Thesis Defense: Adaptive Binary Search Trees

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Thesis Committee:
Daniel Sleator (chair), Guy Blelloch, Gary Miller, Seth Pettie (Michigan)

The Search Problem

- Membership-testing, dictionary, successor/predecessor, etc.
- *Sequence* of queries $\sigma_1 \cdots \sigma_m$
- Assume each $\sigma_j \in \{1, \ldots, n\}$

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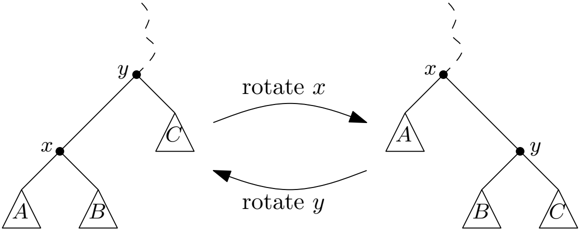

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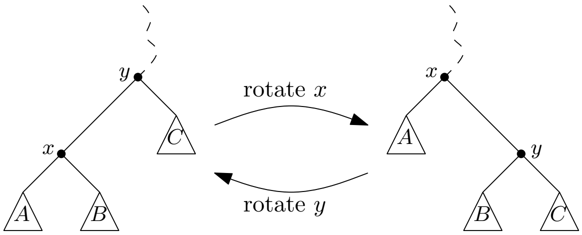

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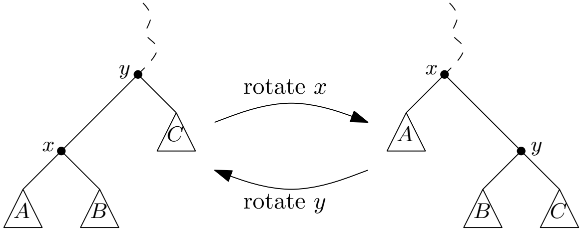

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- Can adapt to sequences with patterns
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- Invalidate the computational model
- Understand the model/problem

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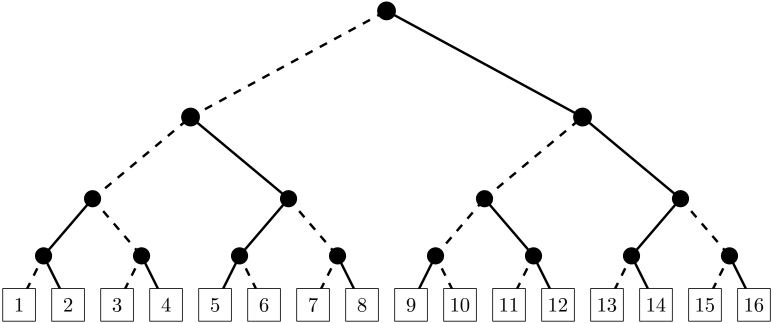

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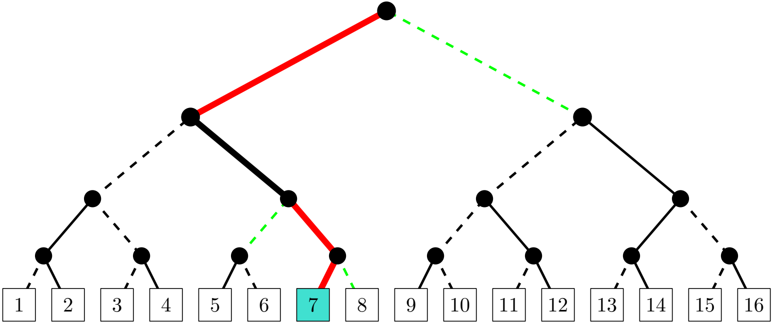

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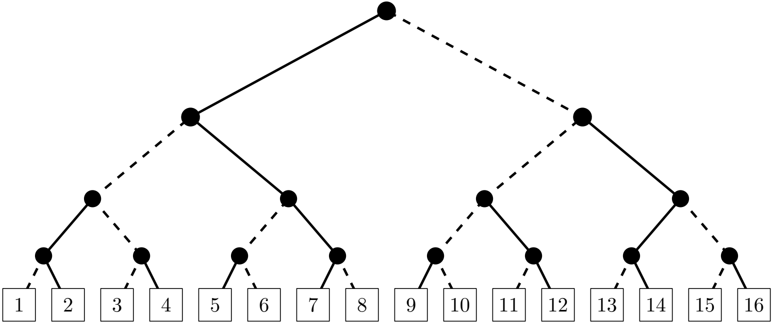

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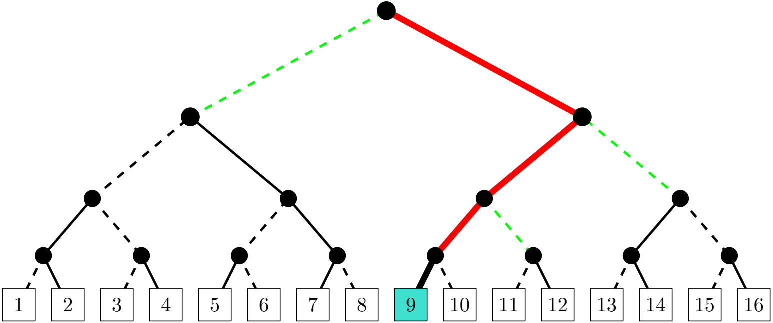

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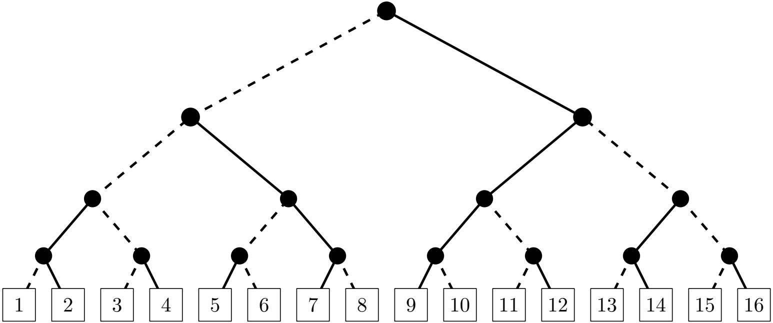
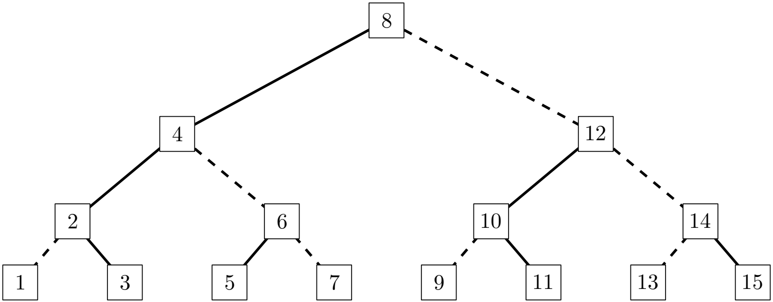


Interleave Bound [DHIP04]



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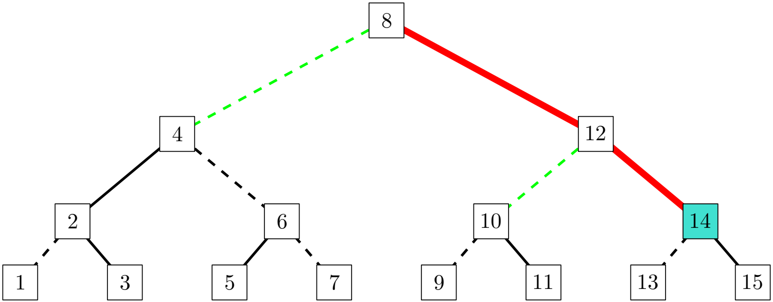

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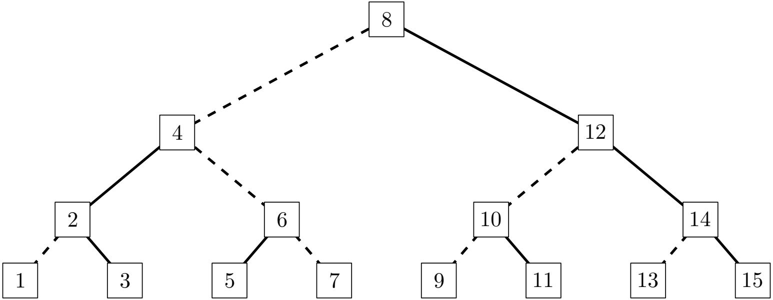

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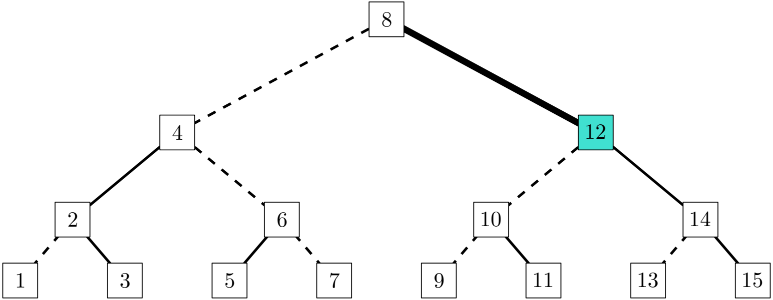

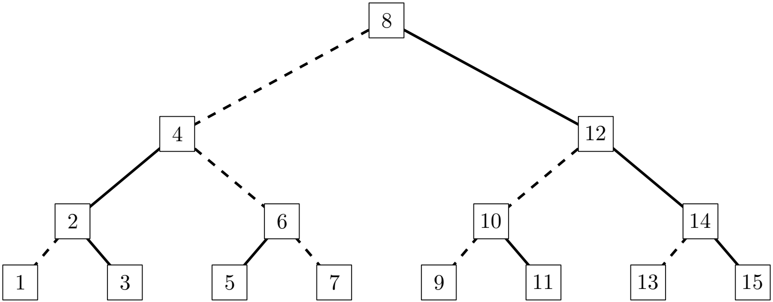
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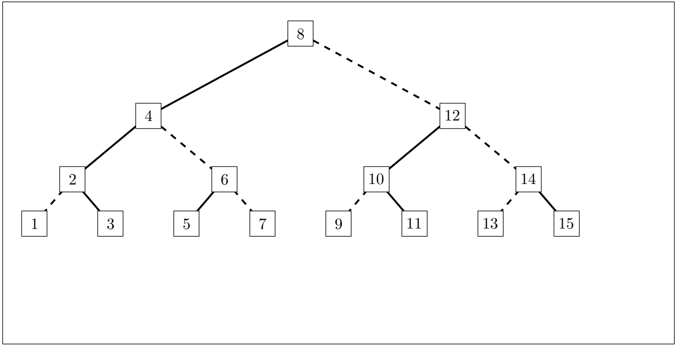
15

Uses of the Interleave Bound



Implications

- Balanced BSTs dynamically optimal for random sequences
- Rotate paths (only in BST) into a red-black tree [DHIP04]
- Swap red-black trees for splay trees and gain [WDS06]
- Can also rotate *lower bound tree* to allow updates [WDS06]

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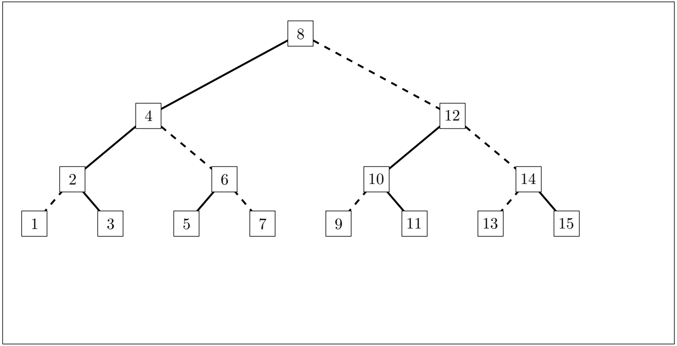

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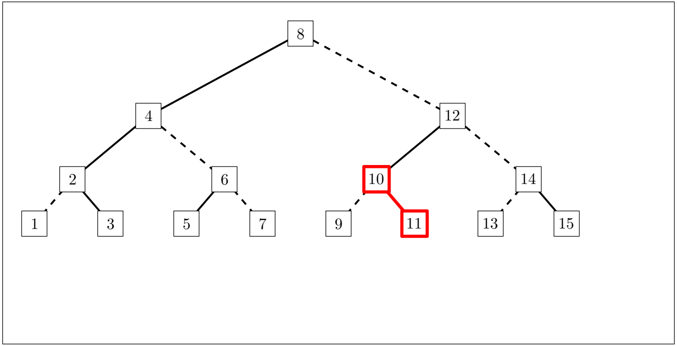

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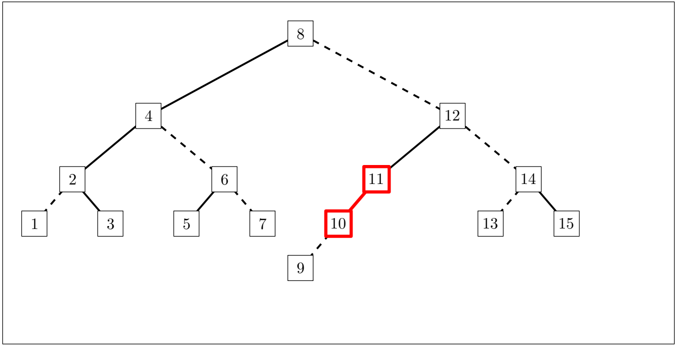

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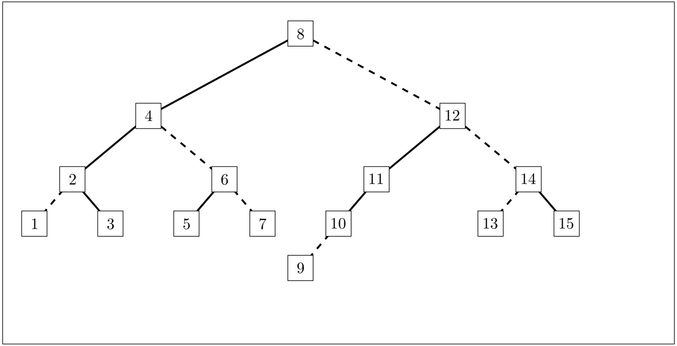

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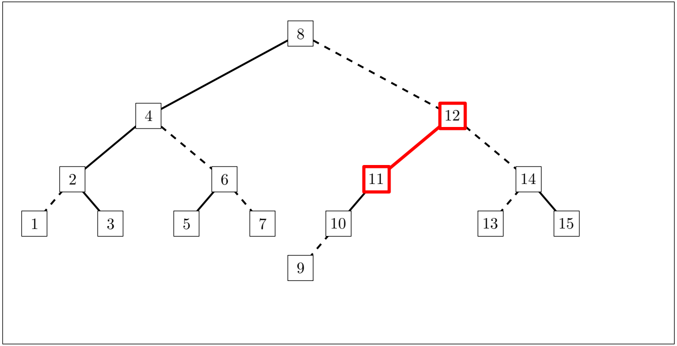

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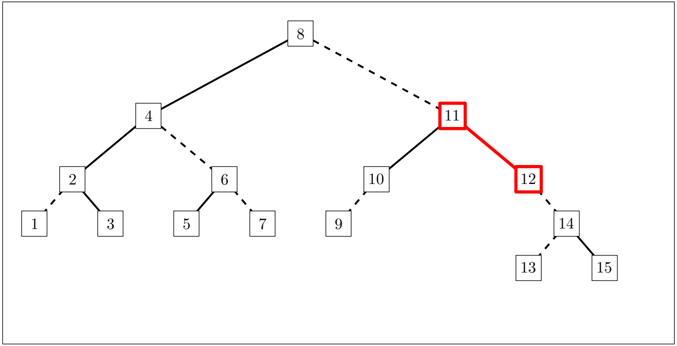

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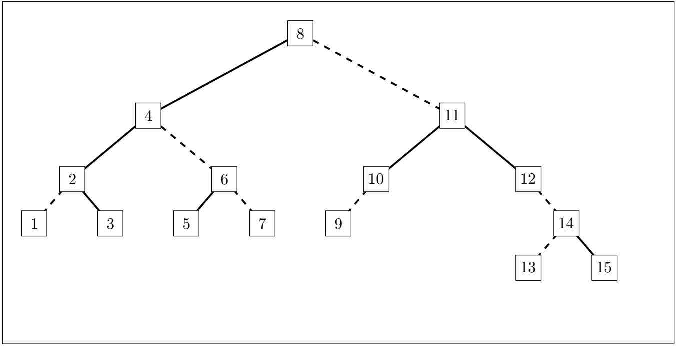

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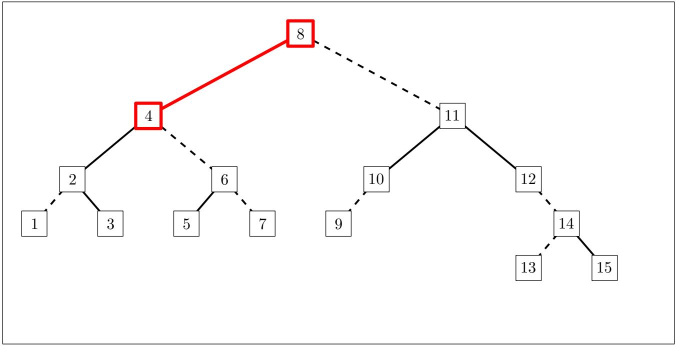

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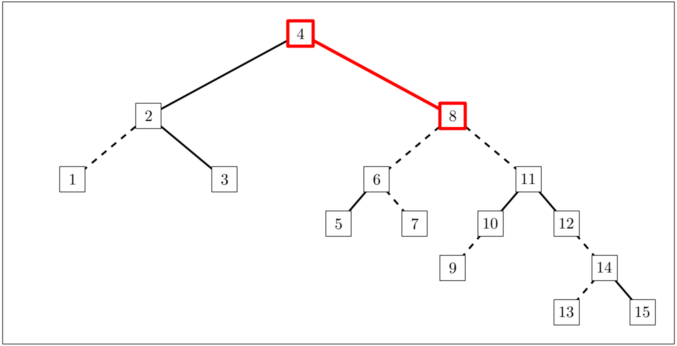

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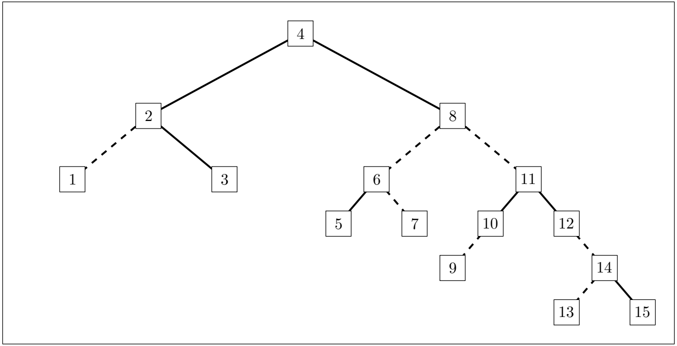

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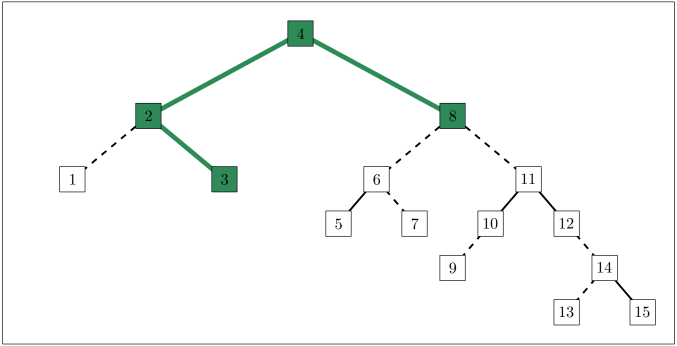

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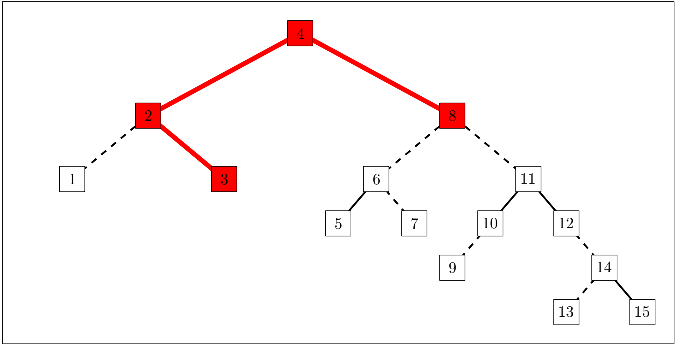

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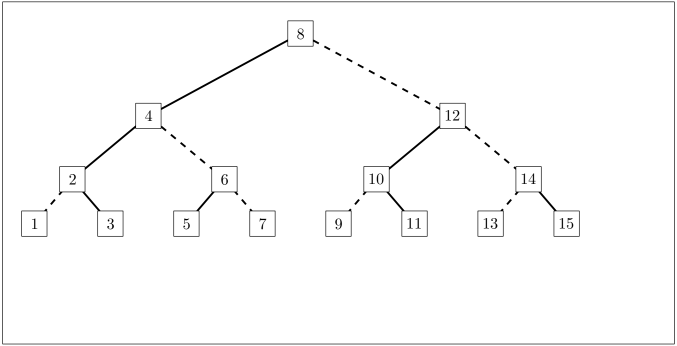

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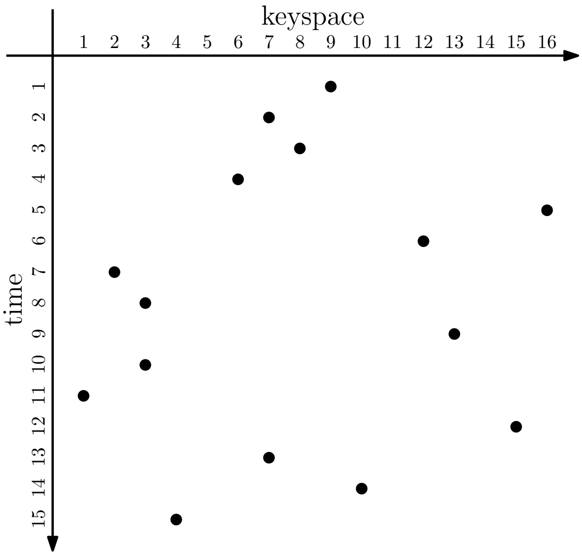

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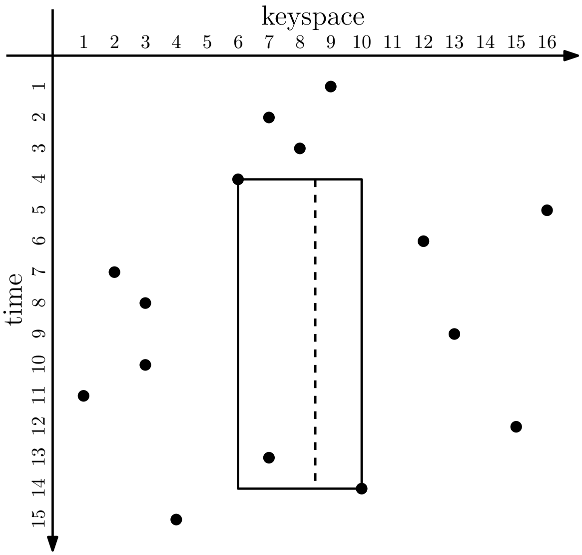

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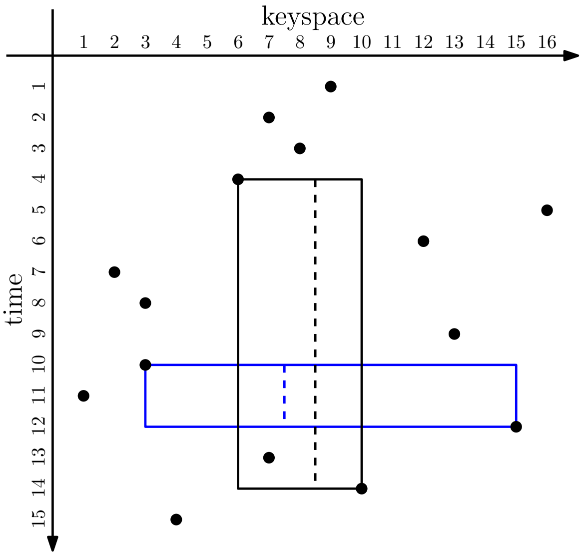

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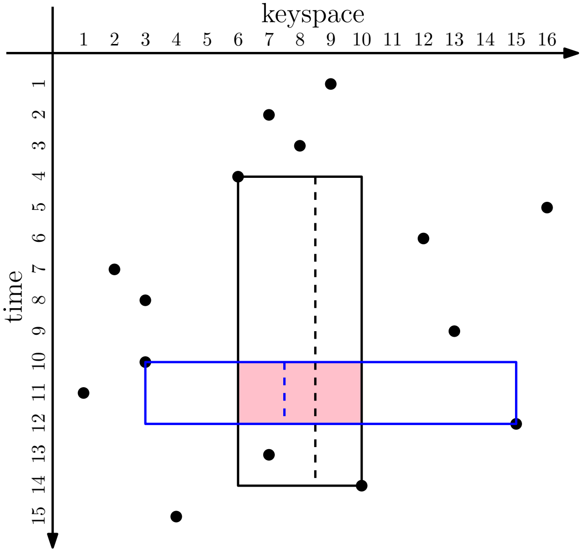

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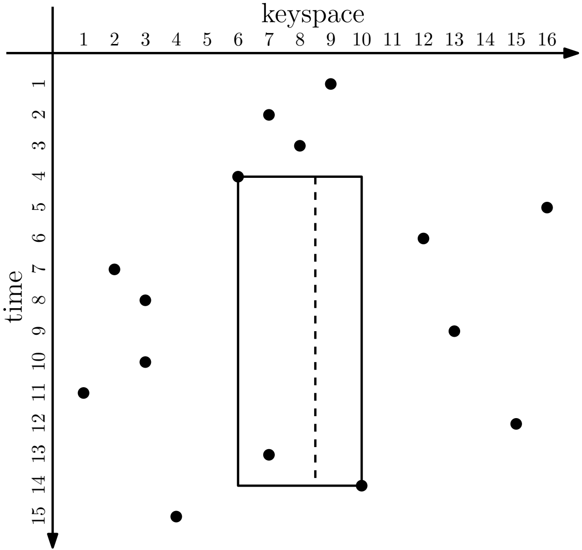

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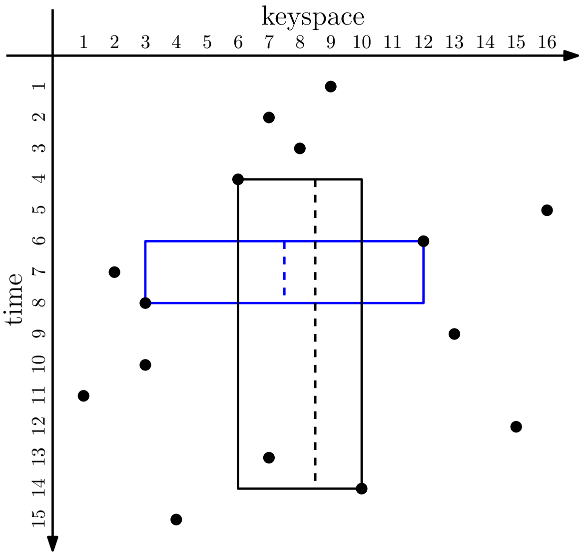

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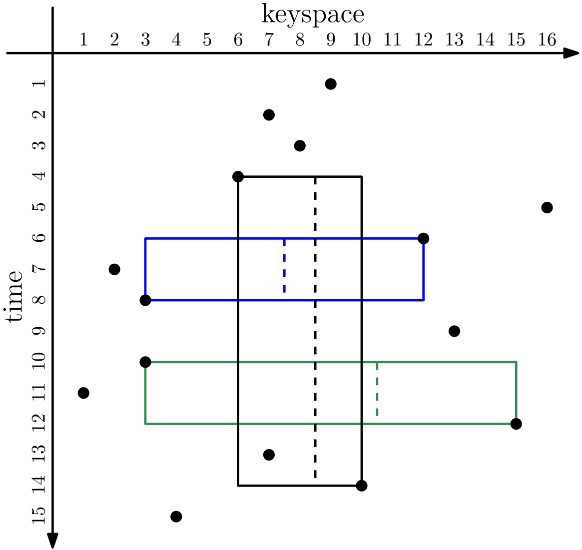

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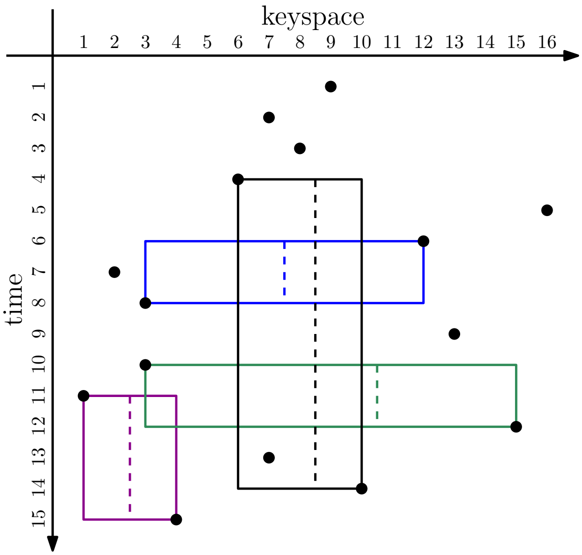

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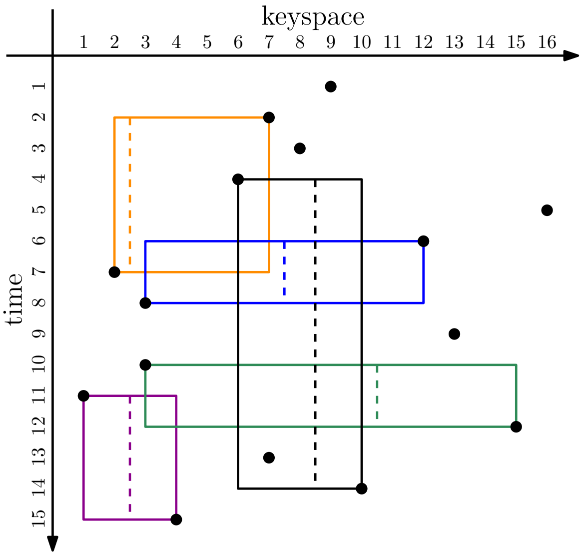

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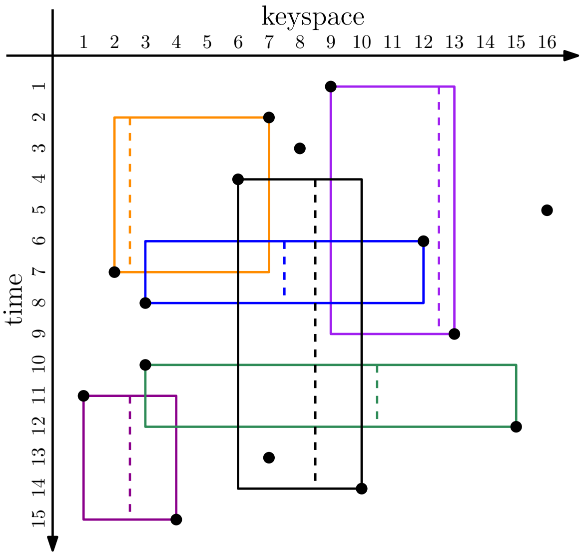
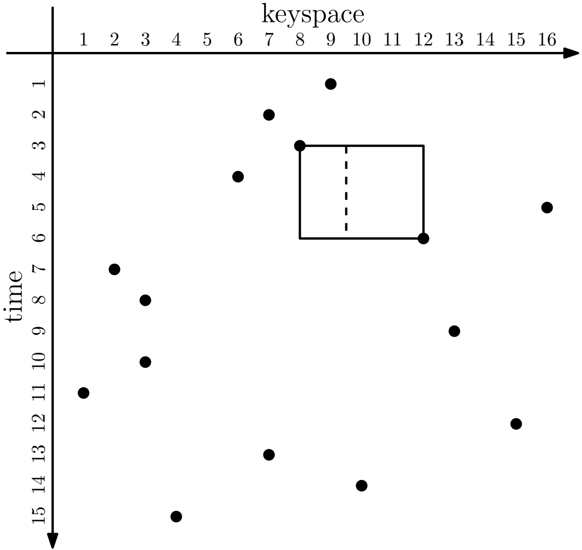


The MIBS Bound

Theorem

The number of boxes is a lower bound on $\mathrm{OPT}(\sigma)$.

Proof of MIBS Bound



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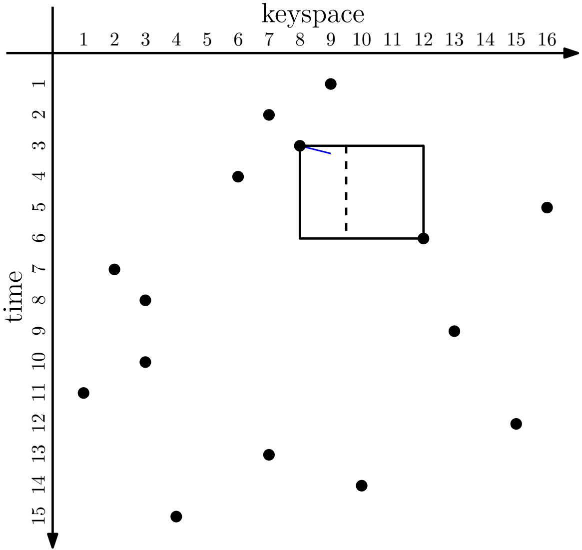

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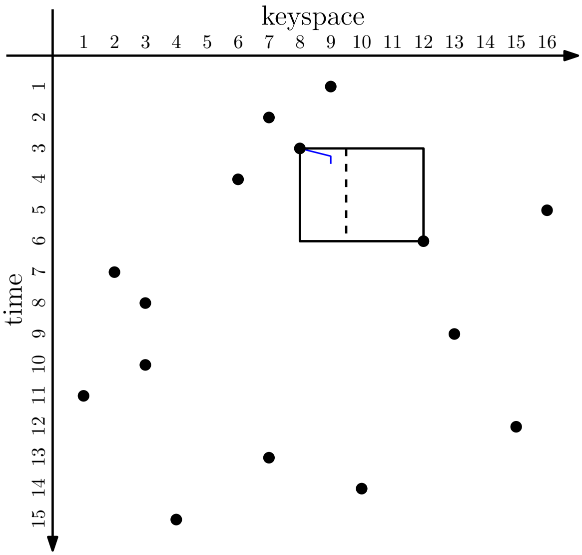

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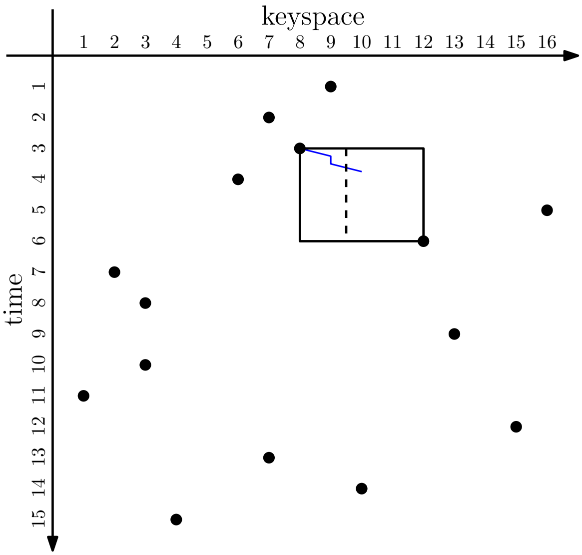

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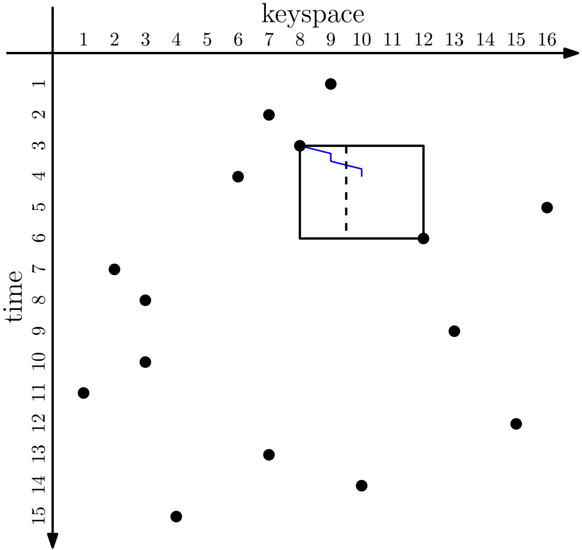

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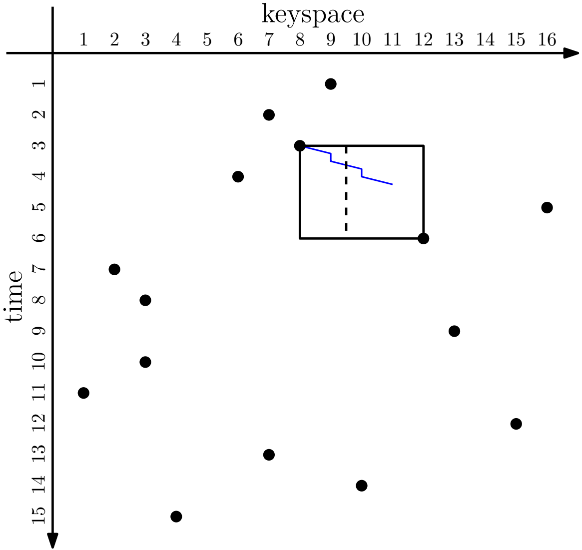

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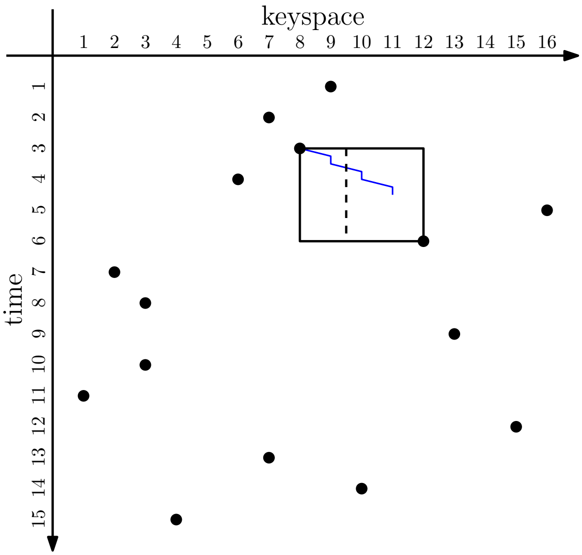

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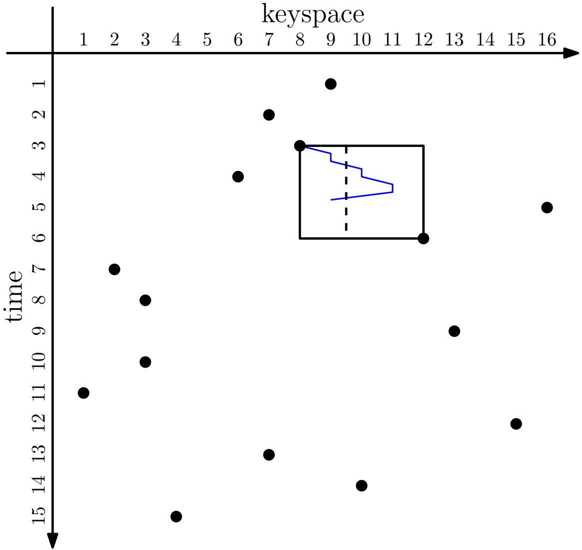

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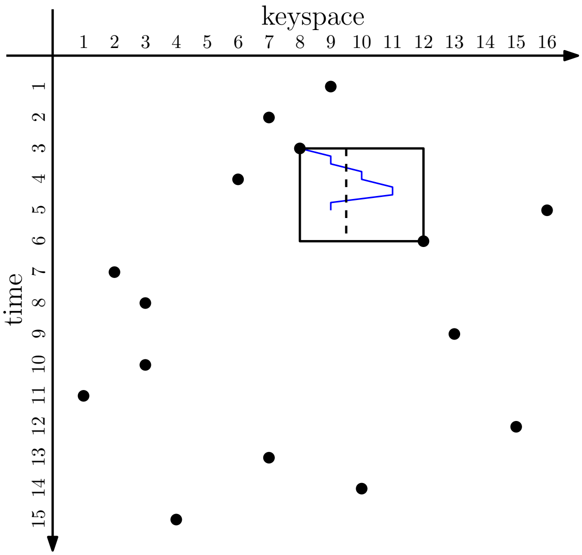

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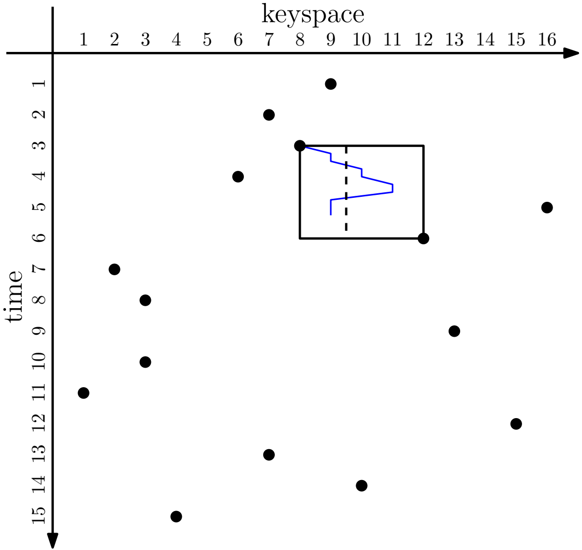

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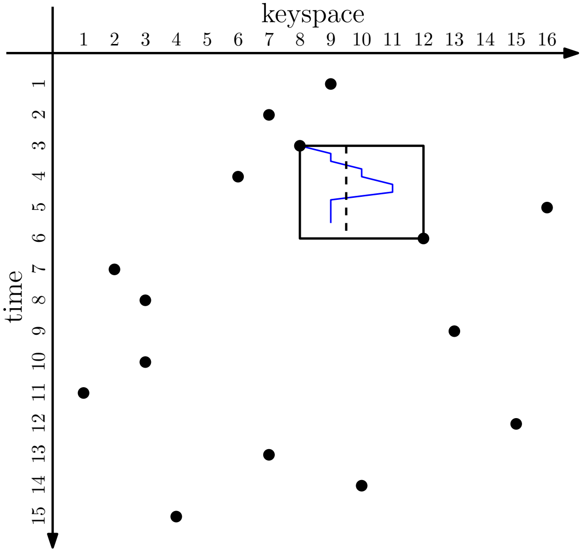

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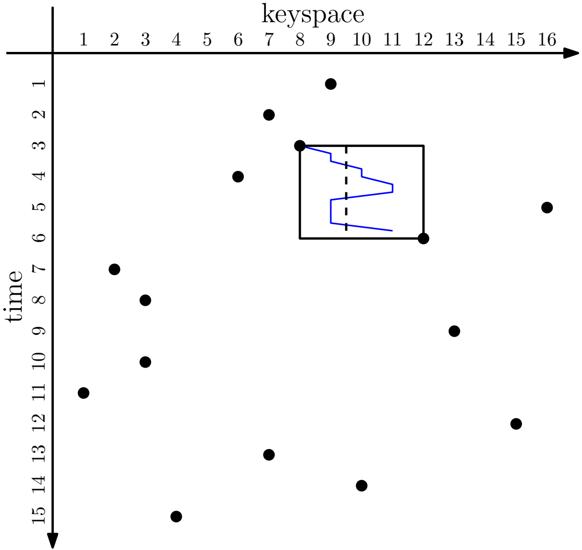

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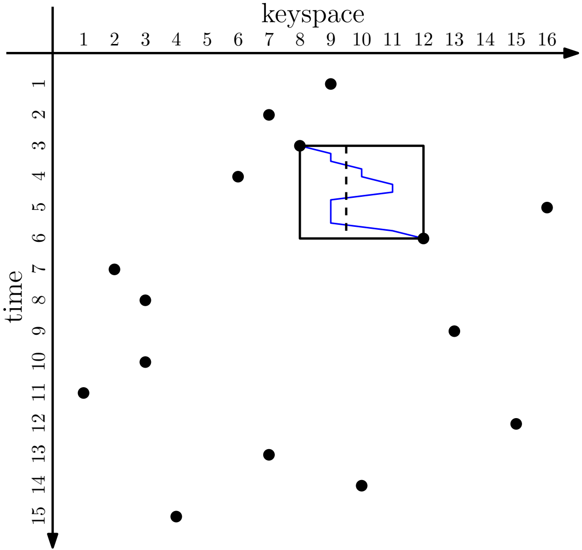

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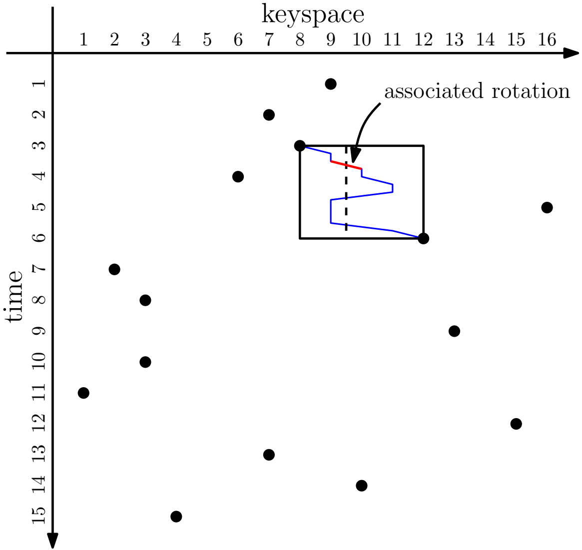

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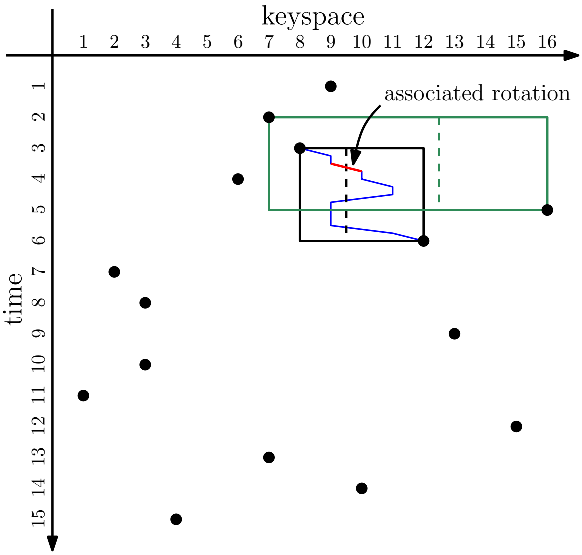

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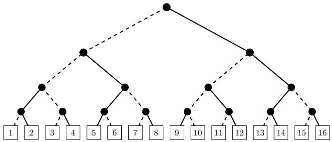

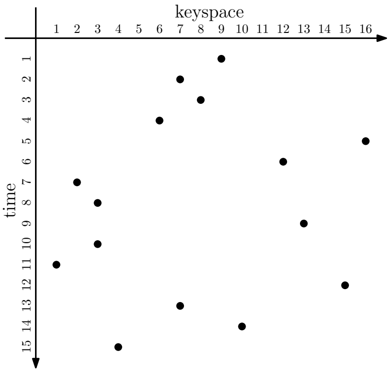

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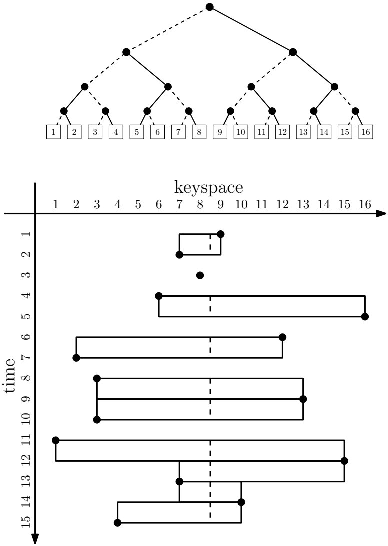

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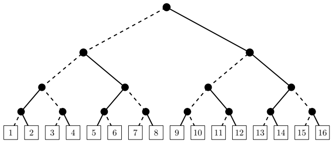

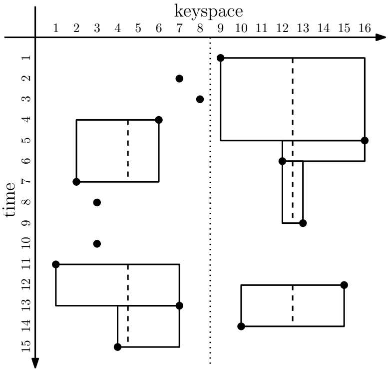

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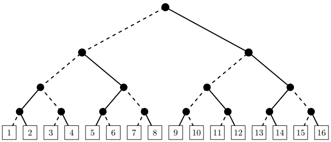

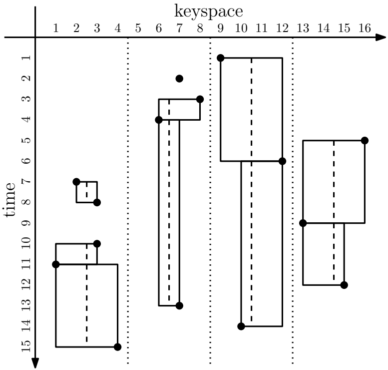

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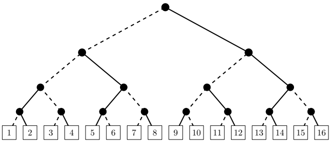

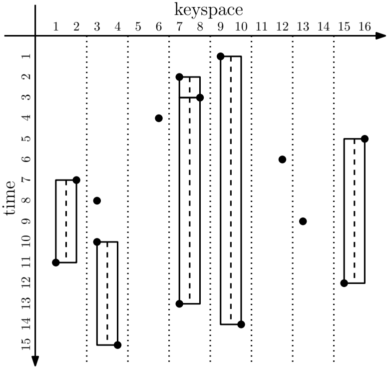

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Need two fingers

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- Generalizes working set and dynamic finger bounds
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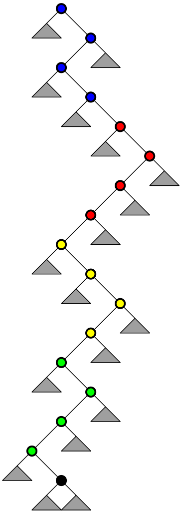

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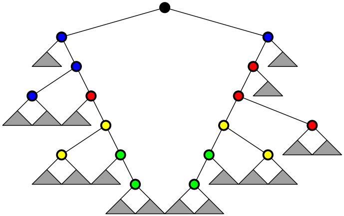

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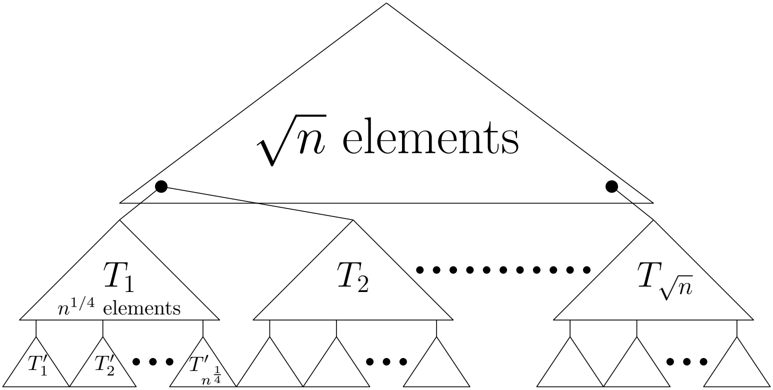

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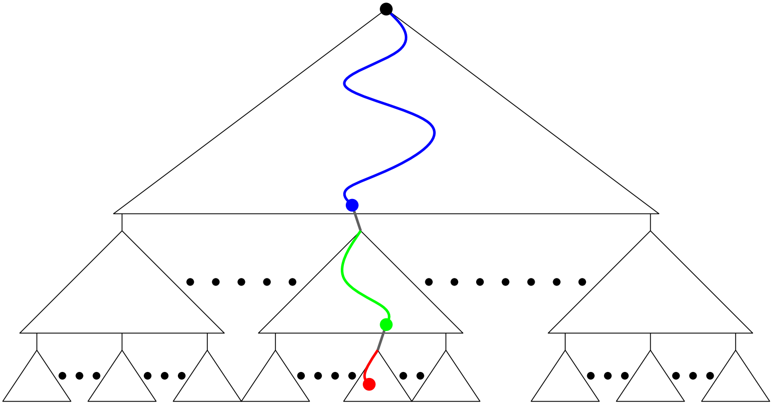

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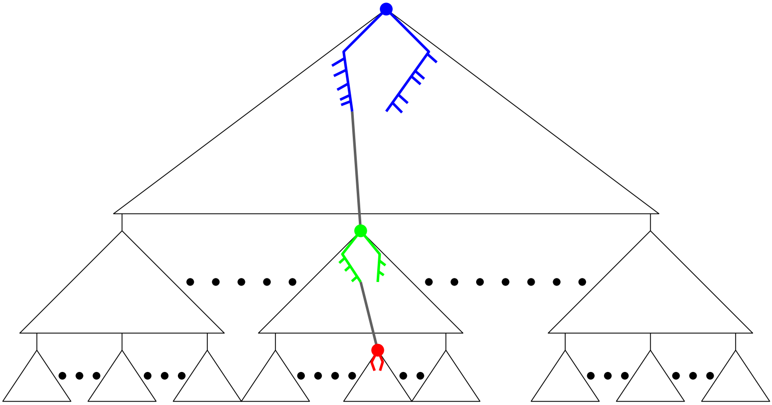

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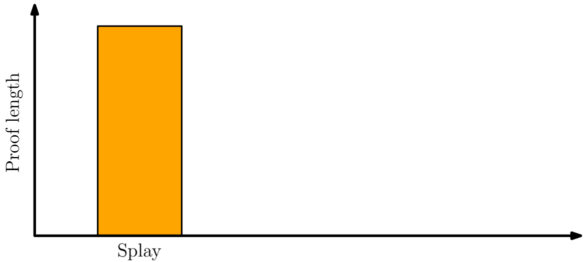

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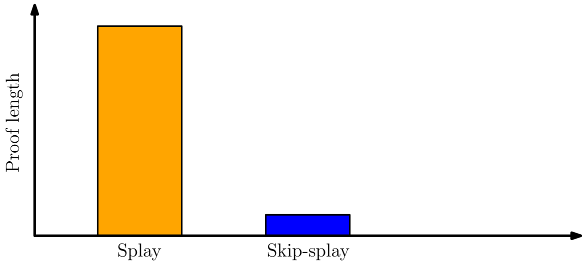

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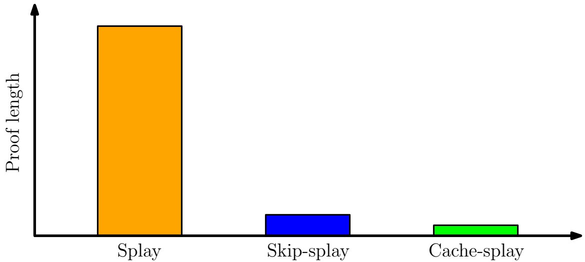
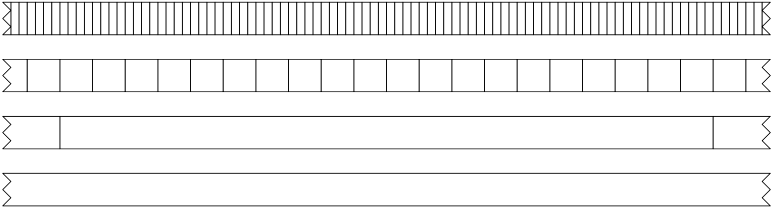
Simple BSTs Versus Simple Proofs



- Splay requires 80-page proof for dynamic finger
- Skip-splay requires 8-page proof for Unified Bound plus $\lg \lg n$
- Cache-splay requires 4-page proof for Unified Bound

Outline

The Cache-Splay Hierarchy of Keys



- Level-1 block contains $b_1 = 4$ keys
- Level-2 block contains b_1 level-1 blocks (16 keys)
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- Level-$\lg \lg n$ block contains all keys

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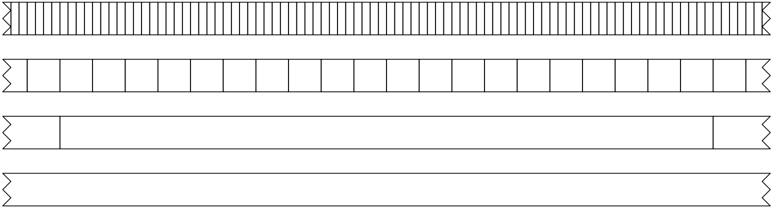

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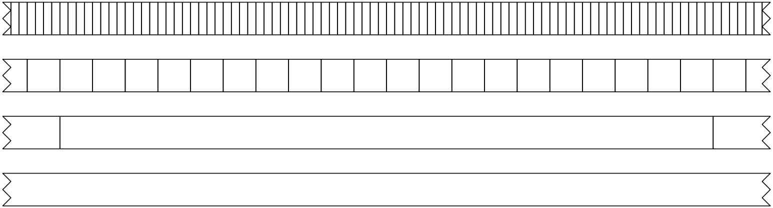

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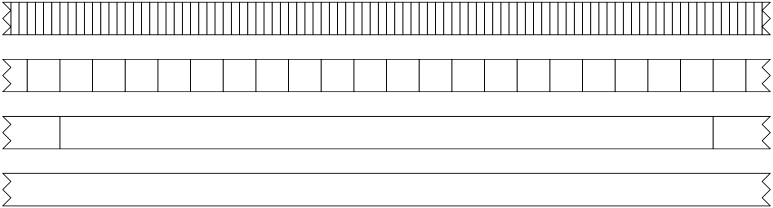

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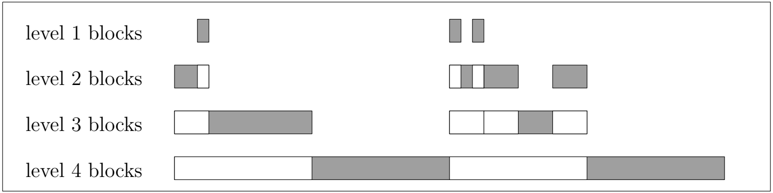
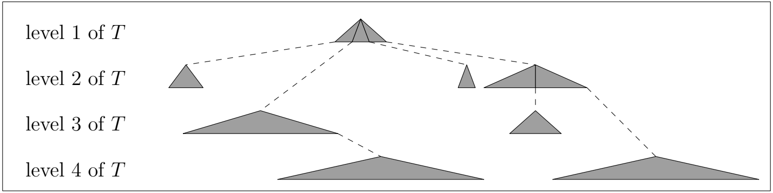
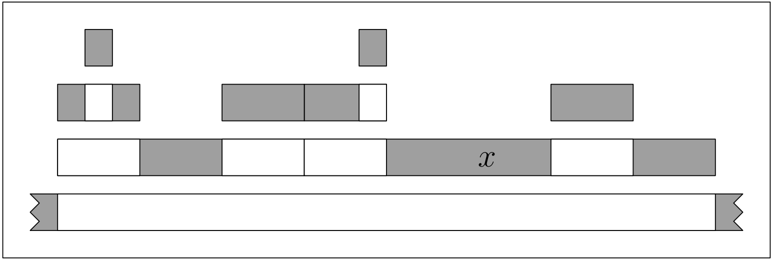
The Cache View and the Tree View





The Cache View of a Query



Cache view of a query to x

- Cache loop, iteration 1
- Cache loop, iteration 2
- Eject loop, iteration 1
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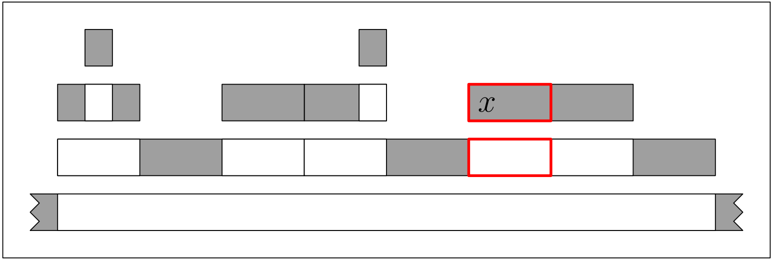
Important Fact

If t keys have been queried since x, then the size of x's current block size is $t^{O(1)}$.

The Cache View of a Query



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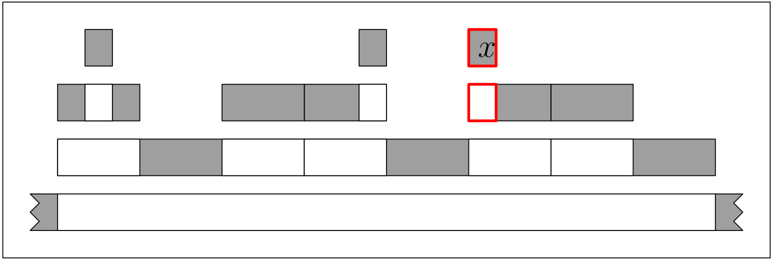

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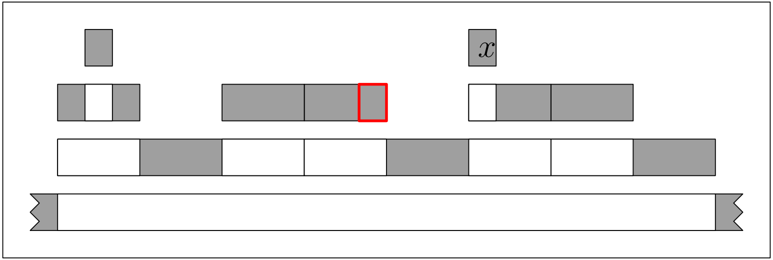

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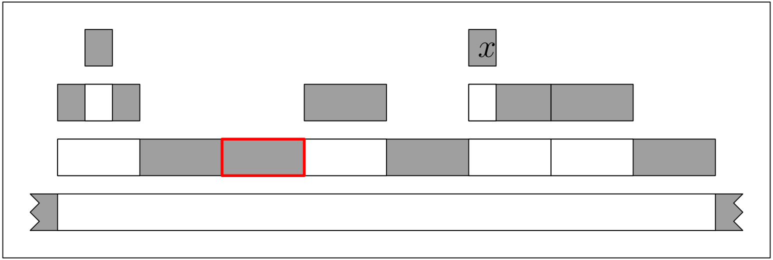

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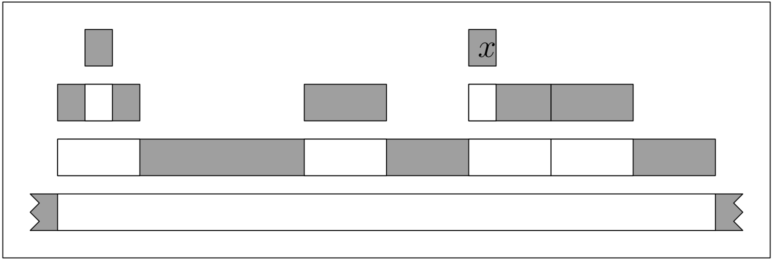

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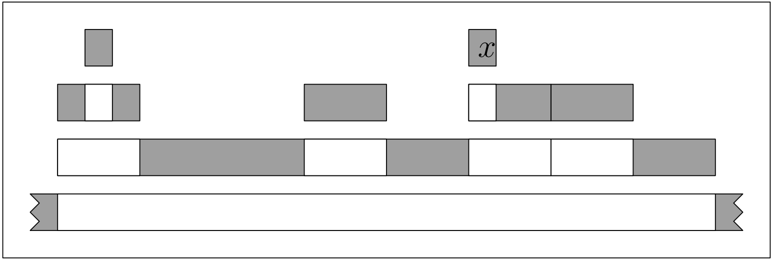

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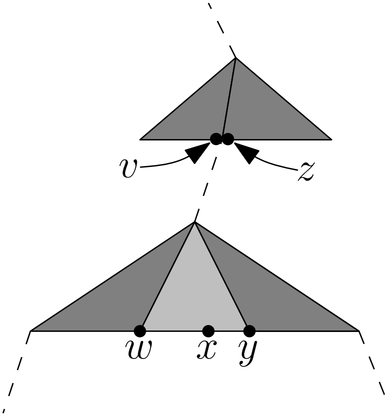

One cache loop iteration

- splay(w)
- splay(y)
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- incRoot(leftChild(w))
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- decRoot(w)

Each operation costs O(lg(block size for lower level))

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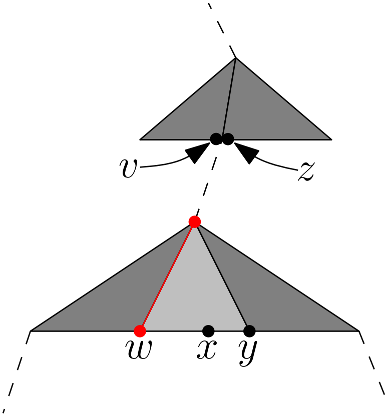

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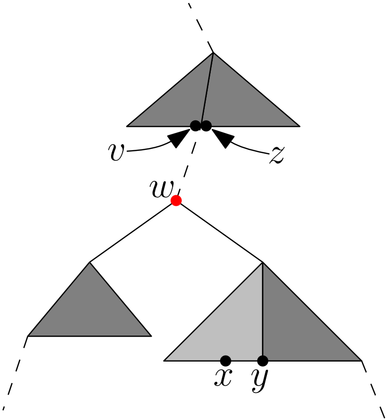

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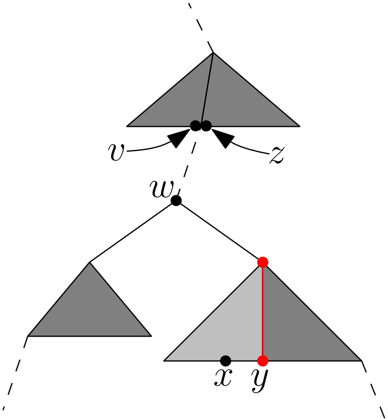

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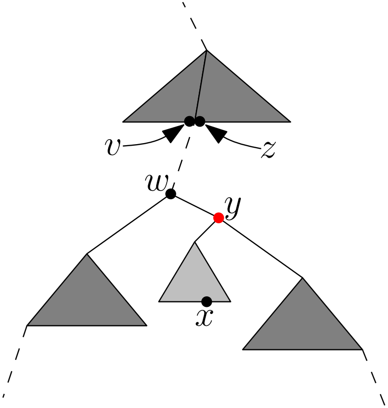

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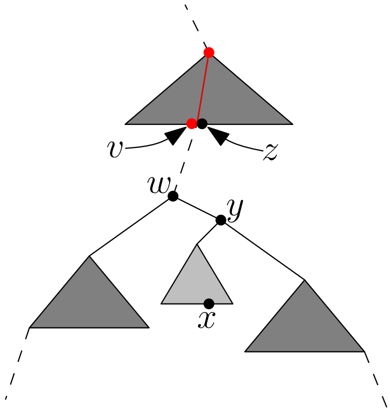

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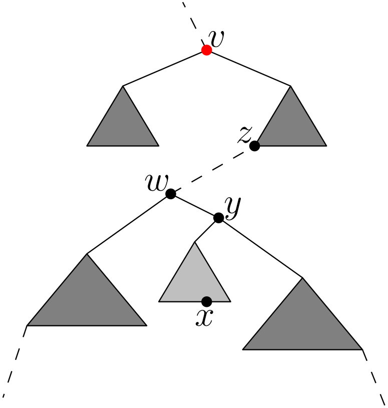

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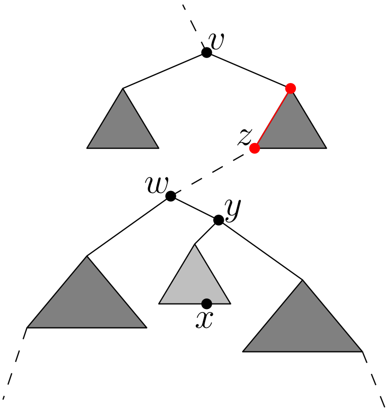

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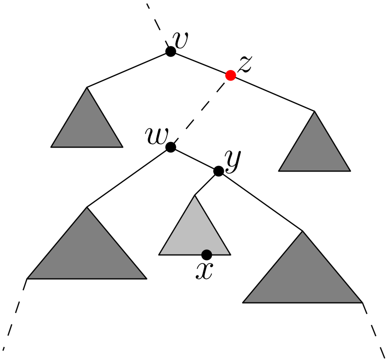

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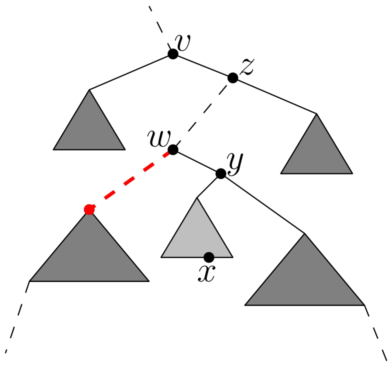

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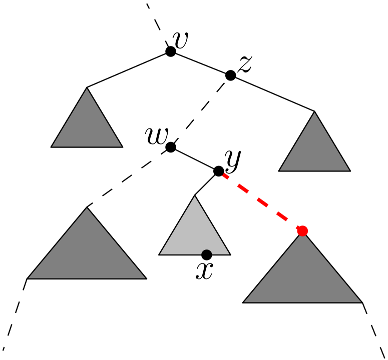

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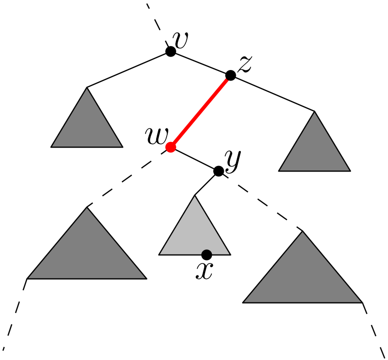

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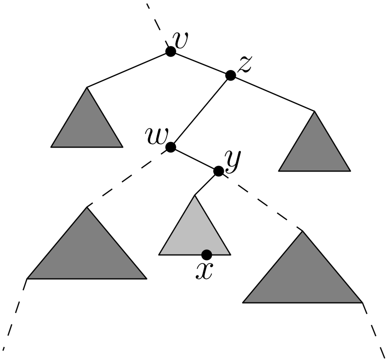

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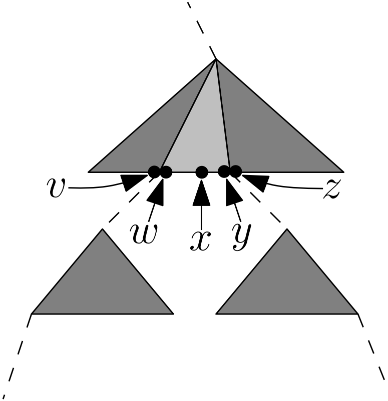

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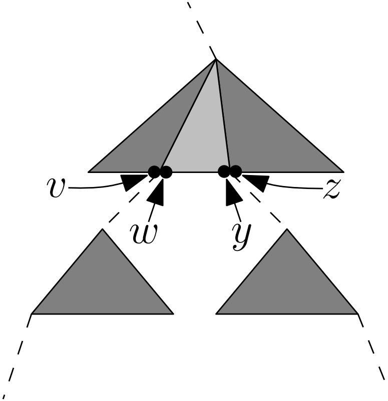

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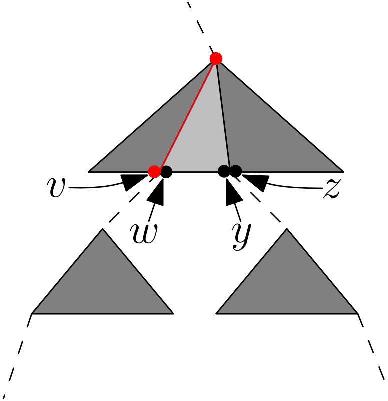

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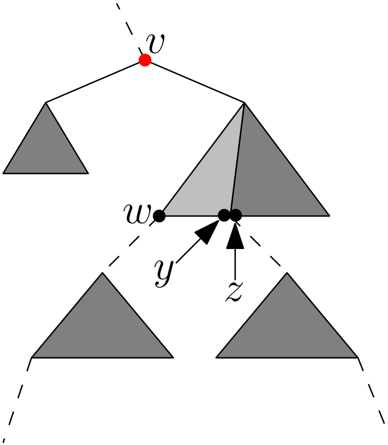

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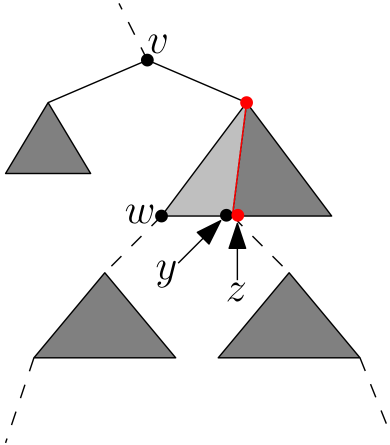

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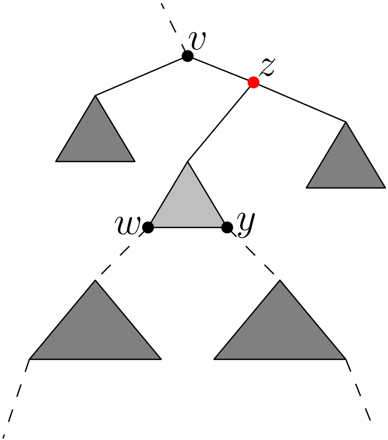

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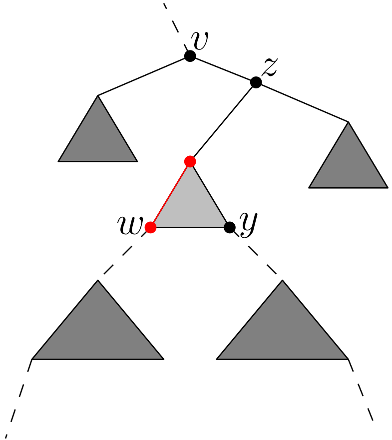

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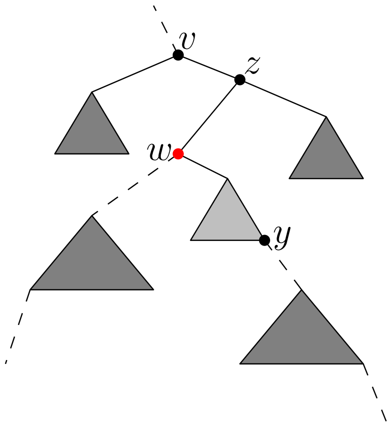

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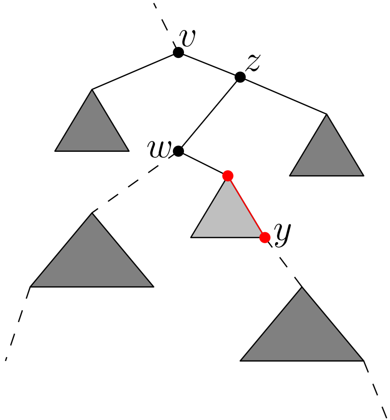

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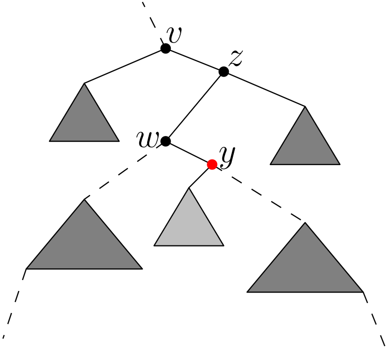

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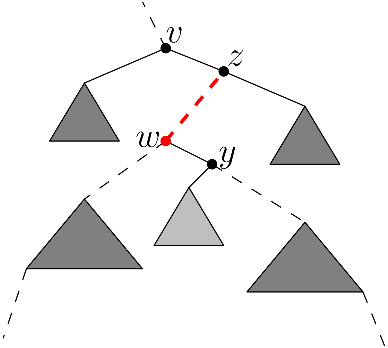

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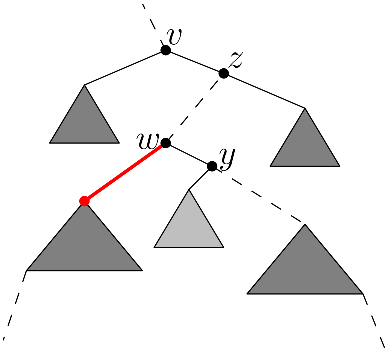

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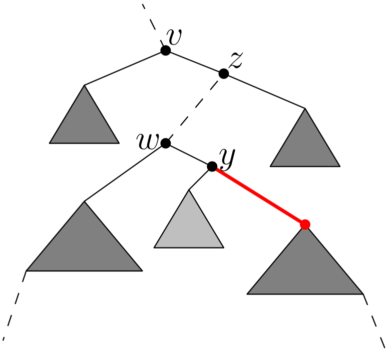

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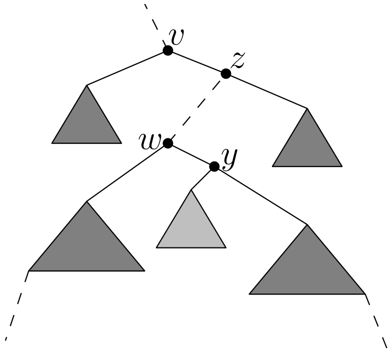

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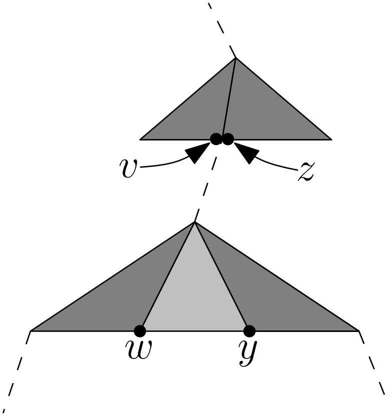

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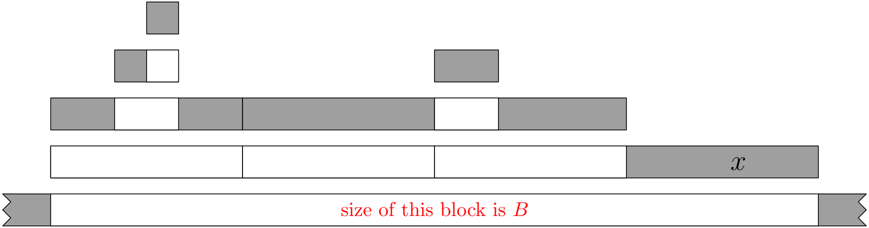
The Cost of a Query



$$\lg B + \tfrac{1}{2}\lg B + \tfrac{1}{4}\lg B + \tfrac{1}{4}\lg B + \tfrac{1}{2}\lg B + \lg B = O(\lg B)$$

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A query to level i costs $O(\lg b_i) = O(\lg(\text{time since queried}))$.

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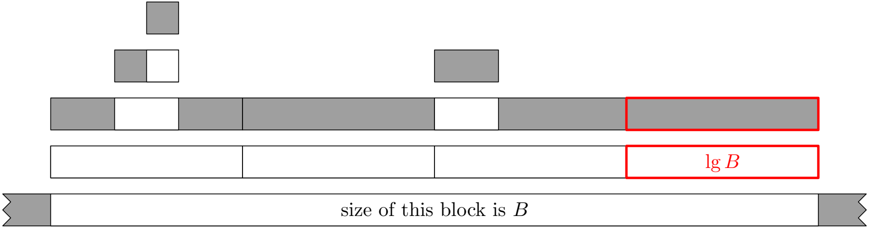

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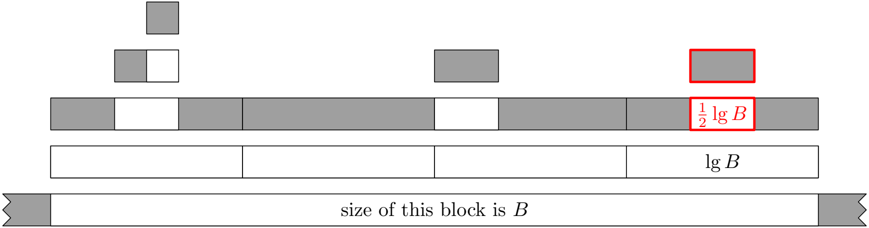

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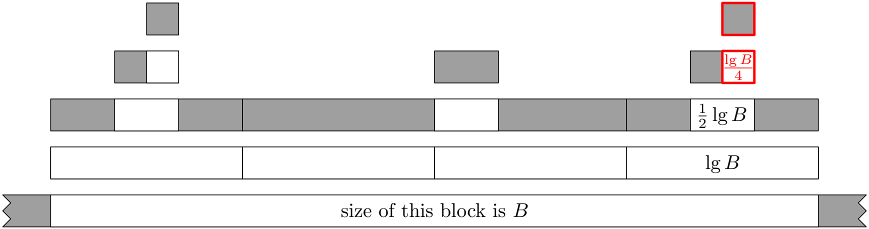

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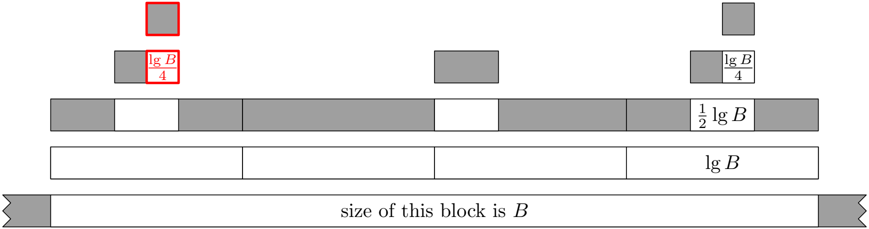

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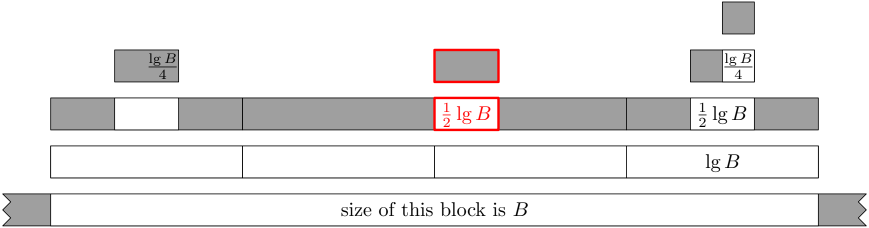

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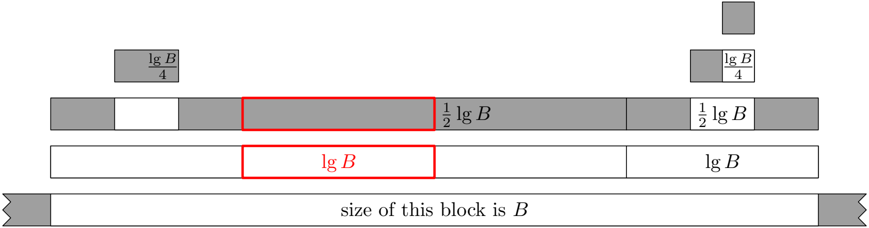

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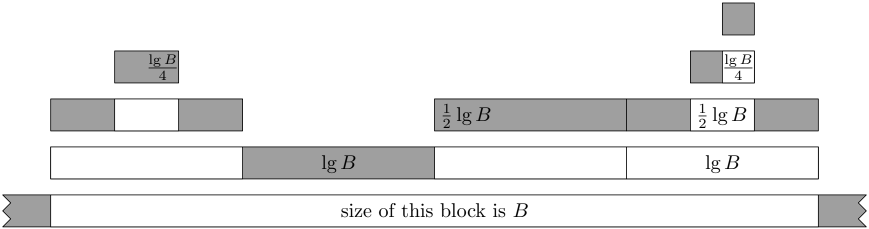

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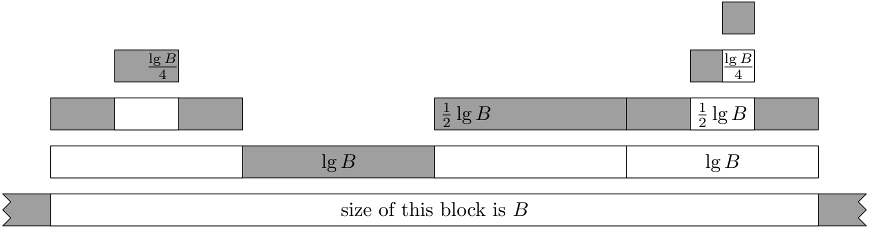

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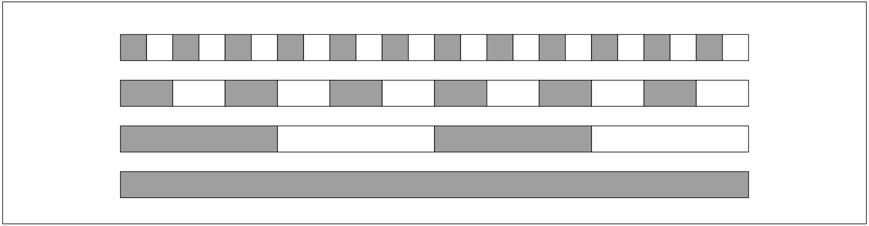

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A query to level i of the "virtual cache" costs amortized $O(\lg b_i)$, which is $O(\lg(\textit{time since virtual block queried}))$.

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Potential of $\lg b_j$ for each level-j block that is overlapped by a level-j virtual block that is at a higher level in the virtual cache. □

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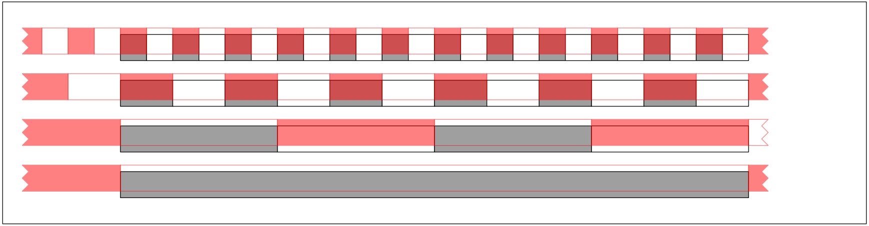

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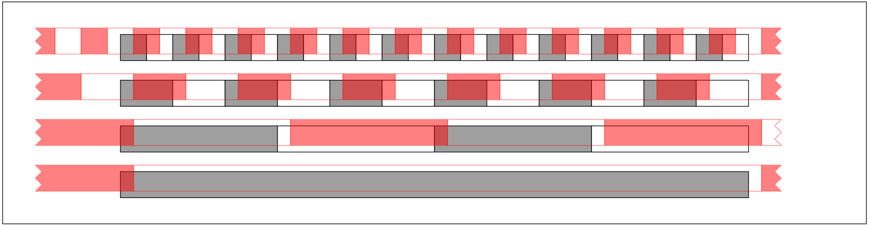

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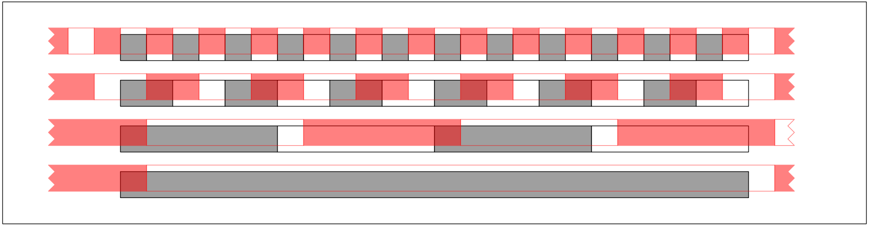

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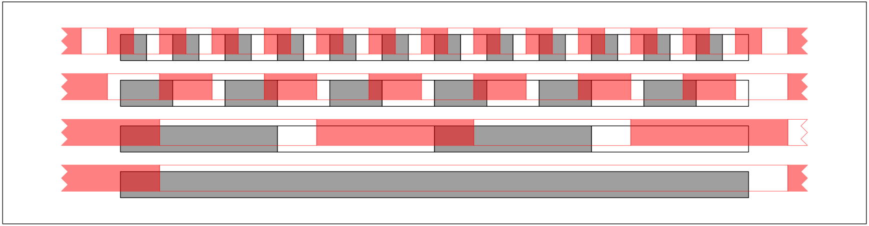

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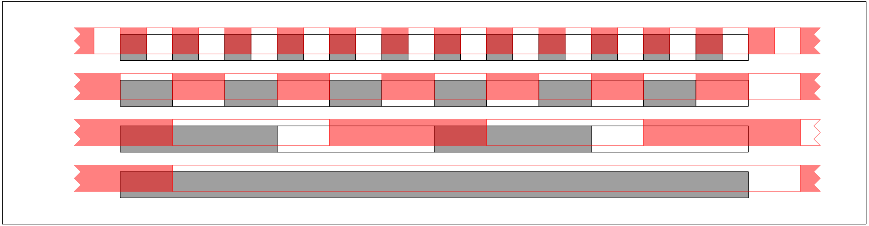

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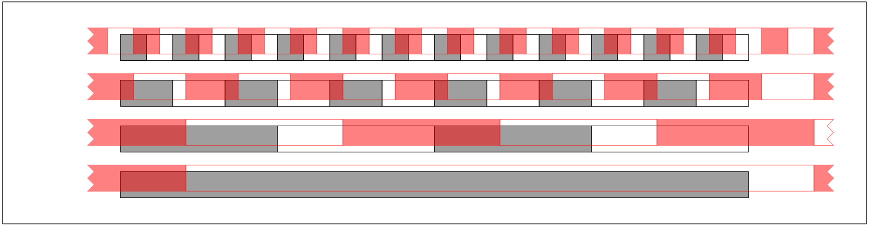

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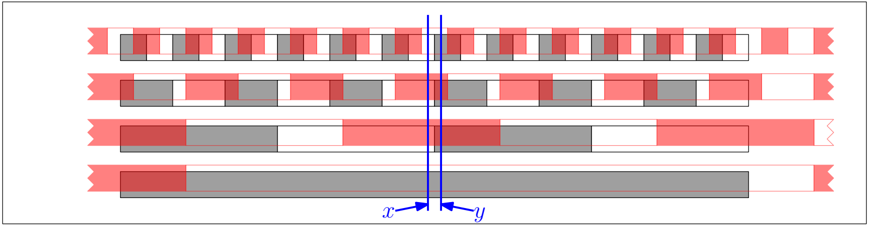

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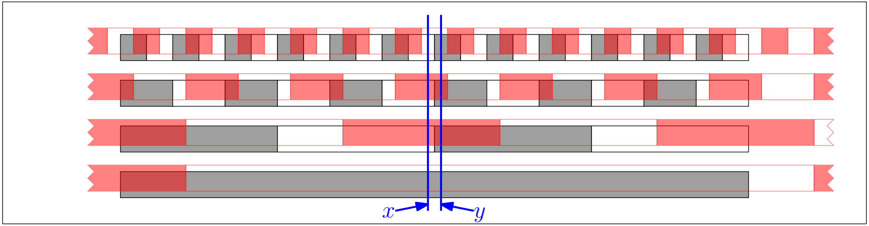

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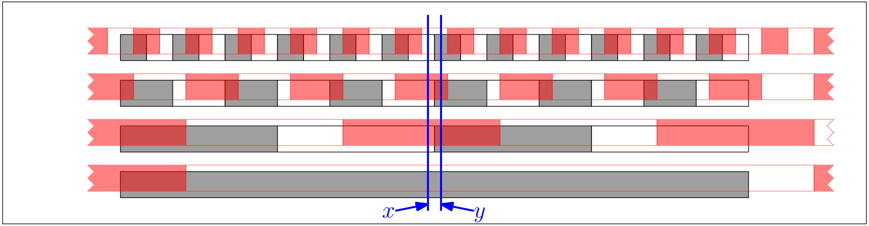

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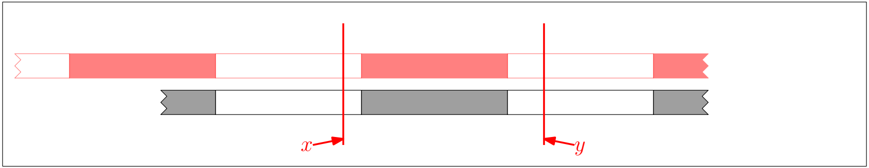

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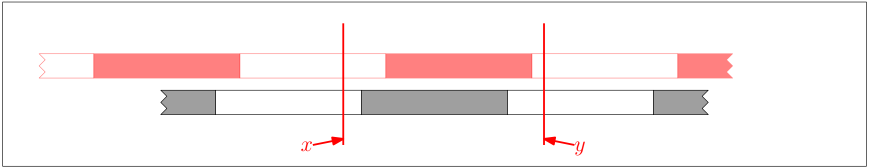

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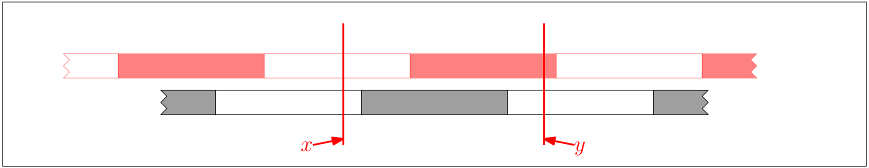

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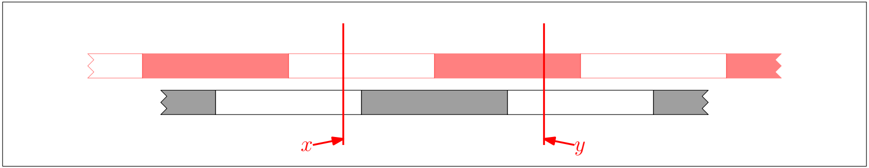

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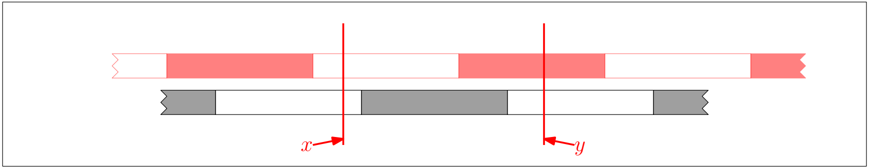

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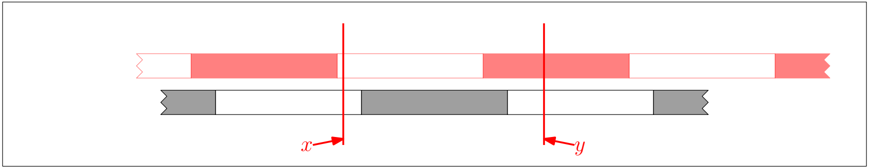

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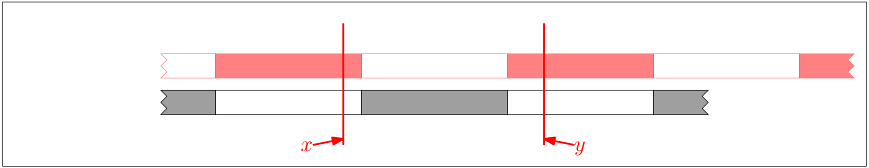

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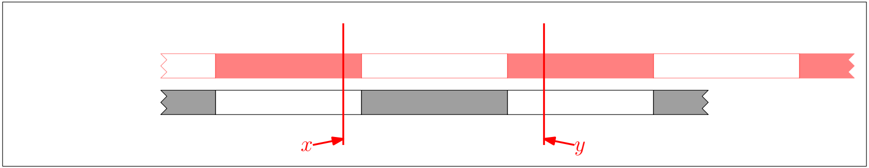

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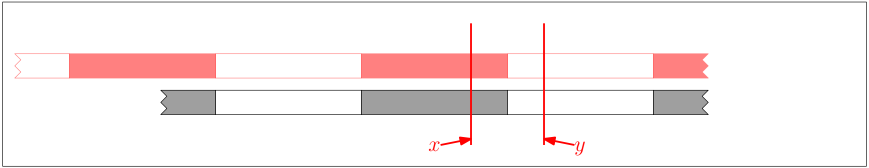

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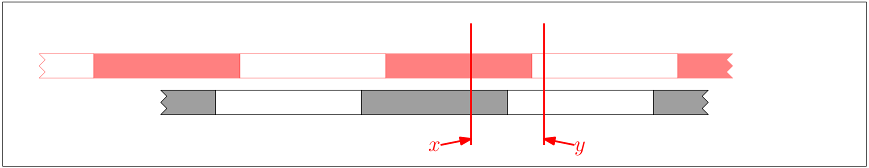

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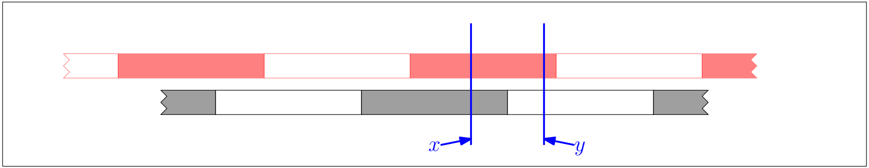

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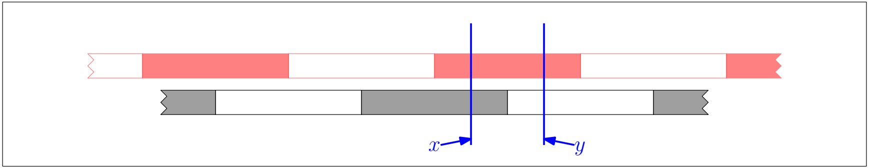

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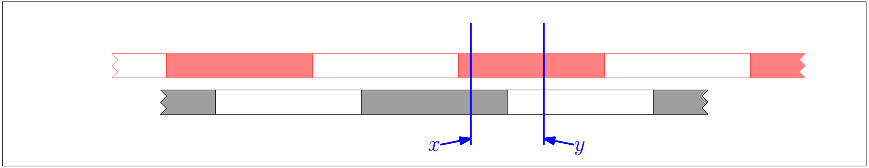

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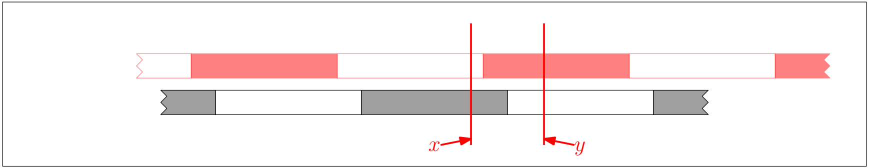

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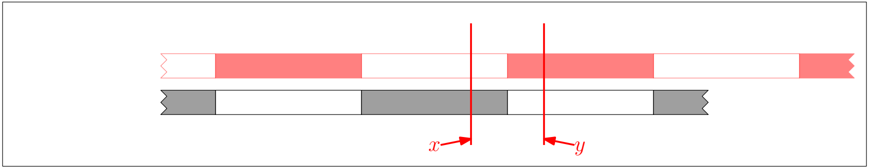

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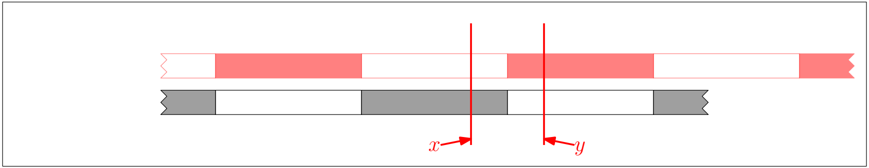

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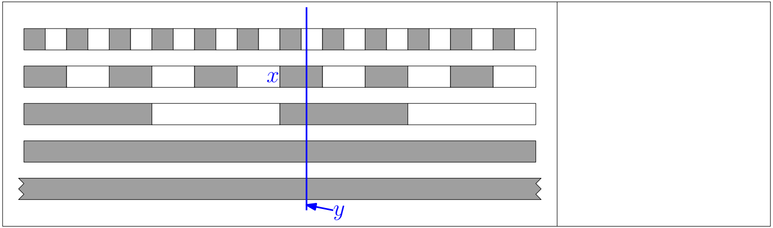
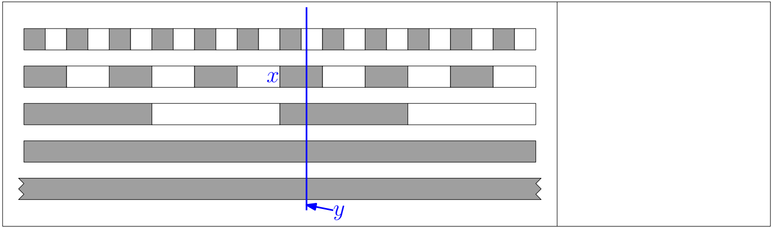
Cache-Splay Satisfies the Unified Bound



Proof.

- Suppose we query x, then t_x other keys, then y.
- Then x's level in virtual cache has block size $b_i = t_x^{O(1)}$.
- If $|x - y| < b_i$, then cost is expected $O(\lg t_x)$.
- If $|x - y| \geq b_i$, then cost is expected $O(\lg |x - y|)$.
- Cost is expected $O(\quad \lg t_x + \lg |x - y| \quad)$.

□



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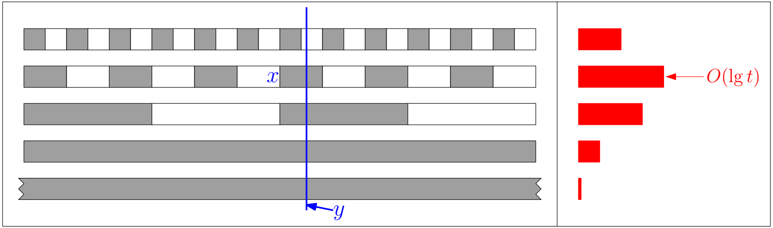

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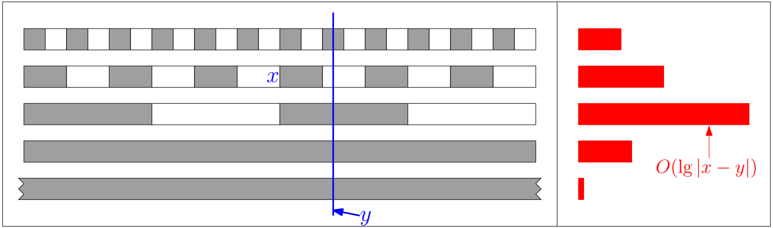

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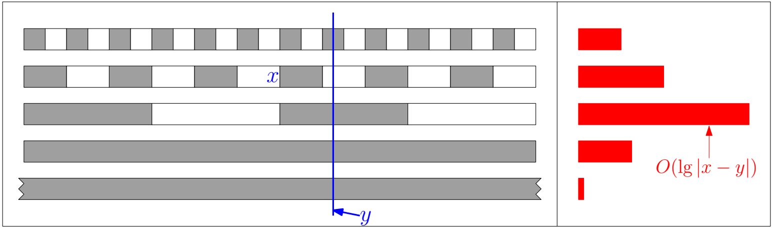

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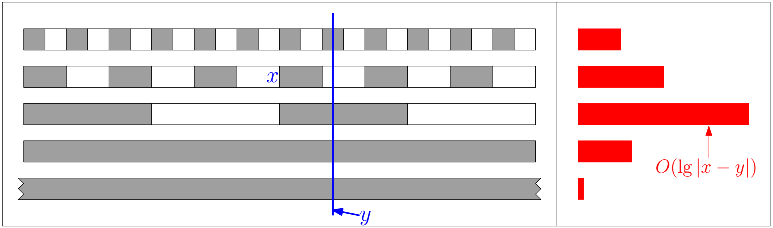

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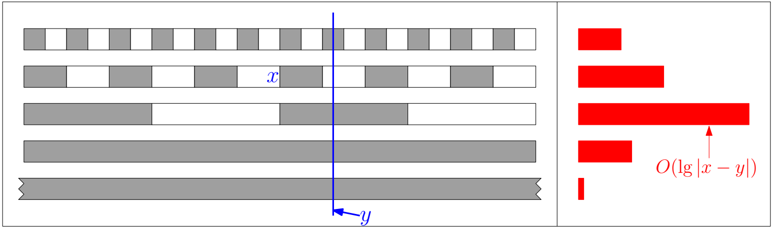
□

Cache-Splay Satisfies the Unified Bound



Proof.

- Suppose we query x, then t_x other keys, then y.
- Then x's level in virtual cache has block size $b_i = t_x^{O(1)}$.
- If $|x - y| < b_i$, then cost is expected $O(\lg t_x)$.
- If $|x - y| \geq b_i$, then cost is expected $O(\lg |x - y|)$.
- Cost is ~~expected~~ $O(\min_x(\lg t_x + \lg |x - y|))$.

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Implications

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- Search for even more general formulaic bounds?

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  - MIBS bound [DSW05]: generalization of Wilber's bounds that is computable in polynomial time
  - ...also shows the strength of the BST model because it is a lower bound for partial-sums (in thesis)
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- Better bounds for *any* BST: use better lower bounds to show $o(\lg \lg n)$-competitiveness for some BST, show that some formulaic bound that implies BST competitiveness to within a $o(\lg n)$ factor
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