Crossing the Bridge between Similar Games

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Outline

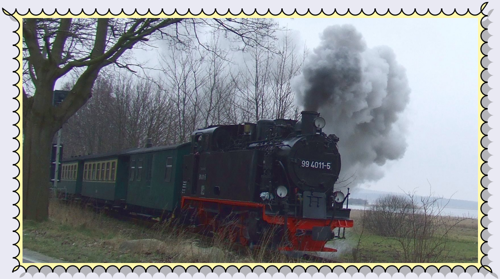
Outline

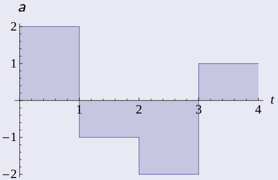
Hybrid Systems

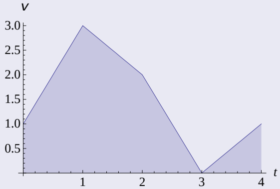
Problem

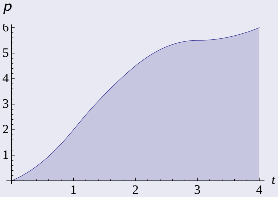
Hybrid System

- Continuous evolutions (differential equations)
- Discrete jumps (control decisions)

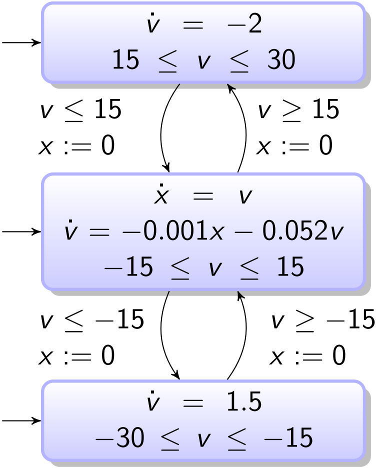


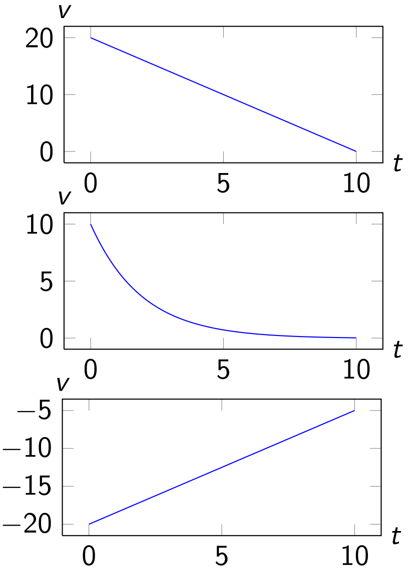






Velocity Controller





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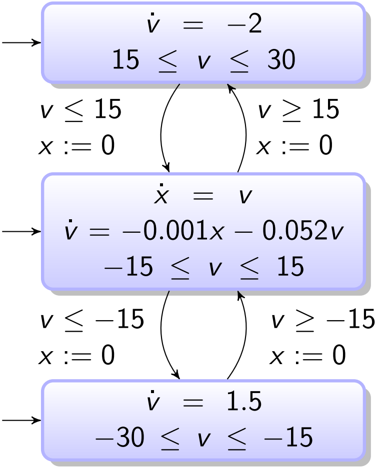

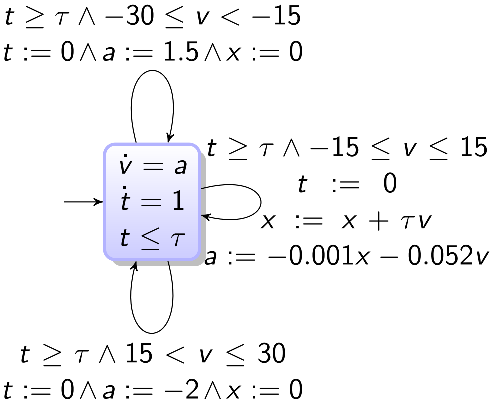

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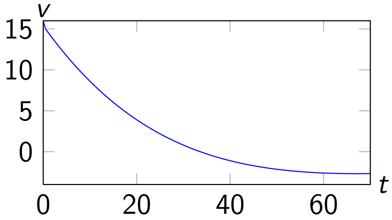
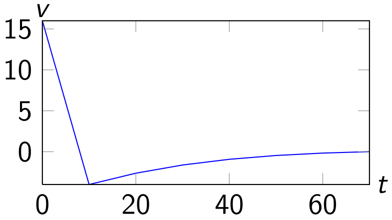
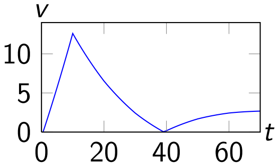


Velocity (specification)

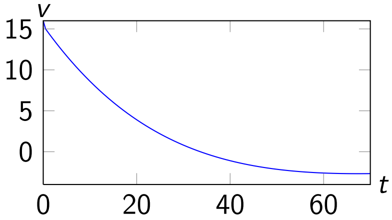


Velocity (implementation)



Velocity differences

Velocity Controller



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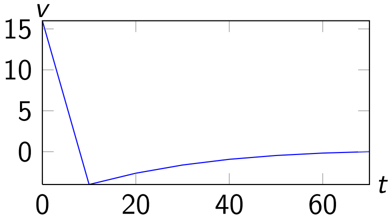

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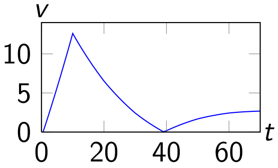

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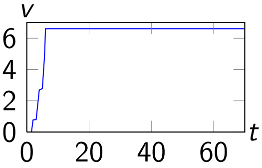
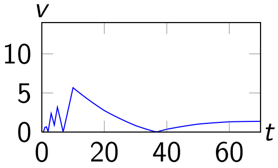


Temporal differences



Velocity differences (retimed)

Outline

Example for the Semantics

Example

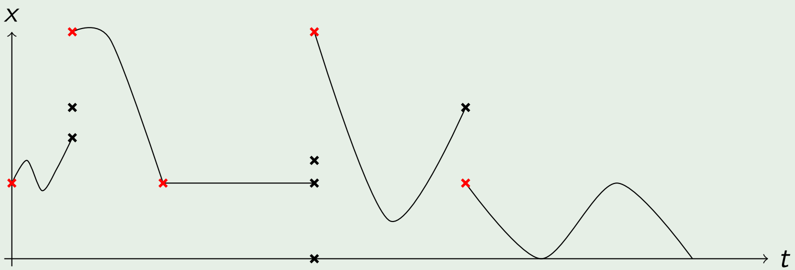
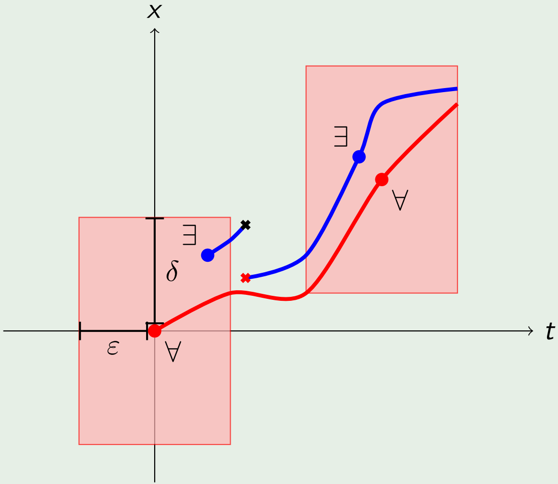


Illustration of the Similarity Notion

Example



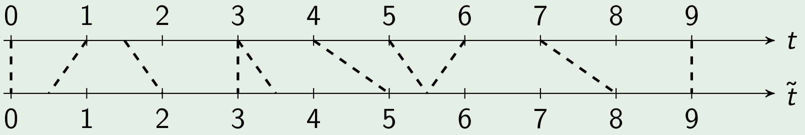
Retiming

Definition (ε-Retiming)

A left-total, surjective relation $\mathfrak{r} \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ is called ε-retiming iff

$$\forall (t, \tilde{t}) \in \mathfrak{r} : |t - \tilde{t}| < \varepsilon \wedge \forall (t', \tilde{t}') \in \mathfrak{r} : (t \leq t' \to \tilde{t} \leq \tilde{t}') \ .$$

Example



Definition of ε-δ-simulation

Definition

For two streams $\sigma_i : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}^p$ with $i \in \{1, 2\}$, given two non-negative real numbers ε, δ, we say that σ_1 is ε-δ-simulated by stream σ_2 (denoted by $\sigma_1 \trianglelefteq^{\varepsilon,\delta} \sigma_2$) iff there is a ε-retiming $\mathfrak{r}$ such that

$$\forall (t, \tilde{t}) \in \mathfrak{r} : ||c(\sigma_1)(t), c(\sigma_2)(\tilde{t})|| < \delta$$

where for $k \in \{1, 2\}$: $c(\sigma_k)$ is defined by $c(\sigma_k)(t) := \lim_{q \to \infty} \sigma_k(t, q)$.

Definition of ε-δ-simulation

Definition

A hybrid system A is ε-δ-simulated by another system B (denoted by $A \trianglelefteq^{\varepsilon,\delta} B$) iff for all input streams ι_A and for all input streams ι_B $\iota_A \trianglelefteq^{\varepsilon,\delta} \iota_B$ implies that for all output streams $\omega_A \in \Xi(\iota_A)$ of A, there is an output stream $\omega_B \in \Xi(\iota_B)$ of B such that $\omega_A \trianglelefteq^{\varepsilon,\delta} \omega_B$ holds.

Outline

Logic $\mathcal{L}\natural$ (Syntax)

Definition (Syntax of $\mathcal{L}\natural$)

The basic formulas are defined by

$$\phi ::= x \in \mathcal{I} \mid f(x_1, \ldots, x_n) \leq 0 \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \mathbb{U}_{\mathcal{J}} \phi_2$$

where $\mathcal{I} \subseteq \mathbb{R}$, $\mathcal{J} \subseteq \mathbb{R}$, f is a Lipschitz continuous function and the x_i are variables.

Logic $\mathcal{L}\natural$ (Semantics)

Definition (Valuation)

We define the valuation of a variable x at time t on a run ξ as

$$\zeta_\xi(t, x) := \lim_{n \to \infty} \xi(t, n)|_x \;\; ,$$

where $y|_x$ denotes the projection of the vector y to its component associated with the variable name x.

Logic $\mathcal{L}\natural$ (Semantics)

Definition (Semantics of $\mathcal{L}\natural$)

We define for a run ξ and some $t \in \mathbb{R}^+$ the semantics of a formula ϕ by:

- $\xi, t \models x \in \mathcal{I}$ iff $\zeta(t, x) \in \mathcal{I}$
- $\xi, t \models f(x_1, \ldots, x_n) \leq 0$ iff $f(\zeta(t, x_1), \ldots, \zeta(t, x_n)) \leq 0$

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- $\xi, t \models \phi \wedge \psi$ iff $\xi, t \models \phi$ and $\xi, t \models \psi$

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- $\xi, t \models \phi \mathbb{U}_{\mathcal{J}} \psi$
  iff $\exists t' \in \mathcal{J} : \xi, max\{t' + t, 0\} \models \psi$ and $\forall t \leq t'' < t' + t : \xi, t'' \models \phi$

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Additionally we define for a set of runs Ξ:

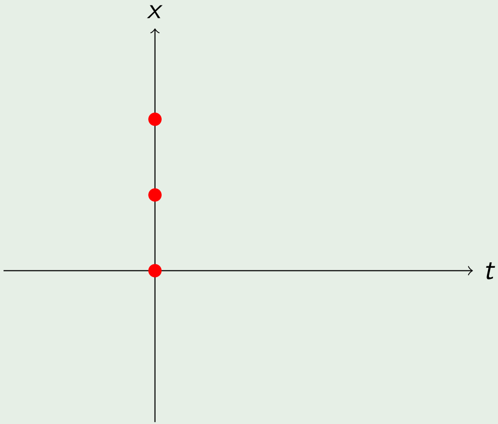
$$\Xi, t \models \phi \text{ iff for all runs } \xi \in \Xi \text{ holds } \xi, t \models \phi$$

A hybrid system H satisfies a formula denoted by $H \models \phi$ iff $\Xi_H, 0 \models \phi$.

Preservation (Informal)

Example

Formula: $x \in \{0, 1, 2\}$



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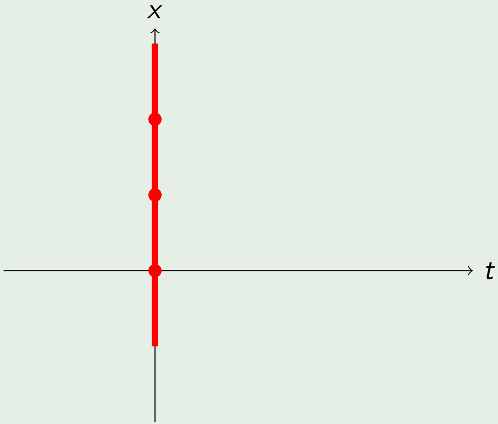
Formula: $x \in \{0, 1, 2\}$, $\delta = 1 \rightsquigarrow x \in [-1, 3]$



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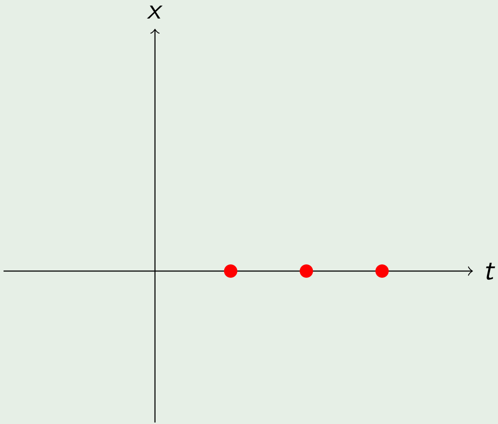

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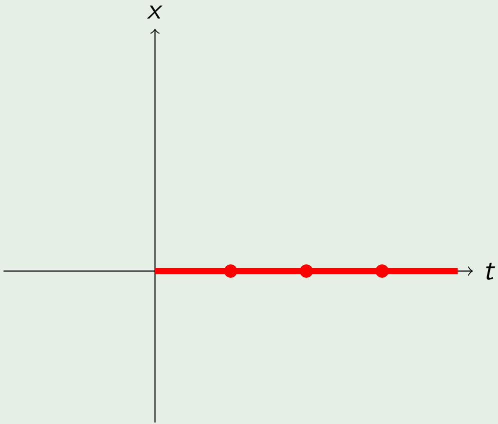
Formula: $\phi \mathbb{U}_{\{1,2,3\}} \psi$, $\varepsilon = 1 \rightsquigarrow \phi' \mathbb{U}_{[0,4]} \psi'$



Preservation (Formal)

Theorem (Preservation of logical properties)

If hybrid systems A and B satisfy $A \trianglelefteq^{\varepsilon,\delta} B$ and $B \models \phi$ then $A \models \phi^{+\delta}_{+\varepsilon}$ where $\phi^{+\delta}_{+\varepsilon} := re_{\varepsilon,\delta}(\phi)$ and $re_{\varepsilon,\delta}$ is defined by:

- $re_{\varepsilon,\delta}(x \in \mathcal{I}) := x \in \mathcal{I}'$, *where* $\mathcal{I}' = \{a \mid \exists b \in \mathcal{I} : a \in [b - \delta, b + \delta]\}$.
- $re_{\varepsilon,\delta}(f(x_1, \dots, x_n) \leq 0) := f(x_1, \dots, x_n) - \delta \cdot M \leq 0$ *where M is the Lipschitz constant for f.*

where $\mathcal{I} \subseteq \mathbb{R}$ and $\mathcal{J} \subseteq \mathbb{R}$ holds.

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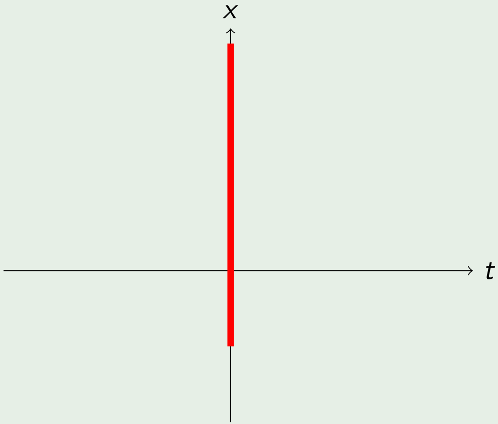

Preservation (Informal)

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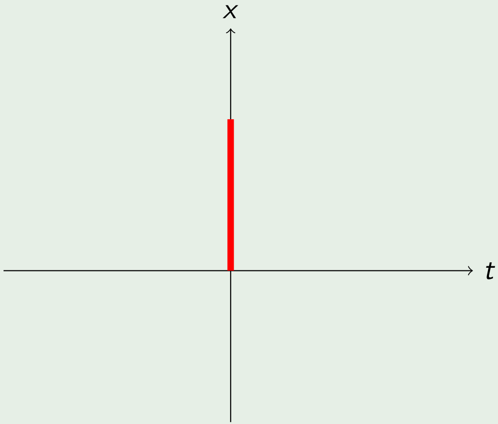
Formula: $\neg x \in [-1, 3]$, $\delta = 1 \rightsquigarrow \neg x \in [0, 2]$



Preservation (Formal)

Theorem (Preservation of logical properties)

The transformation function $ro_{\varepsilon,\delta}$ *is given by:*

- $ro_{\varepsilon,\delta}(x \in \mathcal{I}) := x \in \mathcal{I}'$, *where* $\mathcal{I}' = \{a \mid \forall b \in [a-\delta, a+\delta] : b \in \mathcal{I}\}$.
- $ro_{\varepsilon,\delta}(f(x_1,\ldots,x_n) \le 0) := f(x_1,\ldots,x_n) + \delta \cdot M \le 0$ *where* M *is the Lipschitz constant for* f.

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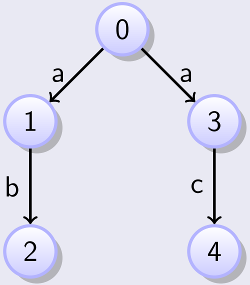
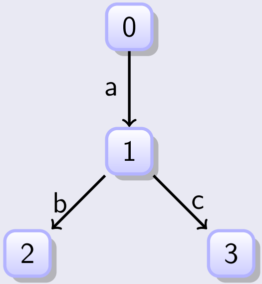
Outline

Classical Relation

Observation

Simulations can be defined in terms of games.





Observation

Controller synthesis is a game as well, i.e. the question whether the controller can win against an malicious environment.

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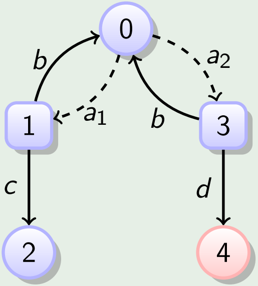


Hybrid Game

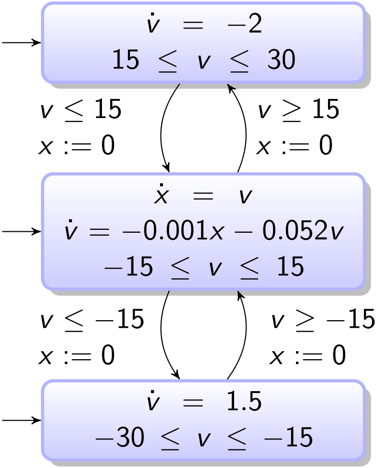
Definition (Hybrid Game)

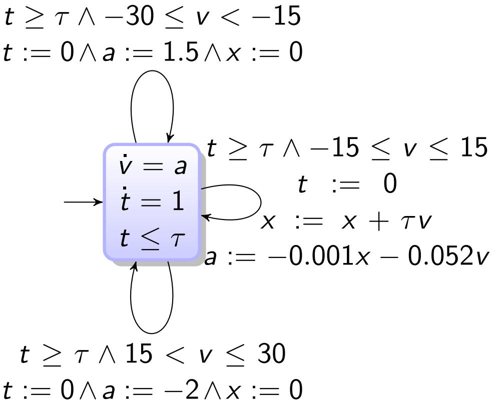
A *hybrid game* $HG = (S, E_c, U_c, l)$ consists of

- a hybrid automaton $S = (U, X, L, E, F, Inv, Init)$,
- a set of controllable transitions $E_c \subseteq E$,
- a set of controllable variables $U_c \subseteq U$,
- and a location $l \in L$.

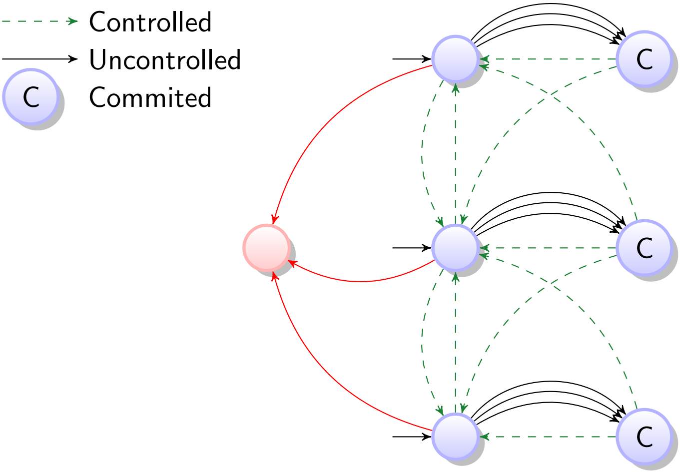
The environment wins, if it can force the game to enter the location l or if the controller does not have any more moves. The controller wins, if he can assert that the location l is avoided.

Velocity Controller





Velocity Controller (Game)



$U_c = \{s\}$ Invariant: $0 \leq s \leq 2$

$\dot{v} = -0.001x - 0.052v \rightsquigarrow \dot{v} = s \cdot (-0.001x - 0.052v)$

$\dot{v} = a \rightsquigarrow \dot{v} = (2 - s) \cdot a$

Similarity and Games

Assumption

The systems that we compare are inputless, i.e. $U = \emptyset$.

Theorem

Given two hybrid systems A and B. If there is a winning strategy for the controller in the game $(A \lessdot B, E_c, \{s\}, bad)$ *then* $A \trianglelefteq^{\varepsilon,\delta} B$ *holds.*

Observation

If system B is deterministic and a retiming strategy is given, model checking can be used to show that the winning strategy exists.

Outline

Summary

We . . .

- . . . defined a notion of similarity for hybrid systems.
- . . . showed properties that are preserved by this notion.
- . . . established the classical relation between simulations and games for this notion.
- . . . established some preliminary results for solving these games.