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Relating Natural Deduction and Sequent Calculus for Intuitionistic Non-Commutative Linear Logic

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Abstract

We present a sequent calculus for intuitionistic non-commutative linear logic (INCLL), show that it satisfies cut elimination, and investigate its relationship to a natural deduction system for the logic. We show how normal natural deductions correspond to cut-free derivations, and arbitrary natural deductions to sequent derivations with cut. This gives us a syntactic proof of normalization for a rich system of non-commutative natural deduction and its associated λ-calculus. INCLL conservatively extends linear logic with means to express sequencing, which has applications in functional programming, logical frameworks, logic programming, and natural language parsing.

1 Introduction

Linear logic [11] has been described as a logic of state because it views linear hypotheses as resources which may be consumed in the course of a deduction. It thereby significantly extends the expressive power of both classical and intuitionistic logics, yet it does not offer means to express sequencing. This is possible in non-commutative linear logic [1,5,2] which goes back to the seminal work by Lambek [16].

In [20] we introduced a system of natural deduction for intuitionistic non-commutative linear logic (INCLL) which conservatively extends intuitionistic linear logic. The associated proof term calculus is a λ-calculus with functions that must use their arguments in a specified order which is useful to capture

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properties of functional programs. This system was constructed from an intuitionistic point of view where the notion of a judgment plays the central role in determining how the logical connectives should behave.

In constructing INCLL we started with a judgment which would distinguish three types of hypotheses—ordered, linear, and unrestricted. In the same manner as linear logic restricts the use of the structural rules of contraction and weakening for linear hypotheses, INCLL further restricts the use of exchange for ordered hypotheses. Thus the non-commutativity of INCLL is not a priori a property of the logical connectives but rather a property of the judgment. This is reflected as a restriction on the mobility of hypotheses: ordered hypotheses must be consumed "in order", while other hypotheses are not subject to such a restriction.

We have found several applications for this logic and believe there are potentially many more. While detailed analysis of these applications is beyond the scope of this paper, we list a few to support our claim that INCLL is useful and worthy of more study. One direct application is a logical explanation for ordering properties of terms in continuation-passing style investigated by Danvy and the second author in [7]. The ordering inherent in non-commutative function arguments can be used to internalize stackability properties of program evaluation in a fragment of INCLL, which is large enough to capture the case of terms resulting from the standard CPS transformation. Another application lies in natural language parsing. Since our logic integrates ordinary, linear, and ordered functions in a consistent manner, we can logically describe more natural language phenomena than can be done by either linear logic or Lambek calculus [14,16]; in fact we can smoothly integrate parsing techniques from both logics. Finally, since INCLL is a conservative extension of intuitionistic linear logic, we can use INCLL as the foundation for a new logic programming language which conservatively extends Lolli [15]. The additional expressive power can be used to eliminate unwanted non-determinism in applications such as parsing, sorting algorithms, and operational semantics as described in [19].

While natural deduction is appropriate for studying functional languages, it does not shed much light on the process of proof search. But proof search is an important aspect of almost every operational use of a logic (e.g., type inference, theorem proving, or logic programming). We therefore present and study a sequent calculus as a foundational calculus of proof search. In ongoing related work we are studying further refinements which allow a logic programming interpretation in the style of uniform derivations [17].

We validate our sequent calculus by showing that it admits cut and by showing a direct mapping between normal natural deductions and cut-free sequent derivations. We further use the sequent system as a vehicle for establishing normalization of the natural deduction system for INCLL by giving a translation from arbitrary deductions into sequent derivations which may include cut. This extends the results from [20] where we gave a logical relations argument in the style of Kripke to show the existence of canonical forms for a fragment of INCLL. Here it is easy to treat the full logic, which leads to a different algorithm for normalization.

2 Natural Deduction

We review the natural deduction formulation of INCLL as given in [20], extended to include quantifiers. For the sake of brevity we elide proof terms here, although strictly speaking they are necessary for an unambiguous specification.

We use Γ, Δ, and Ω to stand for lists of propositions (defined below), where we use juxtaposition both for list concatenation and appending an element to a list on either side. "$\cdot$" denotes the empty list. It will be characteristic for our calculus that we have three different kinds of hypotheses which, for ease of notation, are written in different zones as in [3,12,4]. The unrestricted (sometimes called "intuitionistic") context Γ satisfies contraction, weakening, and exchange and is propagated unchanged from the conclusion to the premises of a rule. In contrast, each hypothesis in the linear context Δ must be distributed to one premise or the other for the multiplicative rules. When viewed top-down, the corresponding operation of *non-deterministic merge*, $\Delta_A \bowtie \Delta_B$ allows an arbitrary interleaving of assumptions. So the linear context satisfies only exchange. Finally, for multiplicative rules, the ordered context Ω must be split at some intermediate point, where all the hypotheses to the left keep the same relative order and are propagated to one premise, while the remaining ordered hypotheses are propagated to another premise. We will tacitly use some trivial properties of non-deterministic merge. We consider here the following connectives.

Propositions	$A ::=$	P	atomic propositions
	$\mid$	$A_1 \rightarrow A_2$	intuitionistic implication
	$\mid$	$A_1 \multimap A_2$	linear implication
	$\mid$	$A_1 \twoheadrightarrow A_2$	ordered right implication
	$\mid$	$A_1 \rightarrowtail A_2$	ordered left implication
	$\mid$	$A_1 \bullet A_2$	ordered multiplicative conjunction
	$\mid$	1	multiplicative truth
	$\mid$	$A_1 \,\&\, A_2$	additive conjunction
	$\mid$	$\top$	additive truth
	$\mid$	$A_1 \oplus A_2$	additive disjunction
	$\mid$	0	additive falsehood
	$\mid$	$\forall x.\, A$	universal quantification
	$\mid$	$\exists x.\, A$	existential quantification
	$\mid$	$¡A$	mobility operator
	$\mid$	$!A$	exponential operator

We now present the natural deduction judgments for INCLL omitting the proof terms and variable labels on hypotheses. The judgment has the form

$\Gamma; \Delta; \Omega \vdash A$, where Γ, Δ, Ω are lists of propositions and A is a proposition. Intuitively, Γ is an unrestricted context, Δ is a linear context, and Ω is an ordered linear context. We will take care to construct the deduction system to ensure that the expected structural properties of the contexts hold.

Hypotheses.

$$\frac{}{\Gamma_L A \Gamma_R; \cdot; \cdot \vdash A}\,\mathbf{ivar} \qquad \frac{}{\Gamma; A; \cdot \vdash A}\,\mathbf{lvar} \qquad \frac{}{\Gamma; \cdot; A \vdash A}\,\mathbf{ovar}$$

Unrestricted implication.

$$\frac{\Gamma A; \Delta; \Omega \vdash B}{\Gamma; \Delta; \Omega \vdash A \to B}\,{\to_I} \qquad \frac{\Gamma; \Delta; \Omega \vdash A \to B \qquad \Gamma; \cdot; \cdot \vdash A}{\Gamma; \Delta; \Omega \vdash B}\,{\to_E}$$

Linear implication.

$$\frac{\Gamma; \Delta A; \Omega \vdash B}{\Gamma; \Delta; \Omega \vdash A \multimap B}\,{\multimap_I} \qquad \frac{\Gamma; \Delta; \Omega \vdash A \multimap B \qquad \Gamma; \Delta_A; \cdot \vdash A}{\Gamma; \Delta \bowtie \Delta_A; \Omega \vdash B}\,{\multimap_E}$$

Ordered implications.

$$\frac{\Gamma; \Delta; A\Omega \vdash B}{\Gamma; \Delta; \Omega \vdash A \rightarrowtail B}\,{\rightarrowtail_I} \qquad \frac{\Gamma; \Delta; \Omega \vdash A \rightarrowtail B \qquad \Gamma; \Delta_A; \Omega_A \vdash A}{\Gamma; \Delta \bowtie \Delta_A; \Omega_A\Omega \vdash B}\,{\rightarrowtail_E}$$

$$\frac{\Gamma; \Delta; \Omega A \vdash B}{\Gamma; \Delta; \Omega \vdash A \twoheadrightarrow B}\,{\twoheadrightarrow_I} \qquad \frac{\Gamma; \Delta; \Omega \vdash A \twoheadrightarrow B \qquad \Gamma; \Delta_A; \Omega_A \vdash A}{\Gamma; \Delta \bowtie \Delta_A; \Omega\Omega_A \vdash B}\,{\twoheadrightarrow_E}$$

Ordered conjunction and unit.

$$\frac{\Gamma; \Delta_A; \Omega_L \vdash A \qquad \Gamma; \Delta_B; \Omega_R \vdash B}{\Gamma; \Delta_A \bowtie \Delta_B; \Omega_L\Omega_R \vdash A \bullet B}\,{\bullet_I}$$

$$\frac{\Gamma; \Delta; \Omega \vdash A \bullet B \qquad \Gamma; \Delta_C; \Omega_L AB \Omega_R \vdash C}{\Gamma; \Delta_C \bowtie \Delta; \Omega_L\Omega\Omega_R \vdash C}\,{\bullet_E}$$

$$\frac{}{\Gamma; \cdot; \cdot \vdash 1}\,1_I \qquad \frac{\Gamma; \Delta; \Omega \vdash 1 \qquad \Gamma; \Delta_C; \Omega_L\Omega_R \vdash C}{\Gamma; \Delta_C \bowtie \Delta; \Omega_L\Omega\Omega_R \vdash C}\,1_E$$

Additive conjunction and unit.

$$\frac{\Gamma;\Delta;\Omega\vdash A \qquad \Gamma;\Delta;\Omega\vdash B}{\Gamma;\Delta;\Omega\vdash A\mathbin{\&}B}\,\&_I$$

$$\frac{\Gamma;\Delta;\Omega\vdash A\mathbin{\&}B}{\Gamma;\Delta;\Omega\vdash A}\,\&_{E1} \qquad \frac{\Gamma;\Delta;\Omega\vdash A\mathbin{\&}B}{\Gamma;\Delta;\Omega\vdash B}\,\&_{E2}$$

$$\frac{}{\Gamma;\Delta;\Omega\vdash\top}\,\top_I$$

Additive disjunction and unit.

$$\frac{\Gamma;\Delta;\Omega\vdash A}{\Gamma;\Delta;\Omega\vdash A\oplus B}\,\oplus_{I1} \qquad \frac{\Gamma;\Delta;\Omega\vdash B}{\Gamma;\Delta;\Omega\vdash A\oplus B}\,\oplus_{I2}$$

$$\frac{\Gamma;\Delta;\Omega\vdash A\oplus B \qquad \Gamma;\Delta_C;\Omega_L A\Omega_R\vdash C \qquad \Gamma;\Delta_C;\Omega_L B\Omega_R\vdash C}{\Gamma;\Delta_C\bowtie\Delta;\Omega_L\Omega\Omega_R\vdash C}\,\oplus_E$$

$$\frac{\Gamma;\Delta;\Omega\vdash 0}{\Gamma;\Delta_C\bowtie\Delta;\Omega_L\Omega\Omega_R\vdash C}\,0_E$$

Universal and existential quantification.

$$\frac{\Gamma;\Delta;\Omega\vdash[a/x]A}{\Gamma;\Delta;\Omega\vdash\forall x.\ A}\,\forall_I^a \quad \frac{\Gamma;\Delta;\Omega\vdash\forall x.\ A}{\Gamma;\Delta;\Omega\vdash[t/x]A}\,\forall_E$$

$$\frac{\Gamma;\Delta;\Omega\vdash[t/x]A}{\Gamma;\Delta;\Omega\vdash\exists x.\ A}\,\exists_I \quad \frac{\Gamma;\Delta;\Omega\vdash\exists x.\ A \qquad \Gamma;\Delta_C;\Omega_L[a/x]A\Omega_R\vdash C}{\Gamma;\Delta\bowtie\Delta_C;\Omega_L\Omega\Omega_R\vdash C}\,\exists_E^a$$

Mobility operator.

$$\frac{\Gamma;\Delta;\cdot\vdash A}{\Gamma;\Delta;\cdot\vdash\text{¡}A}\,\text{¡}_I \quad \frac{\Gamma;\Delta;\Omega\vdash\text{¡}A \qquad \Gamma;\Delta_C A;\Omega_L\Omega_R\vdash C}{\Gamma;\Delta_C\bowtie\Delta;\Omega_L\Omega\Omega_R\vdash C}\,\text{¡}_E$$

Exponential operator.

$$\frac{\Gamma;\cdot;\cdot\vdash A}{\Gamma;\cdot;\cdot\vdash !A}\,!_I \quad \frac{\Gamma;\Delta;\Omega\vdash !A \qquad \Gamma A;\Delta_C;\Omega_L\Omega_R\vdash C}{\Gamma;\Delta_C\bowtie\Delta;\Omega_L\Omega\Omega_R\vdash C}\,!_E$$

The rules $\forall_I^a$ and $\exists_E^a$ must obey the usual restrictions on a. Note that the two forms of ordered implication add their new hypothesis either on the left or right, and that this difference is reflected in the order of Ω and Ω_A in the conclusion of the elimination rules.

Lemma 2.1 (Contraction, Weakening, Exchange)

(i) *If* $\Gamma_L A A \Gamma_R; \Delta; \Omega \vdash C$ *then* $\Gamma_L A \Gamma_R; \Delta; \Omega \vdash C$.

(ii) *If* $\Gamma; \Delta; \Omega \vdash C$ *then* $\Gamma A; \Delta; \Omega \vdash C$.

(iii) *If* $\Gamma_L A B \Gamma_R; \Delta; \Omega \vdash C$ *then* $\Gamma_L B A \Gamma_R; \Delta; \Omega \vdash C$.

(iv) *If* $\Gamma; \Delta_L A B \Delta_R; \Omega \vdash C$ *then* $\Gamma; \Delta_L B A \Delta_R; \Omega \vdash C$.

Proof. By structural induction on the given derivation. Note that this induction defines a structure preserving translation—only the hypotheses change for each judgment in a derivation. On proof terms, only contraction has an effect (the renaming of variables). □

We also have substitution principles for the logic.

Lemma 2.2 (Substitution)

(i) *If* $\Gamma_A A \Gamma; \Delta; \Omega \vdash C$ *and* $\Gamma_A; \cdot; \cdot \vdash A$ *then* $\Gamma_A \Gamma; \Delta; \Omega \vdash C$.

(ii) *If* $\Gamma; \Delta A; \Omega \vdash C$ *and* $\Gamma; \Delta_A; \cdot \vdash A$ *then* $\Gamma; \Delta \bowtie \Delta_A; \Omega \vdash C$.

(iii) *If* $\Gamma; \Delta; \Omega_L A \Omega_R \vdash C$ *and* $\Gamma; \Delta_A; \Omega_A \vdash A$ *then* $\Gamma; \Delta \bowtie \Delta_A; \Omega_L \Omega_A \Omega_R \vdash C$.

Proof. By structural induction over the given derivation for C. Again we have a structure preserving translation where every use of A in the deduction of C is replaced by the given deduction of A. □

3 Normal Natural Deduction

We present a refined system of natural deduction for INCLL which only admits normal deductions. This system is based on separating *normal* deductions, characterized by bottom-up reasoning with introduction rules, from *atomic* deductions, characterized by top-down reasoning with elimination rules. These two can meet at any point with a coercion which allows us to view any atomic deduction as normal. Note that a normal deduction in this sense cannot contain an introduction rule immediately followed by an elimination of the same connective (and neither can it contain any so-called *maximal segments*). As we will see from the correspondence to the sequent calculus, any normal derivation has the subformula property.

We have two kinds of judgments:

(i) $\Gamma; \Delta; \Omega \vdash A \uparrow$ (A has a normal derivation), and

(ii) $\Gamma; \Delta; \Omega \vdash A \downarrow$ (A has an atomic derivation).

The arrow indicates the direction of reasoning allowed.

Here are the judgments for the normal natural deduction system.

Hypotheses.

$$\frac{}{\Gamma_L A \Gamma_R; \cdot; \cdot \vdash A \downarrow}\,\mathbf{ivar} \qquad \frac{}{\Gamma; A; \cdot \vdash A \downarrow}\,\mathbf{lvar} \qquad \frac{}{\Gamma; \cdot; A \vdash A \downarrow}\,\mathbf{ovar}$$

Coercion.

$$\frac{\Gamma;\Delta;\Omega\vdash A\downarrow}{\Gamma;\Delta;\Omega\vdash A\uparrow}\,\textbf{coerce}$$

Unrestricted implication.

$$\frac{\Gamma A;\Delta;\Omega\vdash B\uparrow}{\Gamma;\Delta;\Omega\vdash A\to B\uparrow}\to_I \quad \frac{\Gamma;\Delta;\Omega\vdash A\to B\downarrow \qquad \Gamma;\cdot;\cdot\vdash A\uparrow}{\Gamma;\Delta;\Omega\vdash B\downarrow}\to_E$$

Linear implication.

$$\frac{\Gamma;\Delta A;\Omega\vdash B\uparrow}{\Gamma;\Delta;\Omega\vdash A\multimap B\uparrow}\multimap_I \quad \frac{\Gamma;\Delta;\Omega\vdash A\multimap B\downarrow \qquad \Gamma;\Delta_A;\cdot\vdash A\uparrow}{\Gamma;\Delta\bowtie\Delta_A;\Omega\vdash B\downarrow}\multimap_E$$

Ordered implications.

$$\frac{\Gamma;\Delta;A\Omega\vdash B\uparrow}{\Gamma;\Delta;\Omega\vdash A\rightarrowtail B\uparrow}\rightarrowtail_I \quad \frac{\Gamma;\Delta;\Omega\vdash A\rightarrowtail B\downarrow \qquad \Gamma;\Delta_A;\Omega_A\vdash A\uparrow}{\Gamma;\Delta\bowtie\Delta_A;\Omega_A\Omega\vdash B\downarrow}\rightarrowtail_E$$

$$\frac{\Gamma;\Delta;\Omega A\vdash B\uparrow}{\Gamma;\Delta;\Omega\vdash A\twoheadrightarrow B\uparrow}\twoheadrightarrow_I \quad \frac{\Gamma;\Delta;\Omega\vdash A\twoheadrightarrow B\downarrow \qquad \Gamma;\Delta_A;\Omega_A\vdash A\uparrow}{\Gamma;\Delta\bowtie\Delta_A;\Omega\Omega_A\vdash B\downarrow}\twoheadrightarrow_E$$

Ordered conjunction and unit.

$$\frac{\Gamma;\Delta_A;\Omega_L\vdash A\uparrow \qquad \Gamma;\Delta_B;\Omega_R\vdash B\uparrow}{\Gamma;\Delta_A\bowtie\Delta_B;\Omega_L\Omega_R\vdash A\bullet B\uparrow}\bullet_I$$

$$\frac{\Gamma;\Delta;\Omega\vdash A\bullet B\downarrow \qquad \Gamma;\Delta_C;\Omega_L AB\Omega_R\vdash C\uparrow}{\Gamma;\Delta_C\bowtie\Delta;\Omega_L\Omega\Omega_R\vdash C\uparrow}\bullet_E$$

$$\frac{}{\Gamma;\cdot;\cdot\vdash 1\uparrow}1_I \quad \frac{\Gamma;\Delta;\Omega\vdash 1\downarrow \qquad \Gamma;\Delta_C;\Omega_L\Omega_R\vdash C\uparrow}{\Gamma;\Delta_C\bowtie\Delta;\Omega_L\Omega\Omega_R\vdash C\uparrow}1_E$$

Additive conjunction and unit.

$$\frac{\Gamma;\Delta;\Omega\vdash A\uparrow \qquad \Gamma;\Delta;\Omega\vdash B\uparrow}{\Gamma;\Delta;\Omega\vdash A\,\&\,B\uparrow}\&_I$$

$$\frac{\Gamma;\Delta;\Omega\vdash A\,\&\,B\downarrow}{\Gamma;\Delta;\Omega\vdash A\downarrow}\&_{E1} \qquad \frac{\Gamma;\Delta;\Omega\vdash A\,\&\,B\downarrow}{\Gamma;\Delta;\Omega\vdash B\downarrow}\&_{E2}$$

$$\frac{}{\Gamma;\Delta;\Omega\vdash\top\uparrow}\top_I$$

Additive disjunction and unit.

$$\frac{\Gamma; \Delta; \Omega \vdash A \uparrow}{\Gamma; \Delta; \Omega \vdash A \oplus B \uparrow} \oplus_{I1} \qquad \frac{\Gamma; \Delta; \Omega \vdash B \uparrow}{\Gamma; \Delta; \Omega \vdash A \oplus B \uparrow} \oplus_{I2}$$

$$\frac{\Gamma; \Delta; \Omega \vdash A \oplus B \downarrow \quad \Gamma; \Delta_C; \Omega_L A \Omega_R \vdash C \uparrow \quad \Gamma; \Delta_C; \Omega_L B \Omega_R \vdash C \uparrow}{\Gamma; \Delta \bowtie \Delta_C; \Omega_L \Omega \Omega_R \vdash C \uparrow} \oplus_E$$

$$\frac{\Gamma; \Delta; \Omega \vdash 0 \downarrow}{\Gamma; \Delta_C \bowtie \Delta; \Omega_L \Omega \Omega_R \vdash C \uparrow} 0_E$$

Universal and existential quantification.

$$\frac{\Gamma; \Delta; \Omega \vdash [a/x]A \uparrow}{\Gamma; \Delta; \Omega \vdash \forall x.\ A \uparrow} \forall_I^a \quad \frac{\Gamma; \Delta; \Omega \vdash \forall x.\ A \downarrow}{\Gamma; \Delta; \Omega \vdash [t/x]A \downarrow} \forall_E$$

$$\frac{\Gamma; \Delta; \Omega \vdash [t/x]A \uparrow}{\Gamma; \Delta; \Omega \vdash \exists x.\ A \uparrow} \exists_I \quad \frac{\Gamma; \Delta; \Omega \vdash \exists x.\ A \downarrow \quad \Gamma; \Delta_C; \Omega_L [a/x] A \Omega_R \vdash C \uparrow}{\Gamma; \Delta \bowtie \Delta_C; \Omega_L \Omega \Omega_R \vdash C \uparrow} \exists_E^a$$

Mobility operator.

$$\frac{\Gamma; \Delta; \cdot \vdash A \uparrow}{\Gamma; \Delta; \cdot \vdash ¡A \uparrow} ¡_I \quad \frac{\Gamma; \Delta; \Omega \vdash ¡A \downarrow \quad \Gamma; \Delta_C A; \Omega_L \Omega_R \vdash C \uparrow}{\Gamma; \Delta_C \bowtie \Delta; \Omega_L \Omega \Omega_R \vdash C \uparrow} ¡_E$$

Exponential operator.

$$\frac{\Gamma; \cdot; \cdot \vdash A \uparrow}{\Gamma; \cdot; \cdot \vdash !A \uparrow} !_I \quad \frac{\Gamma; \Delta; \Omega \vdash !A \downarrow \quad \Gamma A; \Delta_C; \Omega_L \Omega_R \vdash C \uparrow}{\Gamma; \Delta_C \bowtie \Delta; \Omega_L \Omega \Omega_R \vdash C \uparrow} !_E$$

We remark that this system enjoys the analagous structural properties as the previous system. Furthermore we have the following substitution principles. Notice that only an atomic derivation may be substituted for a hypothesis since the uses of assumptions are considered atomic deductions (the **ovar**, **lvar**, **ivar** rules).

Lemma 3.1 (Substitution)

(i) *If* $\Gamma_A A \Gamma; \Delta; \Omega \vdash C \uparrow$ *and* $\Gamma_A; \cdot; \cdot \vdash A \downarrow$ *then* $\Gamma_A \Gamma; \Delta; \Omega \vdash C \uparrow$.

(ii) *If* $\Gamma; \Delta A; \Omega \vdash C \uparrow$ *and* $\Gamma; \Delta_A; \cdot \vdash A \downarrow$ *then* $\Gamma; \Delta \bowtie \Delta_A; \Omega \vdash C \uparrow$.

(iii) *If* $\Gamma; \Delta; \Omega_L A \Omega_R \vdash C \uparrow$ *and* $\Gamma; \Delta_A; \Omega_A \vdash A \downarrow$
then $\Gamma; \Delta \bowtie \Delta_A; \Omega_L \Omega_A \Omega_R \vdash C \uparrow$.

(iv) *If* $\Gamma_A A \Gamma; \Delta; \Omega \vdash C \downarrow$ *and* $\Gamma_A; \cdot; \cdot \vdash A \downarrow$ *then* $\Gamma_A \Gamma; \Delta; \Omega \vdash C \downarrow$.

(v) *If* $\Gamma; \Delta A; \Omega \vdash C \downarrow$ *and* $\Gamma; \Delta_A; \cdot \vdash A \downarrow$ *then* $\Gamma; \Delta \bowtie \Delta_A; \Omega \vdash C \downarrow$.

(vi) *If* $\Gamma; \Delta; \Omega_L A \Omega_R \vdash C \downarrow$ *and* $\Gamma; \Delta_A; \Omega_A \vdash A \downarrow$ *then* $\Gamma; \Delta \bowtie \Delta_A; \Omega_L \Omega_A \Omega_R \vdash C \downarrow$.

Proof. By structural induction over the given derivations for C. □

Since the structure of the rules in the normal system has not changed, we can easily see that the normal system simply rules out some valid deductions of the first system. Therefore we have the following soundness theorem.

Theorem 3.2

(i) *If* $\Gamma; \Delta; \Omega \vdash A \uparrow$ *then* $\Gamma; \Delta; \Omega \vdash A$

(ii) *If* $\Gamma; \Delta; \Omega \vdash A \downarrow$ *then* $\Gamma; \Delta; \Omega \vdash A$

Proof. By simple structural induction. Coercions are simply eliminated. □

The converse, that every provable proposition has a normal deduction, does indeed hold and could be proved by a Kripke logical relations argument [9]. The proof for a fragment of the system above is a minor modification of the proof of the existence of canonical forms given in [20]. Instead, we will prove it indirectly by going through a sequent calculus presentation of INCLL, taking advantage of the cut elimination theorem. This will allow us to better focus on the connection between sequent calculus and natural deduction. It will also give further validation to our sequent system by showing that it exactly proves the propositions which have natural deductions.

Before introducing the sequent system, we introduce a third natural deduction system for INCLL which is obviously equivalent to the original. This system is based on the preceding normal system and has two judgments standing for normal and atomic derivations. However, in order to recover arbitrary deductions, it also allows an additional coercion from normal to atomic derivations. We write $\Gamma; \Delta; \Omega \vdash^+ A \uparrow$ and $\Gamma; \Delta; \Omega \vdash^+ A \downarrow$ which is defined by exactly the same rules as the normal and atomic judgments above, plus the rule

$$\frac{\Gamma; \Delta; \Omega \vdash^+ A \uparrow}{\Gamma; \Delta; \Omega \vdash^+ A \downarrow}\,\textbf{lemma}$$

Theorem 3.3

(i) $\Gamma; \Delta; \Omega \vdash^+ A \uparrow$ *iff* $\Gamma; \Delta; \Omega \vdash A$

(ii) $\Gamma; \Delta; \Omega \vdash^+ A \downarrow$ *iff* $\Gamma; \Delta; \Omega \vdash A$

Proof. In each direction, by simple structural induction on the given derivation. In the forward direction coercions are simply eliminated by the translation. In the backwards direction they are introduced if the last inference is not of the right kind. Note that these translations do not form a bijection since redundant coercions collapse. □

4 Sequent Calculus

We now present a sequent calculus for INCLL. This sequent calculus is a conservative extension of both associative Lambek calculus [16] and the sequent system for non-commutative intuitionistic linear logic given in [5].

Similar to natural deduction judgments, our sequents have the form:

$$\Gamma; \Delta; \Omega \longrightarrow A$$

where Γ, Δ, Ω are lists of propositions, and A is a proposition. Again, Γ, Δ, Ω are meant to denote an intuitionistic, linear, and ordered context respectively. In the sequent setting, one may logically think of the three antecedent contexts as one big context where the ordered hypotheses are in a fixed relative order while the other linear and unrestricted propositions may “float”. The intuitionistic propositions may also be copied or ignored in the initial sequents. In a sequent calculus, the logical connectives are characterized by *right rules* and *left rules* which, as we shall see, correspond to the introduction and elimination rules of natural deduction. In addition we have initial sequents and two structural rules.

Hypotheses.

$$\frac{}{\Gamma; \cdot; A \longrightarrow A}\,\mathbf{init}$$

$$\frac{\Gamma_L A \Gamma_R; \Delta; \Omega_L A \Omega_R \longrightarrow B}{\Gamma_L A \Gamma_R; \Delta; \Omega_L \Omega_R \longrightarrow B}\,\mathbf{copy} \qquad \frac{\Gamma; \Delta_L \Delta_R; \Omega_L A \Omega_R \longrightarrow B}{\Gamma; \Delta_L A \Delta_R; \Omega_L \Omega_R \longrightarrow B}\,\mathbf{place}$$

Unrestricted implication.

$$\frac{\Gamma A; \Delta; \Omega \longrightarrow B}{\Gamma; \Delta; \Omega \longrightarrow A \to B}\,{\to_R} \qquad \frac{\Gamma; \Delta; \Omega_L B \Omega_R \longrightarrow C \qquad \Gamma; \cdot; \cdot \longrightarrow A}{\Gamma; \Delta; \Omega_L (A \to B) \Omega_R \longrightarrow C}\,{\to_L}$$

Linear implication.

$$\frac{\Gamma; \Delta A; \Omega \longrightarrow B}{\Gamma; \Delta; \Omega \longrightarrow A \multimap B}\,{\multimap_R} \qquad \frac{\Gamma; \Delta_B; \Omega_L B \Omega_R \longrightarrow C \qquad \Gamma; \Delta_A; \cdot \longrightarrow A}{\Gamma; \Delta_B \bowtie \Delta_A; \Omega_L (A \multimap B) \Omega_R \longrightarrow C}\,{\multimap_L}$$

Ordered implications.

$$\frac{\Gamma; \Delta; A\Omega \longrightarrow B}{\Gamma; \Delta; \Omega \longrightarrow A \rightarrowtail B}\,{\rightarrowtail_R} \qquad \frac{\Gamma; \Delta_B; \Omega_L B \Omega_R \longrightarrow C \qquad \Gamma; \Delta_A; \Omega_A \longrightarrow A}{\Gamma; \Delta_B \bowtie \Delta_A; \Omega_L \Omega_A (A \rightarrowtail B) \Omega_R \longrightarrow C}\,{\rightarrowtail_L}$$

$$\frac{\Gamma; \Delta; \Omega A \longrightarrow B}{\Gamma; \Delta; \Omega \longrightarrow A \twoheadrightarrow B}\,{\twoheadrightarrow_R} \qquad \frac{\Gamma; \Delta_B; \Omega_L B \Omega_R \longrightarrow C \qquad \Gamma; \Delta_A; \Omega_A \longrightarrow A}{\Gamma; \Delta_B \bowtie \Delta_A; \Omega_L (A \twoheadrightarrow B) \Omega_A \Omega_R \longrightarrow C}\,{\twoheadrightarrow_L}$$

Ordered conjunction and unit.

$$\frac{\Gamma; \Delta_A; \Omega_L \longrightarrow A \qquad \Gamma; \Delta_B; \Omega_R \longrightarrow B}{\Gamma; \Delta_A \bowtie \Delta_B; \Omega_L \Omega_R \longrightarrow A \bullet B} \bullet_R \quad \frac{\Gamma; \Delta; \Omega_L A B \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L (A \bullet B) \Omega_R \longrightarrow C} \bullet_L$$

$$\frac{}{\Gamma; \cdot; \cdot \longrightarrow 1} 1_R \quad \frac{\Gamma; \Delta; \Omega_L \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L 1 \Omega_R \longrightarrow C} 1_L$$

Additive conjunction and unit.

$$\frac{\Gamma; \Delta; \Omega \longrightarrow A \qquad \Gamma; \Delta; \Omega \longrightarrow B}{\Gamma; \Delta; \Omega \longrightarrow A \,\&\, B} \&_R$$

$$\frac{\Gamma; \Delta; \Omega_L A \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L (A \,\&\, B) \Omega_R \longrightarrow C} \&_{L1} \qquad \frac{\Gamma; \Delta; \Omega_L B \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L (A \,\&\, B) \Omega_R \longrightarrow C} \&_{L2}$$

$$\frac{}{\Gamma; \Delta; \Omega \longrightarrow \top} \top_R$$

Additive disjunction and unit.

$$\frac{\Gamma; \Delta; \Omega \longrightarrow A}{\Gamma; \Delta; \Omega \longrightarrow (A \oplus B)} \oplus_{R1} \qquad \frac{\Gamma; \Delta; \Omega \longrightarrow B}{\Gamma; \Delta; \Omega \longrightarrow (A \oplus B)} \oplus_{R2}$$

$$\frac{\Gamma; \Delta; \Omega_L A \Omega_R \longrightarrow C \qquad \Gamma; \Delta; \Omega_L B \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L (A \oplus B) \Omega_R \longrightarrow C} \oplus_L$$

$$\frac{}{\Gamma; \Delta; \Omega_L 0 \Omega_R \longrightarrow C} 0_L$$

Universal and existential quantification.

$$\frac{\Gamma; \Delta; \Omega \longrightarrow [a/x]A}{\Gamma; \Delta; \Omega \longrightarrow \forall x.\ A} \forall_R^a \qquad \frac{\Gamma; \Delta; \Omega_L [t/x] A \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L (\forall x.\ A) \longrightarrow C} \forall_L$$

$$\frac{\Gamma; \Delta; \Omega \longrightarrow [t/x]A}{\Gamma; \Delta; \Omega \longrightarrow \exists x.\ A} \exists_R \qquad \frac{\Gamma; \Delta; \Omega_L [a/x] A \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L (\exists x.\ A) \Omega_R \longrightarrow C} \exists_L^a$$

Mobility operator.

$$\frac{\Gamma; \Delta; \cdot \longrightarrow A}{\Gamma; \Delta; \cdot \longrightarrow ¡A} ¡_R \quad \frac{\Gamma; \Delta A; \Omega_L \Omega_R \longrightarrow C}{\Gamma; \Delta; \Omega_L (¡A) \Omega_R \longrightarrow C} ¡_L$$

Exponential operator.

$$\frac{\Gamma; \cdot; \cdot \;\longrightarrow\; A}{\Gamma; \cdot; \cdot \;\longrightarrow\; !A}\,!_R \qquad \frac{\Gamma A; \Delta; \Omega_L \Omega_R \;\longrightarrow\; C}{\Gamma; \Delta; \Omega_L (!A) \Omega_R \;\longrightarrow\; C}\,!_L$$

We point out a few proof-theoretical aspects of INCLL which are more difficult to see using natural deduction. First of all, it is clear that this system has the subformula property: only instances of subformulas of propositions present in the conclusion can appear in the derivation. Since proof search based on this form of sequent calculus proceeds bottom-up, this is a critical property. It is due, of course, to the absence of any explicit cut rule. The commutative fragment of this logic is identical to intuitionistic linear logic. To give a feel for how the two ordered implications and the ordered context work, we note the following: $A \twoheadrightarrow (A \twoheadrightarrow B) \twoheadrightarrow B$ is not provable while $A \twoheadrightarrow (A \twoheadrightarrow B) \rightarrowtail B$ is provable. Symmetrically, $A \twoheadrightarrow (A \rightarrowtail B) \twoheadrightarrow B$ is provable while $A \twoheadrightarrow (A \rightarrowtail B) \rightarrowtail B$ is not provable. We also remark that there is a fundamental symmetry to the ordered fragment (as in Lambek calculus) which allows for all occurrences of $\twoheadrightarrow$ and $\rightarrowtail$ in a closed formula to be interchanged without affecting the provability of the formula.

Further the mobility modality, ¡, was designed to behave like ! with respect to the ordered hypotheses. Therefore we have that $¡A \twoheadrightarrow B \equiv ¡A \rightarrowtail B \equiv A \multimap B$ and that $¡¡A \equiv ¡A$. We also remark that ! subsumes ¡ so that $¡!A \equiv !¡A \equiv !A$. Therefore our logic has exactly two distinct modalities. Here $A \equiv B$ is the strongest possibly equivalence which requires ordered implications in both directions.

Our sequent system combines the ideas of a multi-zone presentation due to Andreoli [3] with implicit structural rules to permit a proof of cut elimination by structural induction as in [18].

Theorem 4.1 (Admissibility of Cut)

(i) $\mathbf{Cut}_\Omega$: *If* $\Gamma; \Delta_C; \Omega_C \;\longrightarrow\; C$ *and* $\Gamma; \Delta; \Omega_L C \Omega_R \;\longrightarrow\; A$ *then* $\Gamma; \Delta_C \bowtie \Delta; \Omega_L \Omega_C \Omega_R \;\longrightarrow\; A$.

(ii) $\mathbf{Cut}_\Delta$: *If* $\Gamma; \Delta_C; \cdot \;\longrightarrow\; C$ *and* $\Gamma; \Delta_L C \Delta_R; \Omega \;\longrightarrow\; A$ *then* $\Gamma; \Delta_L \bowtie \Delta_C \bowtie \Delta_R; \Omega \;\longrightarrow\; A$.

(iii) $\mathbf{Cut}_\Gamma$: *If* $\Gamma; \cdot; \cdot \longrightarrow C$ *and* $\Gamma_L C \Gamma_R; \Delta; \Omega \longrightarrow A$ *then* $\Gamma_L \Gamma \Gamma_R; \Delta; \Omega \longrightarrow A$.

Proof. By induction on the structure of the cut formula, the type of cut where $\mathbf{Cut}_\Gamma > \mathbf{Cut}_\Delta > \mathbf{Cut}_\Omega$, and the derivations of the premises. Therefore we may apply the induction hypothesis in the following cases: 1) the cut formula gets smaller; 2) the same cut formula but we move from $\mathbf{Cut}_\Gamma$ to $\mathbf{Cut}_\Delta$ or $\mathbf{Cut}_\Omega$; 3) the same cut formula but we move from $\mathbf{Cut}_\Delta$ to $\mathbf{Cut}_\Omega$; 4) the cut formula and type of cut stay the same but one of the derivations of the induction hypothesis gets smaller.

There are 4 basic cases to consider: **init** cases where one of the premises is an **init** rule, essential cases where the principal formula of both premises is cut, commutative cases where the cut formula is a side formula on the first or second premise. Note that these cases are not mutually exclusive. □

This lets us define a second sequent system with cut which is equivalent to the previous system. We write $\Gamma; \Delta; \Omega \xrightarrow{+} A$ to denote a sequent derivation of A which may contain the three types of cut in addition to all of the previous sequent rules. Then cut elimination follows directly.

Theorem 4.2 (Cut Elimination) *If* $\Gamma; \Delta; \Omega \xrightarrow{+} A$ *then* $\Gamma; \Delta; \Omega \longrightarrow A$.

Proof. By structural induction on the given derivation. In the case of a cut we appeal to the induction hypothesis on both premises and then to admissibility of cut on the resulting cut-free derivations. □

We now show a sample derivation which sketches how INCLL can be used for natural language parsing. Suppose $\Gamma = [\mathtt{np} \twoheadrightarrow \mathtt{vp} \twoheadrightarrow \mathtt{snt}, \mathtt{tv} \twoheadrightarrow \mathtt{np} \twoheadrightarrow \mathtt{vp}, \mathtt{loves} \twoheadrightarrow \mathtt{tv}, \mathtt{mary} \twoheadrightarrow \mathtt{np}, \mathtt{bob} \twoheadrightarrow \mathtt{np}]$ where all the words and grammatical abbreviations are atomic formulas. We may think of the formulas in Γ as a grammar for simple English sentences. The phrase to be parsed with the grammar is in the ordered context. The succedent contains the grammatical pattern with which we are trying to classify the input. Thus to parse the sentence: `mary loves bob`, we would prove: $\Gamma; \cdot; \mathtt{mary\ loves\ bob} \Longrightarrow \mathtt{snt}$.

$$\dfrac{\dfrac{\dfrac{\Theta \qquad \dfrac{}{\Gamma;\cdot;\mathtt{bob} \Longrightarrow \mathtt{bob}}\,\textbf{init}}{\Gamma;\cdot;\mathtt{np\ \ tv}\ (\mathtt{bob} \twoheadrightarrow \mathtt{np})\ \mathtt{bob} \Longrightarrow \mathtt{snt}}\,{\twoheadrightarrow_L} \qquad \dfrac{}{\Gamma;\cdot;\mathtt{loves} \Longrightarrow \mathtt{loves}}\,\textbf{init}}{\Gamma;\cdot;\mathtt{np}\ (\mathtt{loves} \twoheadrightarrow \mathtt{tv})\ \mathtt{loves}\ (\mathtt{bob} \twoheadrightarrow \mathtt{np})\ \mathtt{bob} \Longrightarrow \mathtt{snt}}\,{\twoheadrightarrow_L} \qquad \dfrac{}{\Gamma;\cdot;\mathtt{mary} \Longrightarrow \mathtt{mary}}\,\textbf{init}}{\dfrac{\Gamma;\cdot;(\mathtt{mary} \twoheadrightarrow \mathtt{np})\ \mathtt{mary}\ (\mathtt{loves} \twoheadrightarrow \mathtt{tv})\ \mathtt{loves}\ (\mathtt{bob} \twoheadrightarrow \mathtt{np})\ \mathtt{bob} \Longrightarrow \mathtt{snt}}{\Gamma;\cdot;\mathtt{mary\ \ loves\ \ bob} \Longrightarrow \mathtt{snt}}\,\textbf{copy} * \mathbf{3}}\,{\twoheadrightarrow_L}$$

where $\Theta =$

$$\dfrac{\dfrac{\dfrac{\Psi \qquad \dfrac{}{\Gamma;\cdot;\mathtt{np} \Longrightarrow \mathtt{np}}\,\textbf{init}}{\Gamma;\cdot;(\boldsymbol{\mathtt{np}} \twoheadrightarrow \mathtt{vp} \twoheadrightarrow \mathtt{snt})\ \mathtt{np}\ (\mathtt{np} \twoheadrightarrow \mathtt{vp})\ \mathtt{np} \Longrightarrow \mathtt{snt}}\,{\twoheadrightarrow_L} \qquad \dfrac{}{\Gamma;\cdot;\mathtt{tv} \Longrightarrow \mathtt{tv}}\,\textbf{init}}{\Gamma;\cdot;(\mathtt{np} \twoheadrightarrow \mathtt{vp} \twoheadrightarrow \mathtt{snt})\ \mathtt{np}\ (\mathtt{tv} \twoheadrightarrow \mathtt{np} \twoheadrightarrow \mathtt{vp})\ \mathtt{tv\ \ np} \Longrightarrow \mathtt{snt}}\,{\twoheadrightarrow_L}}{\Gamma;\cdot;\mathtt{np\ \ tv\ \ np} \Longrightarrow \mathtt{snt}}\,\textbf{copy} * \mathbf{2}$$

and $\Psi =$

$$\dfrac{\dfrac{\dfrac{}{\Gamma;\cdot;\mathtt{snt} \Longrightarrow \mathtt{snt}}\,\textbf{init} \qquad \dfrac{}{\Gamma;\cdot;\mathtt{vp} \Longrightarrow \mathtt{vp}}\,\textbf{init}}{\Gamma;\cdot;(\mathtt{vp} \twoheadrightarrow \mathtt{snt})\ \mathtt{vp} \Longrightarrow \mathtt{snt}}\,{\twoheadrightarrow_L} \qquad \dfrac{}{\Gamma;\cdot;\mathtt{np} \Longrightarrow \mathtt{np}}\,\textbf{init}}{\Gamma;\cdot;(\mathtt{np} \twoheadrightarrow \mathtt{vp} \twoheadrightarrow \mathtt{snt})\ \mathtt{np\ \ vp} \Longrightarrow \mathtt{snt}}\,{\twoheadrightarrow_L}$$

Note that this is not the only way to derive the end-sequent. For instance, we could have moved all instances of **copy** and **place** to the beginning of the derivation; or we could have applied $\twoheadrightarrow_L$ to the formulas in a different order.

5 Correspondences

In this section we will show some correspondences between the systems of natural deduction and sequent calculus. Our methods extend [10] to cover INCLL. Our approach differs from [8] and [13] in that we are not developing a sequent calculus which is in bijective correspondence with natural deductions. Instead, we have introduced a refined analysis of natural deduction by inductively defining normal derivations. These can now be related to the standard formulations of sequent calculus.

We first show how the cut-free sequent system corresponds to the normal natural deduction system. We have already remarked that normal natural deductions are those where the top-down use of elimination rules meets the bottom-up use of introduction rules in the middle. In a sequent system we reason entirely bottom up: the top-down uses of elimination rules are turned around and become bottom-up uses of the left rules. The right rules correspond directly to the introduction rules. They meet, not in the middle, but at the initial sequents.

Theorem 5.1 *If* $\Gamma; \Delta; \Omega \longrightarrow A$ *then* $\Gamma; \Delta; \Omega \vdash A \uparrow$.

Proof. By structural induction on the given derivation. **init** rules are mapped to instances of **coerce**; **place**, **copy** rules are mapped to instances of the substitution principles; right rules are mapped to introduction rules; and left rules are mapped to elimination rules using the substitution principles when necessary. $\Box$

Note that the resulting derivation is *normal*, despite the use of the substitution principles, since we use it in the form of Lemma 3.1. In the opposite direction we first need to generalize the induction hypothesis to make the proper statement about the atomic deduction—otherwise our induction would break down at the first coercion.

Theorem 5.2

(i) *If* $\Gamma; \Delta; \Omega \vdash A \uparrow$ *then* $\Gamma; \Delta; \Omega \longrightarrow A$.

(ii) *If* $\Gamma; \Delta; \Omega \vdash A \downarrow$ *then for any* C, Δ_C, Ω_L *and* Ω_R, $\Gamma; \Delta_C; \Omega_L A \Omega_R \longrightarrow C$ *implies* $\Gamma; \Delta_C \bowtie \Delta; \Omega_L \Omega \Omega_R \longrightarrow C$.

Proof. By structural induction on the given derivations. Instances of **coerce** translate to uses of the **init** rule from the result of the induction hypothesis. Introduction rules are mapped to right rules. Elimination rules are mapped to sequent derivations constructed from the corresponding left rule and the result of an appeal to the induction hypothesis. $\Box$

Our proofs above are constructive and inherently contain a method for translation between sequent derivations in INCLL and natural deductions. This translation could be written out concisely on proof terms (similar to [18]), but this is beyond the scope of this summary.

Clearly, the correspondence is very close, but it is not a bijection, because the order in which left rules are applied in a sequent derivation may be irrelevant to the resulting natural deduction. If one wants to establish a bijection, one has to further restrict the sequent rules. This has been investigated by Herbelin [13] for intuitionistic logic.

We now show that this correspondence extends to arbitrary natural deductions (using the $\vdash^+$ judgments) and arbitrary sequents (using the $\xrightarrow{+}$ sequents). Specifically, coercing a normal derivation into an atomic derivation will correspond to using cut in the sequent calculus.

Theorem 5.3 *If* $\Gamma; \Delta; \Omega \xrightarrow{+} A$ *then* $\Gamma; \Delta; \Omega \vdash^+ A \uparrow$

Proof. By induction on structure of the given derivation. The proof is exactly the same as the proof of Theorem 5.1 with three additional cases. The cut rules are translated into a **lemma** rule followed by an appeal to the substitution principles. □

Theorem 5.4

(i) *If* $\Gamma; \Delta; \Omega \vdash^+ A \uparrow$ *then* $\Gamma; \Delta; \Omega \xrightarrow{+} A$.

(ii) *If* $\Gamma; \Delta; \Omega \vdash^+ A \downarrow$ *then for any* C, Δ_C, Ω_L, *and* Ω_R, $\Gamma; \Delta_C; \Omega_L A \Omega_R \xrightarrow{+} C$ *implies* $\Gamma; \Delta_C \bowtie \Delta; \Omega_L \Omega \Omega_R \xrightarrow{+} C$.

Proof. By induction on structure of the given derivations. The proof is exactly the same as the proof of Theorem 5.2 with one additional case: from the **lemma** coercion we construct a use of the $\mathbf{Cut}_\Omega$ rule. □

These observations also give a syntactic proof of normalization of the natural deduction system.

Theorem 5.5 (Normalization) $\Gamma; \Delta; \Omega \vdash A$ *iff* $\Gamma; \Delta; \Omega \vdash A \uparrow$.

Proof. Given $\Gamma; \Delta; \Omega \vdash A$, we know $\Gamma; \Delta; \Omega \vdash^+ A \uparrow$ from Theorem 3.3. Then $\Gamma; \Delta; \Omega \xrightarrow{+} A$ from Theorem 5.4. Then $\Gamma; \Delta; \Omega \longrightarrow A$ from Theorem 4.2 (Cut Elimination). Then $\Gamma; \Delta; \Omega \vdash A \uparrow$ from Theorem 5.1. The other direction is the contents of Theorem 3.2. □

6 Related Work

INCLL was fundamentally designed from intuitionistic principles of natural deduction with the judgment structure determining the rest of the logic. This has led to INCLL being fundamentally different from some other presentations of non-commutative linear logic (classical and intuitionistic). Indeed one could argue that INCLL appears on the surface to not express anything about non-commutativity so much as non-exchangeability. The notion of commutativity usually concerns binary operators rather than individual properties. However, as we have previously pointed out, INCLL captures non-commutativity at the level of hypotheses rather than connectives.

Both Ruet's intuitionistic non-commutative linear logic [21] and Ruet and Abrusci's classical non-commutative linear logic [2] (or cyclic linear logic) have two context constructors, one of which is commutative. Additionally neither of these systems directly admits the concept of a mobile hypothesis (although the latter system may be able to capture it by not constraining a hypothesis at all in the order variety).

As a result, our system has no way to constrain the ability of the unordered hypotheses to move around among the hypotheses: either the place of a hypothesis is fixed or completely arbitrary. In other words, there is no scoping of mobility in INCLL. While this feature can be easily captured in the systems with a commutative conjunction, INCLL's notion of mobility cannot (at least not directly). We believe that both concepts may be useful in a combined system and plan to investigate how to add a commutative multiplication conjunction to our system which is more restricted than $(¡A) \bullet (¡B)$. From the practical point of view, we have not yet found the need to go beyond the system described here.

7 Conclusions and Future Work

We have presented a sequent calculus which closely corresponds to the system of natural deduction for INCLL proposed by the authors in [20]. The sequent calculus presentation gives better access to the mechanics of proof search. The most important application of refined proof search we have in mind is logic programming.

Based on the sequent system introduced here we have begun work on a logic programming language which extends Lolli [15] with the two ordered implications and the ordered context. We have proved completeness of uniform derivations for a fragment of INCLL and designed an efficient proof search mechanism (based on the input/output model of Lolli [6]) to remove the non-deterministic context splitting in the sequents. With a prototype interpreter for this language we are currently exploring logic programming with non-commutative hypotheses. Preliminary results in natural language parsing and non-deterministic algorithm specification have been promising and may be found in [19].

Other planned work with INCLL includes exploring ordered functional programming and a logical framework based on INCLL. Towards the former we have begun work on an operational semantics for the term calculus associated with INCLL. Towards the latter we have already shown that a stackability of CPS terms can be formalized in a logical framework based on INCLL.

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