

Case Analysis of Higher-Order Data

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Higher-Order Abstract Syntax

- Object-level binders as meta-level binders
 - $\forall x. (x = 1)$ as **forall** $\lambda x. (\text{eq } x \ 1)$
 - $\text{fn } x \Rightarrow x$ as **fn** $\lambda x. x$
- Twelf, λ Prolog, Elphin, Delphin, Abella, Bedwyr, ...

... Beluga



- Data layer (LF) + Computation layer
- A language for programming and writing proofs
- **Explicit** contexts characterized by schemas

Beluga: the usual suspects



- Coverage checking (this paper)
- Termination checking (ongoing work)

Coverage (Closed Data)

Does `case m of Zero ⇒ ... | Suc u ⇒ ...`
cover

`nat`

where `nat : type.`
`Zero : nat.`
`Suc : nat → nat. ?`

- Does every inhabitant of `nat` match some guard in `{Zero, Suc u}`?

Higher-order Data (Open)

- In contextual modal type theory
[Nanevski, Pfenning, Pientka 2008]
- Explicit contexts Ψ :

$$\Psi ::= \cdot \mid \psi \mid \Psi, x:A$$

- Context variables ψ range over contexts
- If $m : \mathbf{nat}[\Psi]$ then $\mathbf{FV}(m) \subseteq \mathbf{dom}(\Psi)$
- If $m : \mathbf{nat}[\cdot]$ then $\mathbf{FV}(m) = \emptyset$
- If $m : \mathbf{nat}[\psi, x:\mathbf{nat}]$ then $\mathbf{FV}(m) \subseteq \mathbf{dom}(\psi) \cup \{x\}$

Contextual Coverage: Variables

Does

`case m of box(x. Zero) ⇒ ... | box(x. Suc u[x]) ⇒ ...`

`cover nat[x:nat]` where `nat : type.`
`Zero : nat.`
`Suc : nat → nat. ?`

- Does every inhabitant of `nat` that is closed with respect to the context `x:nat` match some guard in `{x. Zero, x. Suc u[x]}`?

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`Zero : nat.`
`Suc : nat → nat. ?`

- Does every inhabitant of `nat` that is closed `with respect to the context x:nat` match some guard in `{x. Zero, x. Suc u[x]}`?
- **No:** misses `x`

Contextual Coverage: Variables

Does

case m of $\text{box}(x. \text{Zero}) \Rightarrow \dots \mid \text{box}(x. \text{Suc } u[x]) \Rightarrow \dots$

$\mid \text{box}(x. x) \Rightarrow \dots$

cover $\text{nat}[x:\text{nat}]$ where $\text{nat} : \text{type}.$
 $\text{Zero} : \text{nat}.$
 $\text{Suc} : \text{nat} \rightarrow \text{nat}.$?

- Does every inhabitant of nat that is closed with respect to the context $x:\text{nat}$ match some guard in $\{x. \text{Zero}, x. \text{Suc } u[x], x. x\}$?
- Yes (covers x)

Contextual Coverage: Parameters

Does

$\text{case } m \text{ of } \text{box}(\psi, x. \text{Zero}) \Rightarrow \dots \mid \text{box}(\psi, x. \text{Suc } u[\text{id}_\psi, x]) \Rightarrow \dots$
 $\mid \text{box}(\psi, x. x) \Rightarrow \dots$

cover $\text{nat}[\psi, x:\text{nat}]$ where $\text{nat} : \text{type}.$
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- Does every inhabitant of nat that is closed with respect to the context $\psi, x:\text{nat}$ match some guard in $\{\psi, x. \text{Zero}, \psi, x. \text{Suc } u[x], \psi, x. x\}$?

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- **No:** misses variables in ψ

Contextual Coverage: Parameters

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cover $\text{nat}[\psi, x:\text{nat}]$ where $\text{nat} : \text{type}.$
 $\text{Zero} : \text{nat}.$
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- Does every inhabitant of nat that is closed with respect to the context $\psi, x:\text{nat}$ match some guard in $\{\psi, x. \text{Zero}, \psi, x. \text{Suc } u[x], \psi, x. x, \psi, x. p[\text{id}_\psi]\}$?
- **Yes:** covers variables in ψ

Outline

- Data and contexts
- Patterns
- Theory of coverage checking
- Summary of results
- Related work

Data

Atomic types $P ::= a M_1 \dots M_n$

Types $A, B ::= P \mid \Pi x:A.B \mid \Sigma x:A.B$

Normal terms $M, N ::= \lambda x. M \mid (M, N) \mid R$

Neutral terms $R ::= c \mid x \mid u[\sigma] \mid p[\sigma] \mid R N \mid \mathbf{proj}_k R$

Substitutions $\sigma ::= \cdot \mid \sigma; M \mid \sigma, R \mid \mathbf{id}_\psi$

Data always in canonical form:

$(\lambda x. x)c$ ill-formed

Contexts

Atomic types $P ::= a M_1 \dots M_n$

Types $A, B ::= P \mid \Pi x:A.B \mid \Sigma x:A.B$

Contexts $\Psi ::= \cdot \mid \psi \mid \Psi, x:A$

Meta-contexts $\Delta ::= \cdot \mid \Delta, u::A[\Psi] \mid \Delta, p::A[\Psi]$

Schema contexts $\Omega ::= \cdot \mid \Omega, \psi::W$

Context Schemas

Contexts $\Psi ::= \cdot \mid \psi \mid \Psi, x:A$

Example context

$x:\text{nat}, y:\text{nat}, z:\text{nat}$

Schema W

$(\text{nat})^*$

Context Schemas

Contexts $\Psi ::= \cdot \mid \psi \mid \Psi, x:A$

Example context

$x:\text{nat}, y:\text{nat}, z:\text{nat}$

$x:\text{bool}, y:\text{nat}, z:\text{bool}$

Schema W

$(\text{nat})^*$

$(\text{nat} + \text{bool})^*$

Context Schemas

Contexts $\Psi ::= \cdot \mid \psi \mid \Psi, x:A$

Example context

$x:\text{nat}, y:\text{nat}, z:\text{nat}$

$x:\text{bool}, y:\text{nat}, z:\text{bool}$

$xs:(\text{list } (\text{Suc Zero})), ys:(\text{list } \text{Zero})$

Schema W

$(\text{nat})^*$

$(\text{nat} + \text{bool})^*$

$(\text{all } n:\text{nat}.\text{list } n)^*$

Context Schemas

Contexts $\Psi ::= \cdot \mid \psi \mid \Psi, x:A$

Example context

$x:\text{nat}, y:\text{nat}, z:\text{nat}$

$x:\text{bool}, y:\text{nat}, z:\text{bool}$

$xs:(\text{list } (\text{Suc Zero})), ys:(\text{list } \text{Zero})$

$x:\text{nat}, d:(\text{nd } (\text{Eq } x \ x))$

Schema W

$(\text{nat})^*$

$(\text{nat} + \text{bool})^*$

$(\text{all } n:\text{nat}.\text{list } n)^*$

$(\text{nat} + (\text{all } y:\text{prop}.\text{nd } y))^*$

Context Schemas

Contexts $\Psi ::= \cdot \mid \psi \mid \Psi, x:A$

Example context

$x:\text{nat}, y:\text{nat}, z:\text{nat}$

$x:\text{bool}, y:\text{nat}, z:\text{bool}$

$xs:(\text{list } (\text{Suc Zero})), ys:(\text{list } \text{Zero})$

$x:\text{nat}, d:(\text{nd } (\text{Eq } x \ x))$

$x:(\sum n:\text{nat}.\text{list } n), \dots$

Schema W

$(\text{nat})^*$

$(\text{nat} + \text{bool})^*$

$(\text{all } n:\text{nat}.\text{list } n)^*$

$(\text{nat} + (\text{all } y:\text{prop}.\text{nd } y))^*$

$(\sum n:\text{nat}.\text{list } n)^*$

Context Schemas

Element types $\tilde{A} ::= \Pi x:A.\tilde{A} \mid P$

Schema elements $F ::= \mathbf{all} \ \widetilde{\dots} . \Sigma \widetilde{\dots} . \tilde{A}$

Instance of some F $\Sigma \widetilde{\dots} . \tilde{A}$

Schemas $W ::= (F_1 + \dots + F_n)^*$

Patterns

Names $\hat{\Psi} ::= \cdot \mid \psi \mid \hat{\Psi}, x$

Patterns $\zeta ::= \Pi\Delta'. \mathbf{box}(\hat{\Psi}. M') : A'[\Psi']$

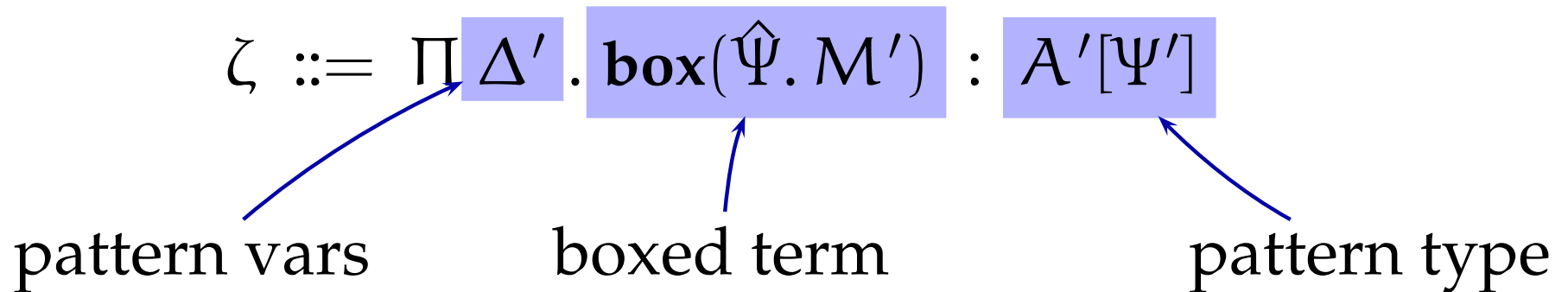
Expressions $e ::= \mathbf{case } e \mathbf{ of } \zeta_1 \Rightarrow e_1 \mid \dots \mid \zeta_n \Rightarrow e_n$
| ...

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 $\mid \dots$

$\zeta ::= \Pi \Delta'. \mathbf{box}(\hat{\Psi}. M') : A'[\Psi']$

pattern vars

boxed term

pattern type

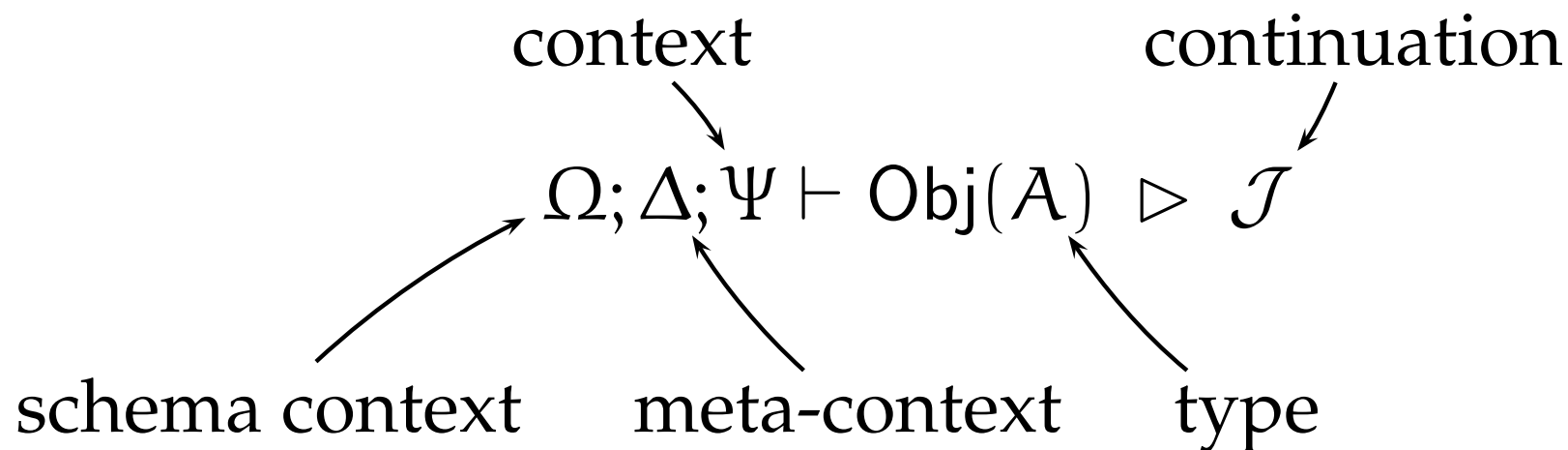
$\Pi u::\mathbf{nat}[\psi, x:\mathbf{nat}] . \mathbf{box}(\psi, x. \mathbf{Suc} u[\mathbf{id}_\psi, x]) : \mathbf{nat}[\psi, x:\mathbf{nat}]$

Higher-Order Pattern Restriction

$$\zeta ::= \Pi\Delta'. \mathbf{box}(\hat{\Psi}. M') : A'[\Psi']$$

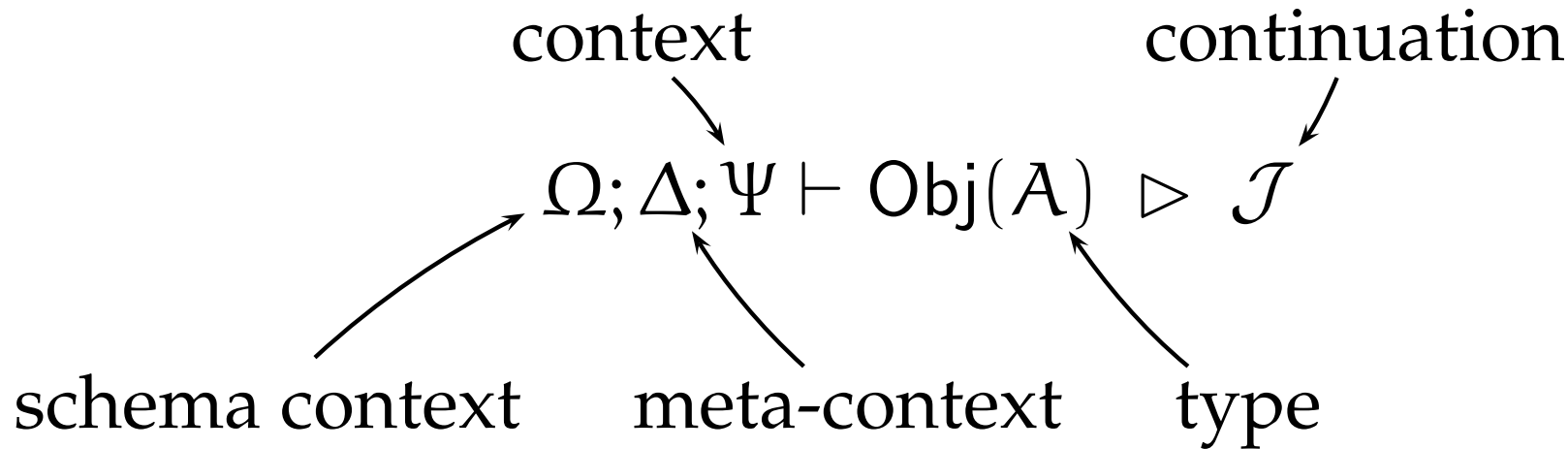
- M' is a higher-order pattern (Miller [1991]):
 - pattern variables declared in Δ'
applied to distinct sets of bound variables
- hence, pattern variables associated with a renaming substitution

Coverage Judgments



- $\text{Obj}(A) \triangleright \mathcal{J}$: Analyze A , then satisfy \mathcal{J}

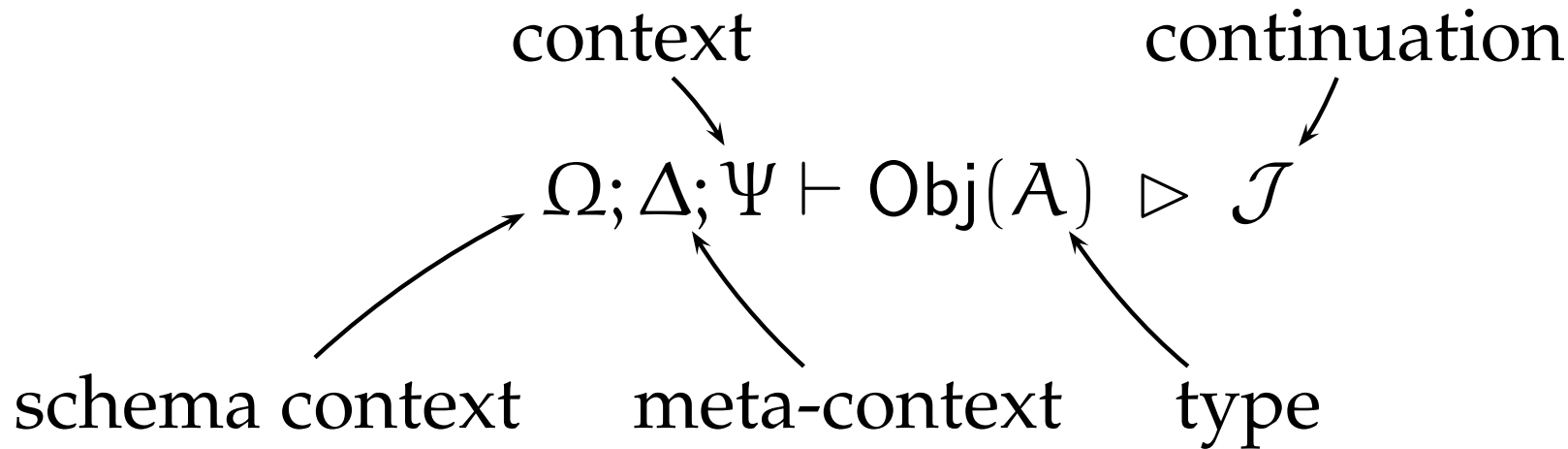
Coverage Judgments



- $\text{Obj}(A) \triangleright \mathcal{J}$: Analyze A , then satisfy \mathcal{J}
- $\text{nat}[\psi, x:\text{nat}]$ covered by $\{\zeta_1, \dots, \zeta_n\}$:

$\psi:(\text{nat}+\text{bool})^*; \cdot; \psi, x:\text{nat} \vdash \text{Obj}(\text{nat}) \triangleright \text{COVERED-BY} \{\zeta_1, \dots, \zeta_n\}$

Coverage Judgments



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- $\text{nat}[\psi, x:\text{nat}]$ covered by $\{\zeta_1, \dots, \zeta_n\}$:

$\psi:(\text{nat}+\text{bool})^*; \cdot; \psi, x:\text{nat} \vdash \text{Obj}(\text{nat}) \triangleright \text{COVERED-BY} \{\zeta_1, \dots, \zeta_n\}$

- Rules for $\text{Obj}(\text{nat}) \triangleright \mathcal{J}$ will lead to subderivations $\dots \vdash M : \text{nat} \triangleright \mathcal{J}$ for various terms M of type nat

Example

- Rules for $\text{Obj}(\text{nat}) \triangleright \mathcal{J}$ will lead to subderivations $M : A \triangleright \mathcal{J}$ for various terms M of type nat

$$\Omega = \psi:(\text{nat} + \text{bool})^*$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{PVars}\langle \psi : \text{nat} \rangle > \text{nat} \triangleright \mathcal{J}$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{PVars}\langle \psi : \text{bool} \rangle > \text{nat} \triangleright \mathcal{J}$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{App}\langle x \rangle (\text{nat} > \text{nat}) \triangleright \mathcal{J}$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{App}\langle \text{Zero} \rangle (\text{nat} > \text{nat}) \triangleright \mathcal{J}$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{App}\langle \text{Suc} \rangle (\text{nat} \rightarrow \text{nat} > \text{nat}) \triangleright \mathcal{J}$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{App}\langle \text{False} \rangle (\text{bool} > \text{nat}) \triangleright \mathcal{J}$$

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$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{Obj}(\text{nat}) \triangleright \mathcal{J}$$

Obj-split

Example

- Rules for $\text{Obj}(\text{nat}) \triangleright \mathcal{J}$ will lead to subderivations $M : A \triangleright \mathcal{J}$ for various terms M of type nat

$$\Omega = \psi : (\text{nat} + \text{bool})^*$$

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$$\frac{}{\Omega; \Delta; \psi, x : \text{nat} \vdash \text{Obj}(\text{nat}) \triangleright \mathcal{J}}$$

Obj-split

Example

$$\Omega \vdash (\Pi \Delta. \mathbf{box}(\psi, x. x) : \mathbf{nat}[\psi, x:\mathbf{nat}])$$
$$\doteq (\Pi \cdot \mathbf{box}(\psi, x. x) : \mathbf{nat}[\psi, x:\mathbf{nat}])$$

———— Covered-By- ζ

⋮

$$\Omega; \Delta; \psi, x:\mathbf{nat} \vdash \mathbf{App}\langle x \rangle(\mathbf{nat} > \mathbf{nat}) \triangleright \mathbf{COVERED-BY} Z$$

- Coverage derivation generates terms that **match** some **pattern** $\zeta \in Z$

Example: Contextual Counting

Checking if x occurs in $e : \text{nat}[\psi, x:\text{nat}]$
(sequential matching)

case e of

$\text{box}(\psi, x. u[\text{id}_\psi]) \Rightarrow \text{false}$
 $| \text{box}(\psi, x. u[x]) \Rightarrow \text{true}$

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- Terms must be at least as precise as user's patterns

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- Coverage derivation generates terms that **match** some **pattern ζ**
- Terms must be at least as precise as user's patterns
- Coverage rule MVars generates meta-variables $v[\sigma]$ for various σ
 - including $v[\text{id}_\psi]$ and $v[x]$

Summary

- Declarative description of coverage checking for contextual open data
- Algorithm follows from description
 - modulo deciding when to split
- Coverage Soundness:
If $\dots \Psi \vdash \text{Obj}(A) \triangleright \text{COVERED-BY } Z$ derivable,
then every $M : A$ under Ψ is matched by some
pattern in Z
- Because we use contextual types
with explicit contexts, coverage is **local**

Related Work

Coverage (in related settings)

- Closed data [Coquand 1992,
Schürmann & Pfenning 2003]
- Open data: regular worlds [Schürmann 2000],
Delphin worlds [Poswolsky & Schürmann]

Beluga

- [Pientka 2008], [Pientka & Dunfield 2008]

Contextual Modal Type Theory

- [Nanevski, Pfenning, Pientka 2008]

Ongoing Work

- An implementation of coverage checking for Beluga

The End

complogic.cs.mcgill.ca/beluga

Acknowledgments:

- The TPHOLs and LFMTP reviewers

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Example

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{nat} \doteq \text{nat} / (\theta, \cdot)$$
$$\Omega; \Delta; \llbracket \theta \rrbracket(\psi, x:\text{nat}) \vdash \llbracket \theta \rrbracket x : \llbracket \theta \rrbracket \text{nat} \triangleright \llbracket \theta \rrbracket \mathcal{J}$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{App}\langle x \rangle(\text{nat} > \text{nat}) \triangleright \mathcal{J} \quad \text{App-}\doteq$$

Example

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{nat} \doteq \text{nat} / (\cdot, \cdot)$$
$$\Omega; \Delta; \llbracket \cdot \rrbracket (\psi, x:\text{nat}) \vdash \llbracket \cdot \rrbracket x : \llbracket \cdot \rrbracket \text{nat} \triangleright \llbracket \cdot \rrbracket \mathcal{J}$$
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Example

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{nat} \doteq \text{nat} / (\cdot, \cdot)$$
$$\Omega; \Delta; \psi, x:\text{nat} \vdash x : \text{nat} \triangleright \mathcal{J}$$

$$\Omega; \Delta; \psi, x:\text{nat} \vdash \text{App}\langle x \rangle(\text{nat} > \text{nat}) \triangleright \mathcal{J} \quad \text{App-}\doteq$$

Example: Contextual Matching

Counting occurrences of x in formulas

(**Eq** : formula \rightarrow formula \rightarrow formula)

case e of

- | **box**($\psi, x. \mathbf{Eq} \ u[\mathbf{id}_\psi] \ v[\mathbf{id}_\psi] \) \Rightarrow 0$
- | **box**($\psi, x. \mathbf{Eq} \ u[\mathbf{id}_\psi, x] \ v[\mathbf{id}_\psi] \) \Rightarrow$ recurse on u
- | **box**($\psi, x. \mathbf{Eq} \ u[\mathbf{id}_\psi] \ v[\mathbf{id}_\psi, x] \) \Rightarrow$ recurse on v
- | **box**($\psi, x. \mathbf{Eq} \ u[\mathbf{id}_\psi, x] \ v[\mathbf{id}_\psi, x] \) \Rightarrow$
recurse on u and v , sum results

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Counting occurrences of x in formulas

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case e of

- | **box**(ψ , x . **Eq** $u[\mathbf{id}_\psi]$ $v[\mathbf{id}_\psi]$) \Rightarrow 0
- | **box**(ψ , x . **Eq** $u[\mathbf{id}_\psi, x]$ $v[\mathbf{id}_\psi]$) \Rightarrow recurse on u
- | **box**(ψ , x . **Eq** $u[\mathbf{id}_\psi]$ $v[\mathbf{id}_\psi, x]$) \Rightarrow recurse on v
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recurse on u and v , sum results

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- Terms must be at least as precise as user's patterns

Example: Contextual Matching

ValidWk($\text{nat}[\psi, x:\text{nat}]$)

=

$$\frac{\begin{array}{l} \{ (\cdot), \quad \Omega; \Delta, v::\text{nat}[\cdot] \quad ; \Psi \vdash (v[\cdot] \quad : \text{nat}) \triangleright \mathcal{J} \\ (\psi), \quad \Omega; \Delta, v::\text{nat}[\psi] \quad ; \Psi \vdash (v[\mathbf{id}_\psi] \quad : \text{nat}) \triangleright \mathcal{J} \\ (x:\text{nat}), \quad \Omega; \Delta, v::\text{nat}[x:\text{nat}] \quad ; \Psi \vdash (v[x] \quad : \text{nat}) \triangleright \mathcal{J} \\ (\psi, x:\text{nat}) \} \quad \Omega; \Delta, v::\text{nat}[\psi, x:\text{nat}]; \Psi \vdash (v[\mathbf{id}_\psi, x] : \text{nat}) \triangleright \mathcal{J} \end{array}}{\Omega; \Delta; \psi, x:\text{nat} \vdash \text{Obj}(\text{nat}) \triangleright \mathcal{J}}$$