Greedy Bidirectional Polymorphism

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Why Bidirectional Typechecking?

- Type inference (without annotations) often undecidable
  - Intersection types
  - Dependent types
  - Impredicative polymorphism (instantiating $\forall$ with $\forall$)

Bidirectional Typechecking

$\Gamma \vdash e \Uparrow A$	e synthesizes type A
$\Gamma \vdash e \Downarrow A$	e checks against type A

- Many typing problems now decidable (with annotations)
- Can report type errors earlier

Why Polymorphism?

- Am I in the right room?

Why Polymorphism?

- Am I in the right room?
- Embarrassingly absent from my dissertation
- Adding polymorphism from scratch
  - First-class polymorphism accidental!

Annotations

- In its **simplest formulation**, bidirectional typing needs annotations on **redexes**: mostly, **let fun** and $(\lambda x.\, e_1)e_2$

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- Annotations are good documentation.
- Thinking about the types is good for you.

Annotations

- In its **simplest formulation**, bidirectional typing needs annotations on **redexes**: mostly, **let fun** and $(\lambda x.\, e_1)\, e_2$
- **Objection:** I don't want to!
- Annotations are good documentation.
- Thinking about the types is good for you.
- Just because the **type system** needs an annotation doesn't mean the user must write it.
  When inference is decidable, a tool could write it, keeping complexity out of the type system.

Outline

- Bidirectional simple typing
- Oracular polymorphic typing
- Greedy polymorphism with ordered contexts
- Completeness
- (Hints)
- (Intersections and unions)
- Future work
- Related work

Simply-Typed Language: $\mathbf{1}, \to$

$$
\begin{array}{rrcl}
\text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, x{:}A \\
\text{Types} & A, B, C & ::= & \mathbf{1} \mid A \to B \\
\\
\text{Terms} & e & ::= & x \mid () \mid \lambda x.\, e \mid e_1\, e_2 \mid (e : A) \\
\\
\text{Values} & v & ::= & x \mid () \mid \lambda x.\, e \mid (v : A)
\end{array}
$$

Bidirectional Typing

- Two judgment forms

$$\text{Synthesis} \quad \Gamma^{+} \vdash e^{+} \Uparrow A^{-}$$

$$\text{Checking} \quad \Gamma^{+} \vdash e^{+} \Downarrow A^{+}$$

$+$ input

$-$ output

Bidirectional Typing

- Using assumption

$$\frac{\Gamma(x) = A}{\Gamma \vdash x : A} \text{ var}$$

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$$\frac{\Gamma(x) = A}{\Gamma \vdash x \Uparrow A}\ \mathsf{var}$$

Bidirectional Typing: $\rightarrow$

- Introduction rules **check**,
  elimination rules **synthesize**

$$\frac{\Gamma, x{:}A \vdash e \ : \ B}{\Gamma \vdash \lambda x.\, e \ : \ A \rightarrow B}\ {\rightarrow}\mathrm{I}$$

$$\frac{\Gamma \vdash e_1 \ : \ A \rightarrow B \qquad \Gamma \vdash e_2 \ : \ A}{\Gamma \vdash e_1 e_2 \ : \ B}\ {\rightarrow}\mathrm{E}$$

- e_1 usually a variable or another application

Bidirectional Typing: $\rightarrow$

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Bidirectional Typing: $\rightarrow$

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$$\frac{\Gamma \vdash e_1 \Uparrow A \rightarrow B \qquad \Gamma \vdash e_2 \Downarrow A}{\Gamma \vdash e_1 e_2 \Uparrow B} \rightarrow\mathrm{E}$$

- e_1 usually a variable or another application

Change of Direction

- In syntax-directed rules,
  intros check, elims synthesize
- Checking a synthesizing form:

$$\frac{\Gamma \vdash e \Uparrow A' \qquad A' = A}{\Gamma \vdash e \Downarrow A}\,\textsf{sub}$$

  (Trivial $=$-"subsumption", for now!)
- Synthesizing a checking form: annotation

$$\frac{\Gamma \vdash e \Downarrow A}{\Gamma \vdash (e : A) \Uparrow A}\,\textsf{anno}$$

Synthesizing and Checking

- Synthesizing terms: $x \quad e_1\, e_2$ (elim)
- Checked terms: $\lambda x.\, e \quad (\,)$ (intro)

- Synthesizing positions: $\boxed{e_1}\, e_2$
- Checked positions: $e_1\, \boxed{e_2} \qquad \lambda x.\, \boxed{e} \qquad (\,\boxed{e} : A\,)$

- Annotations only where a checked term is used in a synthesizing position, i.e. on the function part of a redex

$$\boxed{(\lambda x.\, e_1)}\, e_2$$

and on function declarations.

Simply-Typed Bidirectional Rules

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \Uparrow A}\,\textsf{var}$$

$$\frac{\Gamma \vdash e \Uparrow A' \qquad A' = A}{\Gamma \vdash e \Downarrow A}\,\textsf{sub} \qquad \frac{\Gamma \vdash e \Downarrow A}{\Gamma \vdash (e : A) \Uparrow A}\,\textsf{anno}$$

$$\frac{}{\Gamma \vdash () \Downarrow \mathbf{1}}\,\mathbf{1}\mathrm{I}$$

$$\frac{\Gamma, x{:}A \vdash e \Downarrow B}{\Gamma \vdash \lambda x.\, e \Downarrow A \to B}\,{\to}\mathrm{I} \qquad \frac{\Gamma \vdash e_1 \Uparrow A \to B \qquad \Gamma \vdash e_2 \Downarrow A}{\Gamma \vdash e_1\, e_2 \Uparrow B}\,{\to}\mathrm{E}$$

Polymorphic: $\mathbf{1}, \to, \forall$

$$
\begin{array}{rrcl}
\text{Type Variables} & \boxed{\alpha, \beta, \gamma} & & \\
\text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, x{:}A \mid \boxed{\alpha} \\
\text{Types} & A, B, C & ::= & \mathbf{1} \mid A \to B \mid \boxed{\alpha} \mid \boxed{\forall \alpha.\, A} \\
\\
\text{Terms} & e & ::= & x \mid () \mid \lambda x.\, e \mid e_1\, e_2 \mid (e : A)
\end{array}
$$

Polymorphic: $\mathbf{1}, \rightarrow, \forall$

Type Variables	$\boxed{\alpha, \beta, \gamma}$		
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A \mid \boxed{\alpha}$
Types	A, B, C	$::=$	$\mathbf{1} \mid A \rightarrow B \mid \boxed{\alpha} \mid \boxed{\forall \alpha.\, A}$
Terms	e	$::=$	$x \mid () \mid \lambda x.\, e \mid e_1\, e_2 \mid (e : A)$

$$\frac{\Gamma, \alpha \vdash v \Downarrow A}{\Gamma \vdash v \Downarrow \forall \alpha.\, A}\;\forall\mathrm{I} \qquad \frac{\Gamma \vdash e \Uparrow \forall \alpha.\, A \qquad \Gamma \vdash A' \;\mathsf{wf}}{\Gamma \vdash e \Uparrow [A'/\alpha]A}\;\forall\mathrm{E}$$

But we don't know A'... and we're not magicians.

Subtyping for $\forall$

- "$A \leqq B$" read "A is at least as polymorphic as B"

$$\frac{}{\Gamma \vdash \mathbf{1} \leqq \mathbf{1}}\ \mathbf{1}{\leq} \qquad \frac{\Gamma \vdash B_1 \leqq A_1 \qquad \Gamma \vdash A_2 \leqq B_2}{\Gamma \vdash A_1 \rightarrow A_2 \leqq B_1 \rightarrow B_2} \rightarrow\!\leq$$

$$\frac{}{\Gamma \vdash \alpha \leqq \alpha}\ \alpha\mathrm{Refl} \quad \frac{\Gamma \vdash [A'/\alpha]A \leqq B}{\Gamma \vdash \forall \alpha.\, A \leqq B}\ \forall\mathrm{L}{\leq} \quad \frac{\Gamma, \beta \vdash A \leqq B}{\Gamma \vdash A \leqq \forall \beta.\, B}\ \forall\mathrm{R}{\leq}$$

Subtyping for $\forall$

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$$\frac{}{\Gamma \vdash \mathbf{1} \leqq \mathbf{1}}\;\mathbf{1}{\leqq} \qquad \frac{\Gamma \vdash B_1 \leqq A_1 \qquad \Gamma \vdash A_2 \leqq B_2}{\Gamma \vdash A_1 \to A_2 \leqq B_1 \to B_2}\;{\to}{\leqq}$$

$$\frac{}{\Gamma \vdash \alpha \leqq \alpha}\;\alpha\mathrm{Refl} \qquad \frac{\Gamma \vdash [A'/\alpha]A \leqq B}{\Gamma \vdash \forall\alpha.\,A \leqq B}\;\forall\mathrm{L}{\leqq} \qquad \frac{\Gamma, \beta \vdash A \leqq B}{\Gamma \vdash A \leqq \forall\beta.\,B}\;\forall\mathrm{R}{\leqq}$$

- Change sub:

$$\frac{\Gamma \vdash e \Uparrow A \qquad \Gamma \vdash A \leqq B}{\Gamma \vdash e \Downarrow B}\;\mathsf{sub}$$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\,(\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \to \alpha\ \mathsf{list} \qquad \Gamma \vdash \mathsf{int}\ \mathsf{wf}}
          {\Gamma \vdash \mathit{filter} \Uparrow (\mathsf{int} \to \mathsf{bool}) \to \mathsf{int}\ \mathsf{list} \to \mathsf{int}\ \mathsf{list}}\ \forall E
    \qquad
    \dfrac{\Gamma \vdash \mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool} \qquad \Gamma \vdash \mathsf{int} \to \mathsf{bool} \leq \mathsf{int} \to \mathsf{bool}}
          {\Gamma \vdash \mathsf{f} \Downarrow \mathsf{int} \to \mathsf{bool}}\ \mathsf{sub}
  }
  {\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \mathsf{int}\ \mathsf{list} \to \mathsf{int}\ \mathsf{list}}\ {\to}E
  \qquad
  \Gamma \vdash xs \Downarrow \mathsf{int}\ \mathsf{list}
}
{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs \Uparrow \mathsf{int}\ \mathsf{list}}\ {\to}E
$$

Greed

- The "greedy" method [Cardelli '93]:
  Generate existential variable $\widehat{\alpha}$, something like

$$\frac{\Gamma \vdash e \Uparrow \forall \alpha.\, A}{\Gamma \vdash e \Uparrow [\widehat{\alpha}/\alpha]A} \qquad \frac{\Gamma \vdash [\widehat{\alpha}/\alpha]A \leq B}{\Gamma \vdash \forall \alpha.\, A \leq B}$$

  then take the first solution.

Ordered Contexts

- Track existential variables explicitly

$$
\begin{array}{rcll}
\Gamma & ::= & \Gamma, x{:}A \mid \Gamma, \alpha & \\
 & & \mid \Gamma, \widehat{\alpha} & \text{Unsolved existential type var} \\
 & & \mid \Gamma, \widehat{\alpha}{=}A & \text{Solved existential type var} \\
 & & \mid \Gamma, \blacktriangleleft_{\widehat{\alpha}} & \text{Scope marker}
\end{array}
$$

- Thread Γ through:

$$
\frac{\Gamma_1 \vdash e \Uparrow \forall\alpha.\, A \dashv \Gamma_2}{\Gamma_1 \vdash e \Uparrow [\widehat{\alpha}/\alpha]A \dashv \Gamma_2, \widehat{\alpha}}\,\forall\mathrm{E}\widehat{\alpha}
\qquad
\frac{\Gamma_1, \blacktriangleleft_{\widehat{\alpha}}, \widehat{\alpha} \vdash [\widehat{\alpha}/\alpha]A \leqq B \dashv \Gamma_2, \blacktriangleleft_{\widehat{\alpha}}, \Gamma_Z}{\Gamma_1 \vdash \forall\alpha.\, A \leqq B \dashv \Gamma_2}\,\forall\mathrm{L}\widehat{\alpha}\leqq
$$

$$
\frac{\Gamma_1 \vdash e_1 \Uparrow A \rightarrow B \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 \Downarrow A \dashv \Gamma_3}{\Gamma_1 \vdash e_1\, e_2 \Uparrow B \dashv \Gamma_3}\,\rightarrow\!\mathrm{E}
$$

- Greedily instantiate existential variables, e.g.

$$\frac{\Gamma_1 \vdash B \ \mathsf{wf}}{\Gamma_1, \widehat{\alpha}, \Gamma_2 \vdash \widehat{\alpha} \leqq B \dashv \Gamma_1, \widehat{\alpha}{=}B, \Gamma_2}$$

$$\frac{\Gamma_1 \vdash A \ \mathsf{wf}}{\Gamma_1, \widehat{\beta}, \Gamma_2 \vdash A \leqq \widehat{\beta} \dashv \Gamma_1, \widehat{\beta}{=}A, \Gamma_2}$$

Synthesis	$\Gamma_1^{+} \vdash e^{+} \Uparrow A^{-} \dashv \Gamma_2^{-}$	
Checking	$\Gamma_1^{+} \vdash e^{+} \Downarrow A^{+} \dashv \Gamma_2^{-}$	

$+$ input $\quad$ $-$ output

Input and output contexts the same modulo $\widehat{\alpha}$ and $\widehat{\alpha}{=}A$:

Input	Output	
$x{:}\mathbf{1}$	$x{:}\mathbf{1}, \widehat{\alpha}$	✓
$\widehat{\alpha}, x{:}\widehat{\alpha}, \widehat{\beta}$	$\widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, x{:}\widehat{\alpha}, \widehat{\beta}{=}\widehat{\alpha}$	✓
$\widehat{\alpha}, x{:}\widehat{\alpha}, \widehat{\beta}$	$\widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, x{:}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \widehat{\beta}{=}\widehat{\alpha}, y{:}\widehat{\beta}$	✗

We don't replace $\widehat{\alpha}$ in declarations.

Example

$$(\alpha{\to}\text{bool})$$

$$\to\ \alpha\ \text{list}$$

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\ \to\ \alpha\ \text{list} \dashv\ \Gamma}{}\ \boxed{\forall E\widehat{\alpha}}$$

$$\Gamma \vdash \mathit{filter} \Uparrow$$

$$\vdots$$

$$\dfrac{\Gamma \vdash \text{int wf}}{\widehat{\alpha} \leqq \text{int}}\ \boxed{\widehat{\alpha}{=}\text{L}{\leqq}} \qquad \begin{array}{l}\text{bool}\\ \leqq \text{bool}\end{array}$$

$$\dfrac{\dfrac{\Gamma \vdash \text{int wf}}{\widehat{\alpha} \leqq \text{int}}\ \boxed{\widehat{\alpha}{=}\text{L}{\leqq}} \qquad \begin{array}{l}\text{bool}\\ \leqq \text{bool}\end{array}}{\begin{array}{l}\text{int}{\to}\text{bool}\\ \leqq \widehat{\alpha}{\to}\text{bool}\end{array}}\ {\to}{\leqq}$$

$$\dfrac{\mathrm{f} \Uparrow \text{int}{\to}\text{bool} \qquad \begin{array}{l}\text{int}{\to}\text{bool}\\ \leqq \widehat{\alpha}{\to}\text{bool}\end{array}}{\mathrm{f} \Downarrow}\ \text{sub}$$

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \quad \vdots \quad \mathrm{f} \Downarrow}{\Gamma \vdash \mathit{filter}\ \mathrm{f}}\ {\to}\text{E}$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha} \to \mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\hat{\alpha}}}\ \boxed{\forall\mathrm{E}\hat{\alpha}}$$

$$\vdots$$

$$\dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\hat{\alpha} \leqq \mathsf{int}}\ \boxed{\hat{\alpha}^{=}\mathrm{L}\leqq} \qquad \mathsf{bool} \leqq \mathsf{bool}}{\mathsf{int} \to \mathsf{bool} \leqq \hat{\alpha} \to \mathsf{bool}}\ {\to}{\leqq}$$

$$\dfrac{\dfrac{\mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool} \qquad \mathsf{int} \to \mathsf{bool} \leqq \hat{\alpha} \to \mathsf{bool}}{\mathsf{f} \Downarrow}\ \mathsf{sub}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\widehat{\alpha} \to \mathsf{bool}) \to \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\widehat{\alpha}}}\ \boxed{\forall\mathrm{E}\widehat{\alpha}}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\widehat{\alpha} \leqq \mathsf{int}}\ \boxed{\widehat{\alpha}^{=}\mathrm{L}{\leqq}} \qquad \mathsf{bool} \leqq \mathsf{bool}}{\mathsf{int} \to \mathsf{bool} \leqq \widehat{\alpha} \to \mathsf{bool}}\ {\to}{\leqq} \qquad \mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool}}{\Gamma, \widehat{\alpha} \vdash \mathsf{f} \Downarrow \widehat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}$$

$$\dfrac{\vdots \qquad \Gamma, \widehat{\alpha} \vdash \mathsf{f} \Downarrow \widehat{\alpha} \to \mathsf{bool}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall \alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha} \to \mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\hat{\alpha}}}\ \boxed{\forall\mathrm{E}\hat{\alpha}}$$

$$\dfrac{\begin{array}{c}\vdots \\ \\ \Gamma, \hat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool} \qquad \dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\hat{\alpha} \leqq \mathsf{int}}\ \boxed{\hat{\alpha}{=}\mathrm{L}{\leqq}} \qquad \mathsf{bool} \leqq \mathsf{bool}}{\mathsf{int} \to \mathsf{bool} \leqq \hat{\alpha} \to \mathsf{bool}}\ {\to}{\leqq} \end{array}}{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}$$

$$\dfrac{\vdots \qquad \Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

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$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall \alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha} \to \mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\hat{\alpha}}}\ \boxed{\forall \mathrm{E}\hat{\alpha}}$$

$$\vdots$$

$$\dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\hat{\alpha} \leqq \mathsf{int}}\ \boxed{\hat{\alpha}^{=}\mathrm{L}\leqq} \qquad \mathsf{bool} \leqq \mathsf{bool}}{\mathsf{int} \to \mathsf{bool} \leqq \hat{\alpha} \to \mathsf{bool}}\ {\to}{\leqq}$$

$$\dfrac{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool} \dashv \Gamma, \hat{\alpha} \qquad \mathsf{int} \to \mathsf{bool} \leqq \hat{\alpha} \to \mathsf{bool}}{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}$$

$$\dfrac{\vdots \qquad \Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\,(\alpha\to\mathsf{bool}) \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\widehat{\alpha}\to\mathsf{bool}) \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}}\ \forall E\widehat{\alpha}$$

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash \mathit{filter} \Uparrow (\widehat{\alpha}\to\mathsf{bool}) \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}\end{matrix} \qquad \dfrac{\Gamma, \widehat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int}\to\mathsf{bool} \dashv \Gamma, \widehat{\alpha} \qquad \dfrac{\dfrac{\Gamma \vdash \mathsf{int\ wf}}{\widehat{\alpha} \leqq \mathsf{int}}\ \widehat{\alpha}^{=}\mathrm{L}{\leqq} \qquad \mathsf{bool} \leqq \mathsf{bool}}{\Gamma, \widehat{\alpha} \vdash \mathsf{int}\to\mathsf{bool} \leqq \widehat{\alpha}\to\mathsf{bool}}\ {\to}{\leqq}}{\Gamma, \widehat{\alpha} \vdash \mathsf{f} \Downarrow \widehat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$
\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall \alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha} \to \mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\hat{\alpha}}}\ \boxed{\forall\mathrm{E}\hat{\alpha}}
$$

$$
\vdots
$$

$$
\dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\Gamma, \hat{\alpha} \vdash \hat{\alpha} \leqq \mathsf{int}}\ \boxed{\hat{\alpha}^{=}\mathrm{L}{\leqq}} \qquad \mathsf{bool} \leqq \mathsf{bool}}{\mathsf{int} \to \mathsf{bool}}\ {\to}{\leqq}
$$

$$
\dfrac{\Gamma, \hat{\alpha} \vdash f \Uparrow \mathsf{int} \to \mathsf{bool} \dashv \Gamma, \hat{\alpha} \qquad \Gamma, \hat{\alpha} \vdash\ \leqq \hat{\alpha} \to \mathsf{bool}}{\Gamma, \hat{\alpha} \vdash f \Downarrow \hat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}
$$

$$
\dfrac{\qquad}{\Gamma \vdash \mathit{filter}\ f}\ {\to}\mathrm{E}
$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\, (\alpha{\to}\mathsf{bool}) \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha}{\to}\mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\hat{\alpha}}}\ \boxed{\forall\mathrm{E}\hat{\alpha}}$$

$$\vdots$$

$$\dfrac{\dfrac{\Gamma \vdash \mathsf{int\ wf}}{\Gamma, \hat{\alpha} \vdash \hat{\alpha} \leqq \mathsf{int} \dashv \Gamma, \boxed{\hat{\alpha}{=}\mathsf{int}}}\ \boxed{\hat{\alpha}^{=}\mathrm{L}{\leqq}} \qquad \mathsf{bool} \leqq \mathsf{bool}}{\ }\ {\to}{\leqq}$$

$$\dfrac{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int}{\to}\mathsf{bool} \dashv \Gamma, \hat{\alpha} \qquad \Gamma, \hat{\alpha} \vdash \mathsf{int}{\to}\mathsf{bool} \leqq \hat{\alpha}{\to}\mathsf{bool}}{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}$$

$$\dfrac{\quad\vdots\quad}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\,(\alpha\to\mathsf{bool})\to\alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat\alpha\to\mathsf{bool})\to\hat\alpha\ \mathsf{list} \dashv \Gamma, \boxed{\hat\alpha}}\ \boxed{\forall\mathrm{E}\hat\alpha}$$

$$\vdots$$

$$\dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\Gamma, \hat\alpha \vdash \hat\alpha \leqq \mathsf{int} \dashv \Gamma, \boxed{\hat\alpha{=}\mathsf{int}}}\ \boxed{\hat\alpha^{=}\mathrm{L}{\leqq}} \qquad \Gamma, \hat\alpha{=}\mathsf{int} \vdash \mathsf{bool} \leqq \mathsf{bool}}{\Gamma, \hat\alpha \vdash \mathsf{int}\to\mathsf{bool} \leqq \hat\alpha\to\mathsf{bool}}\ {\to}{\leqq}$$

$$\dfrac{\Gamma, \hat\alpha \vdash \mathsf{f} \Uparrow \mathsf{int}\to\mathsf{bool} \dashv \Gamma, \hat\alpha \qquad \Gamma, \hat\alpha \vdash \mathsf{int}\to\mathsf{bool} \leqq \hat\alpha\to\mathsf{bool}}{\Gamma, \hat\alpha \vdash \mathsf{f} \Downarrow \hat\alpha \to \mathsf{bool}}\ \mathsf{sub}$$

$$\dfrac{\vdots}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha} \to \mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}}\ \forall\mathrm{E}\hat{\alpha}$$

$$\vdots$$

$$\dfrac{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool} \dashv \Gamma, \hat{\alpha} \qquad \dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\Gamma, \hat{\alpha} \vdash \hat{\alpha} \leqq \mathsf{int} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}\ \hat{\alpha}{}^{=}\mathrm{L}{\leqq} \qquad \Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \mathsf{bool} \leqq \mathsf{bool} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}{\Gamma, \hat{\alpha} \vdash \mathsf{int} \to \mathsf{bool} \leqq \hat{\alpha} \to \mathsf{bool}}\ {\to}{\leqq}}{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}$$

$$\dfrac{\vdots}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$\dfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall\alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha} \to \mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\hat{\alpha}}}\ \boxed{\forall\mathrm{E}\hat{\alpha}}$$

$$\dfrac{\vdots \qquad \dfrac{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int}\to\mathsf{bool} \dashv \Gamma, \hat{\alpha} \qquad \dfrac{\dfrac{\Gamma \vdash \mathsf{int\ wf}}{\Gamma, \hat{\alpha} \vdash \hat{\alpha} \leqq \mathsf{int} \dashv \Gamma, \boxed{\hat{\alpha}{=}\mathsf{int}}}\ \boxed{\hat{\alpha}^{=}\mathrm{L}\leqq} \qquad \Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \mathsf{bool} \leqq \mathsf{bool} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}{\Gamma, \hat{\alpha} \vdash \hat{\alpha}\to\mathsf{bool} \leqq \mathsf{int}\to\mathsf{bool} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}\ {\to}{\leqq}}{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool}}\ \mathsf{sub}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}$$

Example

$$
\cfrac{
\cfrac{\Gamma \vdash \mathit{filter} \Uparrow \forall \alpha.\, (\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \mathit{filter} \Uparrow (\hat{\alpha} \to \mathsf{bool}) \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \boxed{\hat{\alpha}}}\ \boxed{\forall\mathrm{E}\hat{\alpha}}
\qquad \vdots \qquad
\cfrac{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool} \dashv \Gamma, \hat{\alpha}
\qquad
\cfrac{\cfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\Gamma, \hat{\alpha} \vdash \hat{\alpha} \leqq \mathsf{int} \dashv \Gamma, \boxed{\hat{\alpha} = \mathsf{int}}}\ \boxed{\hat{\alpha}{=}\mathrm{L}{\leqq}}
\qquad
\Gamma, \hat{\alpha} = \mathsf{int} \vdash \mathsf{bool} \leqq \mathsf{bool} \dashv \Gamma, \hat{\alpha} = \mathsf{int}}{\Gamma, \hat{\alpha} \vdash \mathsf{int} \to \mathsf{bool} \leqq \hat{\alpha} \to \mathsf{bool} \dashv \Gamma, \hat{\alpha} = \mathsf{int}}\ {\to}{\leqq}}
{\Gamma, \hat{\alpha} \vdash \mathsf{f} \Downarrow \hat{\alpha} \to \mathsf{bool} \dashv \Gamma, \hat{\alpha} = \mathsf{int}}\ \mathsf{sub}
}{\Gamma \vdash \mathit{filter}\ \mathsf{f}}\ {\to}\mathrm{E}
$$

Example

$$
\dfrac{\Gamma \vdash \textit{filter} \Uparrow \forall\alpha.\,(\alpha \to \mathsf{bool}) \to \alpha\ \mathsf{list} \dashv \Gamma}{\Gamma \vdash \textit{filter} \Uparrow (\widehat{\alpha} \to \mathsf{bool}) \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}}\ \forall\mathrm{E}\widehat{\alpha}
$$

$$
\dfrac{
\begin{array}{c}\vdots\\ \Gamma \vdash \textit{filter} \Uparrow (\widehat{\alpha} \to \mathsf{bool}) \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}\end{array}
\qquad
\dfrac{\Gamma, \widehat{\alpha} \vdash \mathsf{f} \Uparrow \mathsf{int} \to \mathsf{bool} \dashv \Gamma, \widehat{\alpha}
\qquad
\dfrac{\dfrac{\Gamma \vdash \mathsf{int}\ \mathsf{wf}}{\Gamma, \widehat{\alpha} \vdash \widehat{\alpha} \leqq \mathsf{int} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}\ \widehat{\alpha}^{=}\mathrm{L}{\leqq}
\qquad
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{bool} \leqq \mathsf{bool} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}{\Gamma, \widehat{\alpha} \vdash \mathsf{int} \to \mathsf{bool} \leqq \widehat{\alpha} \to \mathsf{bool} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}\ {\to}{\leqq}}{\Gamma, \widehat{\alpha} \vdash \mathsf{f} \Downarrow \widehat{\alpha} \to \mathsf{bool} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}\ \mathsf{sub}
}{\Gamma \vdash \textit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}\ {\to}\mathrm{E}
$$

Example, cont.

$$
\dfrac{
\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
\qquad
\dfrac{
xs \Uparrow \mathsf{int\ list}
\qquad
\dfrac{
\dfrac{
\dfrac{}{\mathsf{int} \leq}
}{
\mathsf{int} \leq \widehat{\alpha}
}\ \text{ExSubstR}{\leq}
}{
\mathsf{int\ list} \leq \widehat{\alpha}\ \mathsf{list}
}
}{
xs \Downarrow \widehat{\alpha}\ \mathsf{list}
}\ \text{sub}
}{
\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs
}\ {\to}\text{E}
$$

Example, cont.

$$\dfrac{\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \hat{\alpha}\ \mathsf{list} \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int} \qquad \dfrac{xs \Uparrow \mathsf{int\ list} \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{int} \leqq}}{\mathsf{int} \leqq \boxed{\hat{\alpha}}}\ \boxed{\mathrm{ExSubstR}{\leqq}}}{\mathsf{int\ list} \leqq \hat{\alpha}\ \mathsf{list}}}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \hat{\alpha}\ \mathsf{list}}\ \mathsf{sub}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs}\ {\to}\mathrm{E}$$

Example, cont.

$$
\dfrac{\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int} \qquad \dfrac{\begin{array}{c}\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \\ xs \Uparrow \mathsf{int\ list}\end{array} \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{int} \leqq}}{\mathsf{int} \leqq \boxed{\widehat{\alpha}}}\ \mathrm{ExSubstR}{\leqq}}{\mathsf{int\ list} \leqq \widehat{\alpha}\ \mathsf{list}}}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list}}\ \mathsf{sub}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs}\ {\to}\mathrm{E}
$$

Example, cont.

$$\dfrac{\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \hat{\alpha}\ \mathsf{list} \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int} \qquad \dfrac{\begin{array}{c}\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \\ xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}\end{array} \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{int} \leqq}\ \text{ExSubstR}\leqq}{\mathsf{int} \leqq \boxed{\hat{\alpha}}}}{\mathsf{int\ list} \leqq \hat{\alpha}\ \mathsf{list}}}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \hat{\alpha}\ \mathsf{list}}\ \mathsf{sub}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs}\ {\to}\mathrm{E}$$

Example, cont.

$$
\dfrac{
\begin{matrix}
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \\
\quad xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
\end{matrix}
\qquad
\dfrac{
\dfrac{
\dfrac{}{\mathsf{int} \leq}
}{
\mathsf{int} \leq \boxed{\widehat{\alpha}}
}\ \text{ExSubstR}{\leq}
}{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leq \widehat{\alpha}\ \mathsf{list}
}
}{
\Gamma \vdash \mathit{filter}\ \mathrm{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
\qquad
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list}
}\ \mathsf{sub}
$$

$$
\dfrac{\qquad}{\Gamma \vdash \mathit{filter}\ \mathrm{f}\ xs}\ {\to}\mathrm{E}
$$

Example, cont.

$$
\dfrac{
\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \rightarrow \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
\qquad
\dfrac{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
\qquad
\dfrac{
\dfrac{
\dfrac{}{\mathsf{int} \leqq}
}{
\Gamma, \boxed{\widehat{\alpha}{=}\mathsf{int}} \vdash \mathsf{int} \leqq \boxed{\widehat{\alpha}}
}\;\boxed{\mathrm{ExSubstR}{\leqq}}
}{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leqq \widehat{\alpha}\ \mathsf{list}
}
}{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list}
}\;\mathsf{sub}
}{
\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs
}\;{\rightarrow}\mathrm{E}
$$

Example, cont.

$$
\dfrac{
\dfrac{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \atop xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
\qquad
\dfrac{
\dfrac{
\dfrac{}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \mathsf{int}}
}{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \widehat{\alpha}
}\ \mathrm{ExSubstR}{\leqq}
}{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leqq \widehat{\alpha}\ \mathsf{list}
}
}{
\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list}
}\ \mathsf{sub}
\qquad
\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
}{
\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs
}\ {\to}\mathrm{E}
$$

Example, cont.

$$
\dfrac{
  \dfrac{
    \Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \mathsf{int} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}
      }{
        \Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \widehat{\alpha}
      }\,\text{ExSubstR}{\leqq}
    }{
      \Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leqq \widehat{\alpha}\ \mathsf{list}
    }
  }{
    \Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list}
  }\,\text{sub}
  \qquad
  \Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \rightarrow \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}
}{
  \Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs
}\,{\rightarrow}\mathrm{E}
$$

Example, cont.

$$\dfrac{\dfrac{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int} \qquad \dfrac{\dfrac{}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leq \boxed{\mathsf{int}} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}}{\Gamma, \boxed{\widehat{\alpha}{=}\mathsf{int}} \vdash \mathsf{int} \leq \boxed{\widehat{\alpha}} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}\,\boxed{\text{ExSubstR}\leq}}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leq \widehat{\alpha}\ \mathsf{list}}}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list}}\ \mathsf{sub}$$

$$\dfrac{\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int} \qquad \Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs}\ {\to}\mathrm{E}$$

Example, cont.

$$
\dfrac{
\dfrac{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}
\qquad
\dfrac{\dfrac{\dfrac{}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \boxed{\mathsf{int}} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}}{\Gamma, \boxed{\hat{\alpha}{=}\mathsf{int}} \vdash \mathsf{int} \leqq \boxed{\hat{\alpha}} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}\ \boxed{\mathrm{ExSubstR}{\leqq}}}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leqq \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}
}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \hat{\alpha}\ \mathsf{list}}\ \mathsf{sub}
\qquad
\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \hat{\alpha}\ \mathsf{list} \rightarrow \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}
}{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs}\ {\rightarrow}\mathrm{E}
$$

Example, cont.

$$\dfrac{\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \widehat{\alpha}\ \mathsf{list} \to \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int} \qquad \dfrac{\begin{array}{c}\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \\ xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}\end{array} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \mathsf{int} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \widehat{\alpha} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}\ \mathrm{ExSubstR}{\leqq}}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leqq \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}}{\Gamma, \widehat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \widehat{\alpha}\ \mathsf{list} \dashv \Gamma, \widehat{\alpha}{=}\mathsf{int}}\ \mathsf{sub}}{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs}\ {\to}\mathrm{E}$$

Example, cont.

$$
\dfrac{
\Gamma \vdash \mathit{filter}\ \mathsf{f} \Uparrow \hat{\alpha}\ \mathsf{list} \to \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}
\qquad
\dfrac{
\begin{array}{c}\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \\ xs \Uparrow \mathsf{int\ list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}\end{array}
\qquad
\dfrac{
\dfrac{
\dfrac{}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \mathsf{int} \leqq \boxed{\mathsf{int}} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}
}{\Gamma, \boxed{\hat{\alpha}{=}\mathsf{int}} \vdash \mathsf{int} \leqq \boxed{\hat{\alpha}} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}\ \boxed{\text{ExSubstR}\leqq}
}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash \mathsf{int\ list} \leqq \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}
}{\Gamma, \hat{\alpha}{=}\mathsf{int} \vdash xs \Downarrow \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}\ \mathsf{sub}
}{\Gamma \vdash \mathit{filter}\ \mathsf{f}\ xs \Uparrow \hat{\alpha}\ \mathsf{list} \dashv \Gamma, \hat{\alpha}{=}\mathsf{int}}\ {\to}\mathrm{E}
$$

Explicit Substitution Rules

$$\frac{\Gamma \vdash e \Downarrow \Gamma(\widehat{\alpha}) \dashv \Gamma'}{\Gamma \vdash e \Downarrow \widehat{\alpha} \dashv \Gamma'}\ \text{ExSubst}\Downarrow \qquad \frac{\Gamma \vdash e \Uparrow \widehat{\alpha} \dashv \Gamma'}{\Gamma \vdash e \Uparrow \Gamma'(\widehat{\alpha}) \dashv \Gamma'}\ \text{ExSubst}\Uparrow$$

$$\frac{\Gamma \vdash \Gamma(\widehat{\alpha}) \leqq B \dashv \Gamma'}{\Gamma \vdash \widehat{\alpha} \leqq B \dashv \Gamma'}\ \text{ExSubstL}\leqq \qquad \frac{\Gamma \vdash A \leqq \Gamma(\widehat{\beta}) \dashv \Gamma'}{\Gamma \vdash A \leqq \widehat{\beta} \dashv \Gamma'}\ \text{ExSubstR}\leqq$$

Example:

$$\frac{\widehat{\alpha_1}{=}\mathsf{int},\ \widehat{\alpha_2}{=}\mathbf{1},\ \widehat{\alpha}{=}\widehat{\alpha_1}{\rightarrow}\widehat{\alpha_2} \vdash x \Uparrow \widehat{\alpha}}{\widehat{\alpha_1}{=}\mathsf{int},\ \widehat{\alpha_2}{=}\mathbf{1},\ \widehat{\alpha}{=}\widehat{\alpha_1}{\rightarrow}\widehat{\alpha_2} \vdash x \Uparrow \widehat{\alpha_1}{\rightarrow}\widehat{\alpha_2}}\ \text{ExSubst}\Uparrow$$

Complete?

- Sound with respect to oracular system
- Is it complete?
  - Can we always find the A' used in $\forall$E and $\forall$L$\leq$?
  - Impredicative counterexample:

$$f{:}\forall\alpha.\ \alpha \longrightarrow \alpha \longrightarrow \mathbf{1}, \quad x{:}(\mathsf{int}{\longrightarrow}\mathsf{int}) \longrightarrow \mathbf{1}, \quad y{:}(\forall\beta.\ \beta{\longrightarrow}\beta) \longrightarrow \mathbf{1}$$

$$\vdash$$

$$f\ x\ y \Uparrow$$

α becomes $\hat{\alpha}$, which is then solved: $(\mathsf{int}{\longrightarrow}\mathsf{int}) \longrightarrow \mathbf{1}$

Need to show: $(\forall\beta.\ \beta{\longrightarrow}\beta) \longrightarrow \mathbf{1} \leqq (\mathsf{int}{\longrightarrow}\mathsf{int}) \longrightarrow \mathbf{1}$

So need to show: $\mathsf{int} \longrightarrow \mathsf{int} \leqq \forall\beta.\ \beta \longrightarrow \beta$

Completeness

Instantiation	∀ location	Complete?
Predicative A^{mono} / α	prenex $\forall\alpha.\ A \to B$	✓
Predicative A^{mono} / α	arbitrary-rank $(\forall\alpha.\ A \to B) \to C$	✓
Impredicative $(\forall\beta.\ \beta \to \beta) / \alpha$	arbitrary-rank $(\forall\alpha.\ A \to B) \to C$	✗

Hints

Annotations can instantiate, but somewhat verbosely:

```
(*[  val  ordinarily_annotated
        :  ( -all 'a- 'a → 'a → unit)
        → ((int → int) → unit)
        → ((-all 'b- 'b → 'b) → unit)
        → unit                              ]*)

fun ordinarily_annotated f x y =
  let  val  f_at = ( f :   ((-all 'c- 'c → 'c) → unit)
                         → ((-all 'c- 'c → 'c) → unit)
                         → unit )
  in  f_at x y
  end
```

Hints

Instead, use a **hint**:

```
(*[ val hinted : (-all 'a- 'a → 'a → unit)
                → ((int → int) → unit)
                → ((-all 'b- 'b → 'b) → unit)
                → unit                              ]*)

fun hinted f x y =
  hint (-all 'c- 'c → 'c) → unit in
    f x y
  end
```

Implementation

- System decidable (see paper), but not syntax-directed

$$\frac{\Gamma, \widehat{\alpha} \vdash [\widehat{\alpha}/\alpha]A \leqq B \dashv \Gamma', \widehat{\alpha}[\ldots], \Gamma_Z}{\Gamma \vdash \forall\alpha.\, A \leqq B \dashv \Gamma'}\ \forall\mathrm{L}\widehat{\alpha}\leqq$$

$$\frac{\Gamma_1 \vdash A \ \mathsf{wf}}{\Gamma_1, \widehat{\beta}, \Gamma_2 \vdash A \leqq \widehat{\beta} \dashv \Gamma_1, \widehat{\beta}{=}A, \Gamma_2}\ \widehat{\alpha}^{=}\mathrm{R}\leqq$$

Either rule might derive $\Gamma_1, \widehat{\beta}, \Gamma_2 \vdash \forall\alpha.\, A_0 \leqq \widehat{\beta}$

$\forall\mathrm{L}\widehat{\alpha}\leqq$ instantiates $\widehat{\beta}$ with $[\widehat{\alpha}/\alpha]A_0$
$\widehat{\alpha}^{=}\mathrm{R}\leqq$ instantiates $\widehat{\beta}$ with $\forall\alpha.\, A_0$

- Backtracking used

Intersection Types

- $v \Downarrow A \wedge B$ if $v \Downarrow A$ and $v \Downarrow B$
- If $e \Uparrow A \wedge B$ then $e \Uparrow A$ and $e \Uparrow B$

$$\mathbf{op}+ : \big(\mathsf{int} \longrightarrow \mathsf{int} \longrightarrow \mathsf{int}\big) \wedge \big(\mathsf{real} \longrightarrow \mathsf{real} \longrightarrow \mathsf{real}\big)$$

Subtyping Needs Unions

- Add "real" subtyping $\leq$

$$\textsf{empty} \leq \textsf{list} \qquad \textsf{nonempty} \leq \textsf{list}$$

- Let *choose* $: \forall\alpha.\ \alpha \rightarrow \alpha \rightarrow \alpha$

$$ne{:}\textsf{nonempty}, xs{:}\textsf{list} \nvdash \textit{choose}\ ne\ xs \Uparrow$$

- Let *choose* $: \forall\alpha_1, \alpha_2.\ \alpha_1 \rightarrow \alpha_2 \rightarrow (\alpha_1 \vee \alpha_2)$

$$ne{:}\textsf{nonempty}, xs{:}\textsf{list} \vdash \textit{choose}\ ne\ xs \Uparrow (\textsf{nonempty} \vee \textsf{list}) \quad \checkmark$$

Type Abuse

$$compose : \forall\alpha, \beta, \gamma.\ (\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$$

- Can't compose $g : (B_1 \to C_1) \wedge (B_2 \to C_2)$
  and $f : A \to (B_1 \vee B_2)$
- So rewrite type, as we did for *choose*:

$$\begin{aligned} compose : \ & \forall\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2. \\ & ((\beta_1 \to \gamma_1) \wedge (\beta_2 \to \gamma_2)) \\ & \to (\alpha \to (\beta_1 \vee \beta_2)) \\ & \to \alpha \\ & \to (\gamma_1 \vee \gamma_2) \end{aligned}$$

- Doesn't scale

$$\frac{\Gamma_1 \vdash B \text{ wf}}{\Gamma_1, \widehat{\alpha}, \Gamma_2 \vdash \widehat{\alpha} \leq B \dashv \Gamma_1, \widehat{\alpha}^{\wedge}{=}B, \Gamma_2}\ \widehat{\alpha}^{=}\mathrm{L}\,\mathit{Auto}$$

$$\frac{\Gamma_1 \vdash A \text{ wf}}{\Gamma_1, \widehat{\beta}, \Gamma_2 \vdash A \leq \widehat{\beta} \dashv \Gamma_1, \widehat{\beta}^{\vee}{=}A, \Gamma_2}\ \widehat{\alpha}^{=}\mathrm{R}\,\mathit{Auto}$$

$$\frac{\Gamma_1 \vdash B \text{ wf}}{\Gamma_1, \widehat{\alpha}^{\wedge}{=}A, \Gamma_2 \vdash \widehat{\alpha} \leq B \dashv \Gamma_1, \widehat{\alpha}^{\wedge}{=}A \wedge B, \Gamma_2}\ \widehat{\alpha}^{\wedge}\mathrm{L}\,\mathit{Auto}$$

$$\frac{\Gamma_1 \vdash A \text{ wf}}{\Gamma_1, \widehat{\beta}^{\vee}{=}A, \Gamma_2 \vdash A \leq \widehat{\beta} \dashv \Gamma_1, \widehat{\beta}^{\vee}{=}A \vee B, \Gamma_2}\ \widehat{\alpha}^{\vee}\mathrm{R}\,\mathit{Auto}$$

Future Work

- Completeness of intersection/union system
- GADTs

Related Work

- [Cardelli ’93]: Greedy method for System $F_{<:}$
- [Pierce & Turner ’00]: Local type inference
- [Davies ’05] ML-style $\forall$, finitely many refinements
- [Peyton Jones et al. ’07]: Higher-rank $\forall$, mixed $:/\Uparrow\Downarrow$
- [Le Botlan & Rémy ’03, Rémy & Yakobowski ’08]: ML^{F}: impred. $\forall$
- [Leijen ’07–’09]: HMF, HML: impred. $\forall$, relatively simple, annotations on impredicative uses only

Summary

- Bidirectional typing
- First-class polymorphism in a bidirectional setting
  - Ordered contexts
  - Complete for predicative polymorphism
- For subtyping: Intersection and union types

More Information

`http://www.cs.cmu.edu/~joshuad/papers/poly/`

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