved that every finite ordinal potential game has a PNE.

Another class of games that always have PNEs is the so-called *quasi-supermodular* game (Topkis, 1998). A finite game (A, B, u_1, u_2) is quasi-supermodular if there are two linear orderings $<_1$ and $<_2$ of A and B, respectively, such that u_1 and u_2 satisfies the so-called single crossing property: $\forall (a_1 <_1 a_2) \in A^2$ and $\forall (b_1 <_2 b_2) \in B^2$, we have

$$u_1(a_2, b_1) > u_1(a_1, b_1) \quad \Rightarrow \quad u_1(a_2, b_2) > u_1(a_1, b_2),$$
$$u_2(a_1, b_2) > u_2(a_1, b_1) \quad \Rightarrow \quad u_2(a_2, b_2) > u_2(a_2, b_1)$$

and

$$u_1(a_2, b_1) \geqslant u_1(a_1, b_1) \quad \Rightarrow \quad u_1(a_2, b_2) \geqslant u_1(a_1, b_2),$$
$$u_2(a_1, b_2) \geqslant u_2(a_1, b_1) \quad \Rightarrow \quad u_2(a_2, b_2) \geqslant u_2(a_2, b_1).$$

For strict games, these two pairs of conditions are equivalent.

Given a 2-person game, a *best-response path* (Voorneveld, 2000) of the game is a sequence of profiles $s_1, \ldots, s_n$ such that for each $1 \leqslant i < n$, s_i and s_{i+1} differ on exactly one coordinate with the deviating player moving to a best response: if $s_i = (a, b)$ and $s_{i+1} = (a', b')$, then either $a = a'$ and $b' \in B_2(a)$ or $b = b'$ and $a' \in B_1(b)$. A best-response cycle is a best-response path whose two end points are the same.

Voorneveld (2000) introduced a slightly different notion of best-response cycles, and showed that a game with countable strategy sets is best-response equivalent to a *potential game* iff it has no best-response cycles under his definition. The

[1] This was suggested by one of the reviewers.

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difference between Voorneveld's notion and ours is that Voorneveld requires his best-response cycle to include at least one edge that strictly increases the payoff of the deviating player. The difference disappears for strict games, and without going into details, we remark here that while an ordinal potential game may not be a potential game, every ordinal potential game is best-response equivalent to a potential game. Thus from Voorneveld's result, we see that a 2-person finite strict game is best-response equivalent to an ordinal potential game iff it has no best-response cycles.

It also follows from the weak finite best response improvement path property[2] (cf. e.g. Kukushkin et al., 2005) of quasi-supermodular games that if a finite 2-person strict game is best-response equivalent to a quasi-supermodular game, then it has no best-response cycle. In this section, we show the other direction: if a finite 2-person strict game has no best-response cycle, then it is best-response equivalent to a quasi-supermodular game. Thus, a finite strict 2-person game is best-response equivalent to an ordinal potential game iff it is best-response equivalent to a quasi-supermodular game:

Theorem 2. *A 2-person strict game is best-response equivalent to a quasi-supermodular game if and only of it has no best-response cycle.*

Corollary 2.1. *A 2-person strict game is best-response equivalent to a quasi-supermodular game iff it is best-response equivalent to an ordinal potential game.*

Notice that best-response paths model Cournot dynamics, thus Theorem 2 also implies that if Cournot dynamics does not cycle, then it must be best-response equivalent to a quasi-supermodular game.

We now prove Theorem 2 through two lemmas. The first one relates quasi-supermodular games to the complementarity of best response functions, and the second the complementarity of best-response function to the acyclicity of best-response paths. While the "only if" part of Theorem 2 is already entailed by the current results, these two lemmas also provide an interesting alternative proof.

Given a 2-person game (A, B, u_1, u_2), and two linear orderings $<_1$ and $<_2$ of A and B, respectively, we say that the best-response function B_1 for player 1 is *non-decreasing* with respect to $<_1$ and $<_2$, if for each pair $b_i <_2 b_j$ $a_i = B_1(b_i)$ and $a_j = B_1(b_j)$, we have $a_i \leqslant_1 a_j$ and similarly for player 2, the best-response function B_2 is non-decreasing with respect to $<_1$ and $<_2$, if for each pair $a_i < a_j$ $1 \leqslant i < j \leqslant m$, $b_i = B_2(a_i)$ and $s'_j = (a_j, b_j)$, we have $b_i \leqslant_2 b_j$.

Essentially, if both players' best-response functions are non-decreasing, then they are strategic complementary.

Lemma 4.1. *A strict game is best-response equivalent to a quasi-supermodular game iff there exist two linear orderings under which the best-response functions for both players are non-decreasing.*

Non-decreasing best-response functions for quasi-supermodular game is a well-known intuition (the only if part), although we still prove it as follows since it is simple.

Proof of Lemma 4.1. Let $G = (A, B, u_1, u_2)$ be a strict game.

$\Rightarrow$: Suppose G is best-response equivalent to a quasi-supermodular game G'. Clearly, the best-response functions of G and G' are the same under any linear orderings. Now let $G' = (A, B, u'_1, u'_2)$. Since G' is quasi-supermodular, there are two linear orderings $<_1$ of A and $<_2$ of B such that u'_1 and u'_2 satisfy the single crossing properties. Then in G', the best-response functions under $<_1$ and $<_2$ are non-decreasing. For otherwise, suppose that B_1 is not non-decreasing in G'. Then there are two profiles (a_1, b_1) and (a_2, b_2) such that

$$a_1 \in B_1(b_1), \qquad a_2 \in B_1(b_2), \qquad b_1 <_2 b_2,$$
$$a_2 <_1 a_1.$$

Thus $u'_1(a_1, b_1) > u'_1(a_2, b_1)$ and $u'_1(a_2, b_2) > u'_1(a_1, b_2)$, which violate the single crossing conditions.

$\Leftarrow$: Suppose the best-response functions of the two players in G are non-decreasing under $<_1$ and $<_2$. We denote in this part that for $a_i, a_j \in A$, $a_i <_1 a_j$ if $i < j$. Similar for $b_i, b_j \in B$. Now consider the following game $G' = (A, B, u'_1, u'_2)$:

- if $s < t$, then $u'_1(a_i, b_s) < u'_1(a_j, b_t)$ for any a_i and a_j in A;
- for any $b_s \in B$, if $B_1(b_s) = \{a_k\}$ in G, then
  - if $i \neq a_k$, then $u'_1(a_i, b_s) < u'_1(a_k, b_s)$;
  - if $i > j > k$, then $u'_1(a_i, b_s) < u'_1(a_j, b_s)$;
  - if $i < j < k$, then $u'_1(a_j, b_s) > u'_1(a_i, b_s)$;
  - if $i < k < j$, then $u'_1(a_j, b_s) > u'_1(a_i, b)$;
- similarly for u'_2.

[2] A game has weak finite best-response improvement path property if every best-response path, in which the deviating player moves to a strictly better best response, is finite.

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Clearly, one can find two such functions u'_1 and u'_2, and that G' is best-response equivalent to G. We show that G' is quasi-supermodular on $<_1$ and $<_2$. Suppose $i < j$, $s < t$, and $u'_1(a_j, b_s) > u'_1(a_i, b_s)$. We show that $u'_1(a_j, b_t) > u'_1(a_i, b_t)$. Let $B_1(b_s) = \{a_k\}$ and $B_1(b_t) = \{a_l\}$. Then $k \leqslant l$ according to the non-decreasing property of player one's best-response function. Given $u'_1(a_j, b_s) > u'_1(a_i, b_s)$, by our construction, there are three cases:

1. $j = k$. We have $i < j \leqslant l$, thus $u'_1(a_j, b_t) > u'_1(a_i, b_t)$.
2. $i < j < k$. We have $i < j < k \leqslant l$, thus $u'_1(a_j, b_t) > u'_1(a_i, b_t)$.
3. $i < k < j$. We have either $i < l \leqslant j$ or $i < j < l$. Either way, we have $u'_1(a_j, b_t) > u'_1(a_i, b_t)$ by our construction.

Thus u'_1 satisfies the single crossing condition. Similarly, u'_2 satisfies the single crossing condition. So G is a quasi-supermodular game. □

Given Lemma 4.1, Theorem 2 then follows from the following lemma, which says non-decreasing best response functions are further equivalent to absence of best-response cycle. Before we start stating the lemma and the proof, we have some brief remarks on a related work.

At first sight, the "only if" part of the lemma below is closely related to an existing result, Echenique's (2004) Proposition 11, which says every non-degenerate ordinal potential game (thus without best-response cycle) is a *game of strategic complementarities*.[3] However, the definition of *game of strategic complementarities* differs from ours in that the orders on strategies are partial and complete lattice while we further require them to be linear. In other words, the set of games that are best-response equivalent to a quasi-supermodular game in our definition is a proper subset of the set of games of strategic complementarities. Therefore, our result generalizes Echenique's result but not vice versa. In addition, Echenique's (2004) proof relies on Theorem 5, which says a game with at least two PNEs is a GSC defined on lattices. However, there are games with at least two PNEs that are not GSC under any linear orderings (see the game given in Fig. 4 in Echenique's paper, for example). This means that Echenique's proof does not work on the games that we consider in our paper.

Lemma 4.2. *A strict 2-person game has no best-response cycle iff there are two linear orderings under which both players' best response functions are non-decreasing.*

Proof. Let $G = (A, B, u_1, u_2)$ be a strict game.

$\Leftarrow$: Let $<_1$ and $<_2$ be two linear orderings of A and B, respectively, under which the best-response functions B_1 and B_2 for player 1 and 2, respectively, are non-decreasing. We show that G has no best-response cycle. We begin with a profile (a_1, b_1) where, with loss of generality, $\{a_1\} = B_1(b_1)$. Let then player 2 and player 1 deviates to each own best-response alternatively in the following rounds. If in the first round, player 2 deviates to an action b_2, there are three cases:

- Case 1: $b_1 <_2 b_2$, then according to the non-decreasing property of B_1, we have $a_1 \leqslant_1 a_2$ where $\{a_2\} = B_1(b_2)$. If $a_1 = a_2$, it means that (a_1, b_2) is a PNE, where the best response path terminates. Otherwise $a_1 <_1 a_2$, this process will continue generating new profiles that alternatively increase in each player's action until it meets a PNE and terminates. This shows that the best-response path generated this way cannot return to where it started, therefore in this case there is no best-response cycle.
- Case 2: $b_2 <_2 b_1$. It is not hard to see that this process will continue generating new profiles that strictly increases in one of its players rank until it meets a PNE. This shows again that there is no best-response cycle.
- Case 3: $b_1 = b_2$, this means that (a_1, b_1) is a PNE, where the best-response path terminates. No cycle exists.

We have proved the $\Leftarrow$ part of the lemma.

$\Rightarrow$: We prove this part constructively. Suppose there is no best-response cycle in G. Our following procedure will produce two ranking functions, $rank_A$ that maps each $a \in A$ to an integer between 1 and $|A|$, and $rank_B$ that maps each $b \in B$ to an integer between 1 and $|B|$. From these two rankings, we get two linear orderings on A and B: $a <_1 a'$ if $rank_A(a) > rank_A(a')$, and $b <_2 b'$ if $rank_B(b) > rank_B(b')$. We show that under these two linear orderings, the best-response functions for both players are non-decreasing. For an informal description of how the following algorithm works, see an illustrative example below:

1. Initially, let $A_r = A$, $B_r = B$, $I_A = I_B = 1$, $A_c = B_c = []$ (the empty list), $rank_A(a) = 0$ for any $a \in A$, and $rank_B(b) = 0$ for any $b \in B$.
2. **while** $A_r \neq \emptyset$ or $B_r \neq \emptyset$ **do**
   - 2.1. Let $(a, b) \in A_r \times B_r$ be a Nash equilibrium of G (there must be such a Nash equilibrium as we show below), and let $A_c = [a]$, $B_c = [b]$.
   - 2.2. **while** $A_c \neq []$ or $B_c \neq []$ **do**
     - (i) **If** $A_c \neq []$ **then**

[3] We thank the advisory editor for exposing us to this work.

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(a) Let a^* be the first element in A_c.
(b) $rank_A(a^*) = I_A$; $I_A = I_A + 1$.
(c) Delete a^* from A_c.
(d) Delete a^* from A_r.
(e) For each $b' \in B$ such that $B_1(b') = \{a^*\}$, add b' to the end of B_c. (If there are more than one such b', the order by which they are added to B_c does not matter.)

(ii) **If** $B_c \neq []$ **then**
(a) Let b^* be the first element in B_c.
(b) $rank_B(b^*) = I_B$; $I_B = I_B + 1$.
(c) Delete b^* from B_c.
(d) Delete b^* from B_r.
(e) For each $a' \in A$ such that $B_2(a') = \{b^*\}$, add a' to the end of A_c. (If there are more than one such a', the order by which they are added to A_c does not matter.)

We now show the correctness of step 2.1 and that the best-response sequences for both players under the orderings output by the procedure are non-decreasing.

Let A_{r_i} and B_{r_i} be the A_r and B_r at the beginning of the ith loop and let $G_i = (A_{r_i}, B_{r_i}, u_1, u_2)$, we have the following properties of our procedure that guarantee the precondition of 2.1 can always be satisfied.

Proposition 4.1. *For each $G_i = (A_{r_i}, B_{r_i}, u_1, u_2)$, we have:*

1. *For all $b_j \in B_{r_i}$, $B_1(b_j) \subseteq A_{r_i}$.*
2. *For all $a_j \in A_{r_i}$, $B_2(a_j) \subseteq B_{r_i}$.*
3. *G_i has no best-response cycle.*
4. *Every Nash equilibrium of G_i is also one of the original game G.*

Proof. We prove this proposition by induction on i.

- Base case: It is easy to verify that $G_1 = G$ satisfies 1–4.
- Inductive case: Suppose G_i satisfies 1–4, we now verify 1–4 for G_{i+1}.
  1. Suppose otherwise, there exists $b_j \in B_{r_{i+1}}$ such that $B_1(b_j) = \{a\} \subseteq A_{r_i} \setminus A_{r_{i+1}}$. According to our procedure, a will be deleted during this loop, which means b_j will been added to B_c and deleted during this loop because B_c will be empty at the end of the loop. This means $b_j \in B_{r_i} \setminus B_{r_{i+1}}$, a contradiction.
  2. Similar to above.
  3. 1–2 tells us the best-responses of G_{i+1} are also best-responses of G. Suppose G_{i+1} has a best-response cycle, this cycle would still be one in G, a contradiction.
  4. This part also follows directly from 1 to 2. □

Since G_i has no best-response cycle $\Leftrightarrow$ G_i is best-response equivalent to an ordinal potential game $\Rightarrow$ G_i has a Nash equilibrium. According to 1 and 4 of the above proposition, G_i always has a Nash equilibrium. Up to now, we have proved that the precondition of 2.1 in our procedure can always be satisfied.

Proposition 4.2. *The best-response functions under the orderings generated by our procedure are non-decreasing.*

Proof. For B_2 and for all $a_1, a_2 \in A$ with $a_2 <_1 a_1$, let $B_2(a_1) = \{b_1\}$ and $B_2(a_2) = \{b_2\}$, we show in the following that $b_2 \leqslant_1 b_1$.

If $b_1 = b_2$, we get to the conclusion immediately.

Now consider $b_1 \neq b_2$, when b_1 is the first element of B_c, a_1 is added to the end of A_c because $B_2(a_1) = \{b_1\}$. Similarly, when b_2 is the first element of B_c, a_2 is added to the end of A_c because $B_2(a_2) = \{b_2\}$. Since $a_2 <_1 a_1$, we must have a_2 is added to A_c later than a_1. This means b_2 appears as the first element of B_c later than b_1. So we get $b_2 <_2 b_1$.

Similar for player B_1. □

A few more words about the proof above:

- As we can see, our proof is algorithmic thus provides a means to sort acyclic best-response functions into non-decreasing order. In other words, given a strict (more generally, non-degenerate) game without best-response cycle, our procedure tells how to order the strategies to make the game quasi-supermodular.
- Also as a byproduct of our proof, in two-person strict quasi-supermodular game, the set of PNEs is actually linearly ordered (of course, this is partially caused by the fact that we require the orders on the strategy sets to be linear), in contrast to a previous result (Zhou, 1994) that says the set of PNEs is a complete lattice in a supermodular game.

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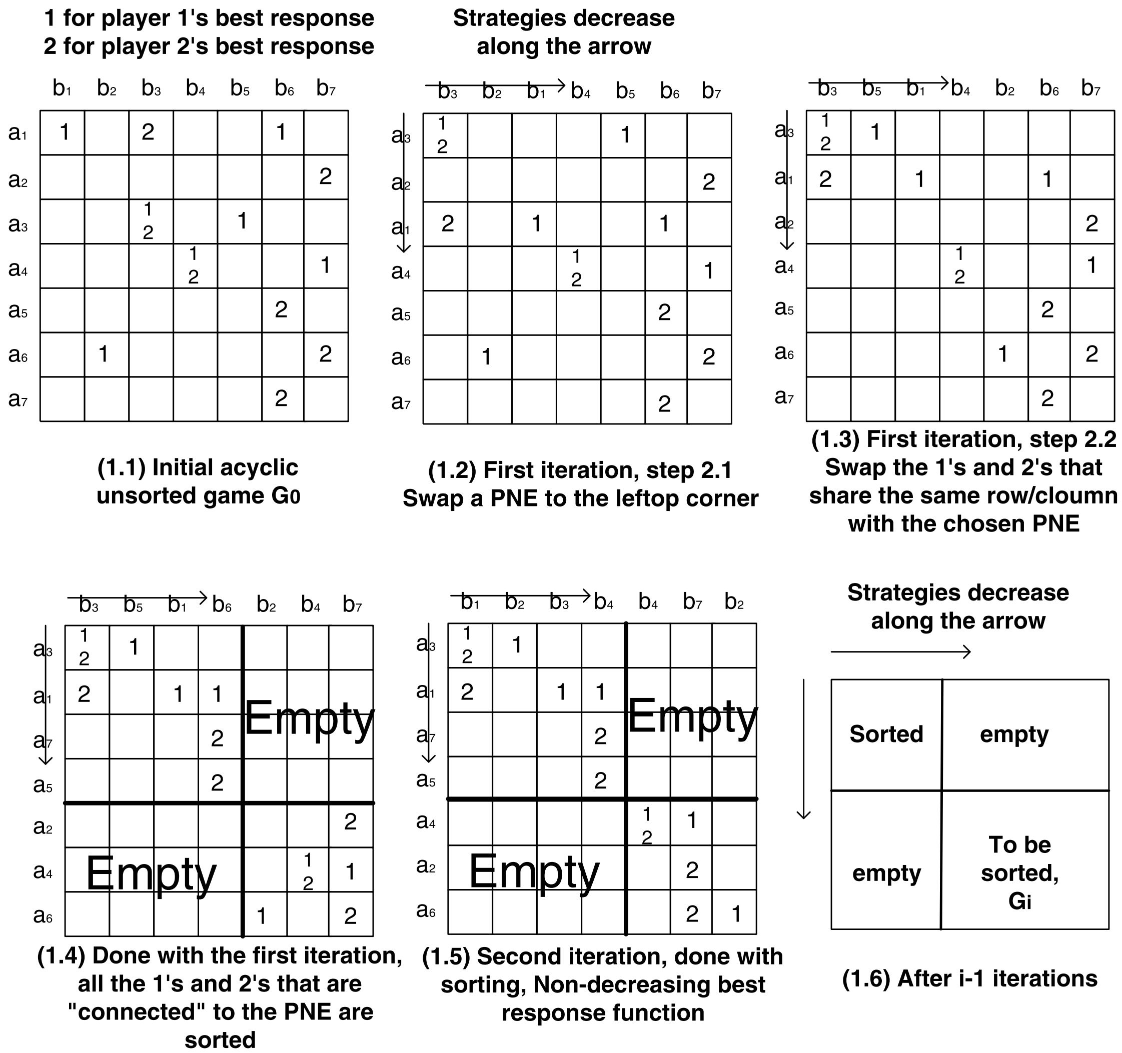

Fig. 1. Sorting the best-response matrix of a 7 × 7 acyclic game.

4.1. An illustrating example of the algorithm

We illustrate the algorithm for proving Lemma 4.2 using an example.

Consider a 7×7 2-player strict game whose best-response matrix is given in Fig. 1(1.1). In the matrix, the 1's indicate the player 1's best response, and 2's the player 2's best response. So the number 1 in the cell (a_1, b_1) means that the player 1's best response to b_1 is a_1, and the number 2 in (a_1, b_3) means that the player 2's best response to a_1 is b_3, and so on. This is a game that has no best-response cycles. However, if we order player 1's strategies from top to down, and player 2's strategies from left to right, then the best response functions are not non-decreasing. For instance, $b_3 <_2 b_6$, but $B_1(b_3) = a_3 >_1 a_1 = B_1(b_6)$. We now show how the procedure in our proof of Lemma 4.2 will re-arrange the two players' strategies so make the best response functions non-decreasing. The procedure is iterative. Step 2.1 finds a PNE, and moves it to the left top corner, as shown in Fig. 1(1.2). Step 2.2, as shown in Fig. 1(1.3) and (1.4), keeps swapping rows and columns until the matrix is divided into four regions: The top-left one is sorted, the top-right and the bottom-left ones have no best response entries in them and the bottom-right one is a sub-matrix that needs to be sorted recursively. For this particular game, this sub-matrix is sorted in one more iteration as shown in Fig. 1(1.5). In general, the number of iterations needed is the same as the number of PNEs in the game.

4.2. Discussions

Theorem 2 still holds for non-degenerate games, since the only assumption we made is that the best-response functions are single valued.

For general two person games, there are three ways to define a best response cycle:

1. Normal cycle. In this definition, as long as any best-response function is multi-valued, there is always a trivial cycle in which one player deviates among multiple best-responses.

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2. Voorneveld's cycle. This definition is essentially the same to above except that it requires at least one edge in the cycle that strictly increases the payoff for the deviating player. In this way, it rules out the trivial cycle in our original definition.
3. Strict cycle. In this definition, every edge in the cycle must strictly increases the payoff for the deviating player.

Three definitions are equivalent when consider strict games only. We now discuss the extension of Theorem 2 to general 2-person games with respect to three types of cycles.

The "only if" part of Theorem 2 does not hold for general 2-person games for normal cycle and Voorneveld's cycle but still holds for strict cycle. For instance, the following general game:

1, 1	2, 1
2, 2	1, 2

is a quasi-supermodular game under the ordering that orders player 1's (row player) strategies from top to down and player 2's action from left to right. However, going counterclockwise starting in any cell will form both a normal cycle as well as Voorneveld's cycle. Thus this game is not best-response equivalent to any ordinal potential game. However, according to Theorem 3 in Kukushkin et al. (2005), there is no strict cycle if a game is quasi-supermodular (therefore no strict cycle for a game that is best-response equivalent to it).

The "if" part of theorem 2 does not hold for general 2-person games for Voorneveld's as well as strict cycles but holds for normal cycles. To show this, consider the following general game:

2, 2	2, 1	2, 2
1, 1	1, 2	1, 2
1, 2	1, 2	1, 1

It has no Voorneveld's nor strict cycles. However it is not best-response equivalent to any quasi-supermodular game since there is no ordering of strategies under which player 2's best-response function satisfies single crossing condition. On the other hand, if we assume that a game has no normal cycle, it implies that its best-response functions are single-valued. Therefore, the if part holds for normal cycles.

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