A note on the Haah et al. tomography algorithm

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Haah et al.'s tomography algorithm [1]. Given $\rho^{\otimes n}$,

1. Perform weak Schur sampling on $\rho^{\otimes n}$, yielding a random $\boldsymbol{\lambda}$. Then $\rho^{\otimes n}$ collapses to $\pi_{\boldsymbol{\lambda}}(\rho)/s_{\boldsymbol{\lambda}}(\alpha)$.
2. Measure within the space $\mathrm{V}^d_{\boldsymbol{\lambda}}$ using the POVM $\dfrac{\dim(\mathrm{V}^d_{\boldsymbol{\lambda}})}{s_{\boldsymbol{\lambda}}(\underline{\boldsymbol{\lambda}})}\pi_{\boldsymbol{\lambda}}(U\mathrm{diag}(\underline{\boldsymbol{\lambda}})U^\dagger)\mathrm{d}U$, for $U \in \mathrm{U}(d)$.
3. Output $\boldsymbol{U}\mathrm{diag}(\underline{\boldsymbol{\lambda}})\boldsymbol{U}^\dagger$.

The weight the POVM in step 2 gives a particular $U \in \mathrm{U}(d)$ is

$$\frac{\dim(\mathrm{V}^d_\lambda)}{s_\lambda(\underline{\lambda})s_\lambda(\alpha)}\mathrm{tr}(\pi_\lambda(\rho)\pi_\lambda(U\mathrm{diag}(\underline{\lambda})U^\dagger))\mathrm{d}U = \frac{\dim(\mathrm{V}^d_\lambda)}{s_\lambda(\underline{\lambda})s_\lambda(\alpha)}s_\lambda(\rho U\mathrm{diag}(\underline{\lambda})U^\dagger)\mathrm{d}U.$$

Because this is a POVM, integrating this quantity over the unitary group yields 1, which (essentially) proves the following well-known equation from representation theory.

$$\int_U s_\lambda(AUBU^\dagger)\mathrm{d}U = \frac{s_\lambda(A)s_\lambda(B)}{\dim(\mathrm{V}^d_\lambda)}, \tag{1}$$

where here the $s_\lambda(\cdot)$'s are applied to the eigenvalues of their arguments.

Computing the error. Our goal is to show that the expected Frobenius-squared error of the Haah et al. algorithm is $(4d-3)/n$, matching the Keyl measurement [2, 3]. To begin,

$$n^2 \mathop{\mathbf{E}}_{\boldsymbol{\lambda},\boldsymbol{U}} \|\rho - \boldsymbol{U}\mathrm{diag}(\underline{\boldsymbol{\lambda}})\boldsymbol{U}^\dagger\|_F^2 = \mathop{\mathbf{E}}_{\boldsymbol{\lambda},\boldsymbol{U}}\left[\sum_{i=1}^d (n\alpha_i)^2 + \sum_{i=1}^d \boldsymbol{\lambda}_i^2 - 2n^2\cdot\mathrm{tr}(\rho\boldsymbol{U}\mathrm{diag}(\underline{\boldsymbol{\lambda}})\boldsymbol{U}^\dagger)\right]. \tag{2}$$

For a fixed λ, we analyze the cross-term as follows:

$$\begin{aligned}
\mathop{\mathbf{E}}_{\boldsymbol{U}} \mathrm{tr}(\rho\boldsymbol{U}\mathrm{diag}(\underline{\lambda})\boldsymbol{U}^\dagger) &= \frac{\dim(\mathrm{V}^d_\lambda)}{s_\lambda(\underline{\lambda})s_\lambda(\alpha)}\int_U \mathrm{tr}(\rho U\mathrm{diag}(\underline{\lambda})U^\dagger)s_\lambda(\rho U\mathrm{diag}(\underline{\lambda})U^\dagger)\mathrm{d}U \\
&= \frac{\dim(\mathrm{V}^d_\lambda)}{s_\lambda(\underline{\lambda})s_\lambda(\alpha)}\int_U \sum_{i=1}^d s_{\lambda+e_i}(\rho U\mathrm{diag}(\underline{\lambda})U^\dagger)\mathrm{d}U && \text{(Pieri's rule)} \\
&= \frac{\dim(\mathrm{V}^d_\lambda)}{s_\lambda(\underline{\lambda})s_\lambda(\alpha)}\sum_{i=1}^d \frac{s_{\lambda+e_i}(\alpha)s_{\lambda+e_i}(\underline{\lambda})}{\dim(\mathrm{V}^d_{\lambda+e_i})}, && \text{(equation (1))} \\
&= \sum_{i=1}^d \frac{\Phi_{\lambda+e_i}(\alpha)}{\Phi_\lambda(\alpha)}\cdot\frac{s_{\lambda+e_i}(\underline{\lambda})}{s_\lambda(\underline{\lambda})} \geq \sum_{i=1}^d \frac{\Phi_{\lambda+e_i}(\alpha)}{\Phi_\lambda(\alpha)}\cdot\left(\frac{\lambda_i}{n}\right).
\end{aligned}$$

Here this last step uses three facts: (i) that the $\Phi_{\lambda+e_i}(\alpha)$'s form a *decreasing* sequence (by a recent result of Sra [4]), (ii) Proposition 2.1 from [3] (applied to $s_{\lambda+e_i}(\underline{\lambda})/s_\lambda(\underline{\lambda})$), and (iii) equation (2) from [3], i.e. the elementary majorization inequality. Plugging this into (2), we see that

$$n^2 \mathop{\mathbf{E}}_{\boldsymbol{\lambda},\boldsymbol{U}} \|\rho - \boldsymbol{U}\mathrm{diag}(\underline{\boldsymbol{\lambda}})\boldsymbol{U}^\dagger\|_F^2 \leq \mathop{\mathbf{E}}_{\boldsymbol{\lambda}}\left[\sum_{i=1}^d (n\alpha_i)^2 + \sum_{i=1}^d \boldsymbol{\lambda}_i^2 - 2n\cdot\sum_{i=1}^d \frac{\Phi_{\boldsymbol{\lambda}+e_i}(\alpha)}{\Phi_{\boldsymbol{\lambda}}(\alpha)}\cdot\boldsymbol{\lambda}_i\right].$$

This equation is analyzed in the proof of Theorem 1.2 in [3], in which it is shown to be at most $4dn - 3n$. Dividing by n^2 gives the desired Frobenius-squared bound.

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References

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