McKay Tantalizers

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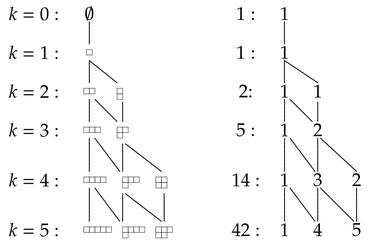
The $SL_2(\mathbb{C})$ tensor power centralizer algebra

We studied subgroups G of the special linear group $SL_2(\mathbb{C})$ and their natural representation on $V = \mathbb{C}^2$. We considered the k-fold tensor product representation $V^{\otimes k}$ and the centralizer algebra of all endomorphisms that commute with G on $V^{\otimes k}$, denoted $C_k^G = \mathrm{End}_G(V^{\otimes k})$.

For $G = SL_2(\mathbb{C})$, the irreducible representations of $SL_2(\mathbb{C})$ are indexed by integer partitions with at most two parts. Let V^λ denote the irreducible representation of $SL_2(\mathbb{C})$ indexed by partition λ. The natural representation $V = V^{(1)}$ is indexed by the partition (1). The rule for tensoring with V is

$$V^\lambda \otimes V = \bigoplus_{\mu=\lambda+(1)} V^\mu$$

Thus, the Bratteli diagram for $SL_2(\mathbb{C})$ is



A **Bratteli diagram** is a graph with the properties that

- the vertices on level k label the irreducible representation of C_k,
- the edges from level $k-1$ to level k encode the "restriction" rules for $C_{k-1} \subseteq C_k$ and
- $\dim C_k$ is the sum of the squares of the labels on level k.

The sums of the squares of the multiplicities on level k are the Catalan numbers. By the above, the dimensions of the irreducible modules of $C_k = C_k^{SL_2(\mathbb{C})}$ are the multiplicities of the irreducible $SL_2(\mathbb{C})$-modules in $V^{\otimes k}$. Thus $\dim C_k$ is the kth Catalan number.

We call C_k the **tantalizer algebra** of $SL_2(\mathbb{C})$, for *tensor power centralizer*.

Subgroups of $SL_2(\mathbb{C})$

Using the McKay correspondence, it can be shown that the Bratteli diagram of C_k^G can be explicitly constructed from the Dynkin diagram of G. Furthermore, every rank 2 complex reflection group in $SL_2(\mathbb{C})$ can be constructed from one of three groups:

- **T**, the binary tetrahedral group, order 24;
- **O**, the binary octahedral group, order 48, and
- **I**, the binary icosahedral group, order 120.

Since each of these is a subgroup of $SL_2(\mathbb{C})$, we know that the centralizer algebras will contain C_k.

The tetrahedral group

The tetrahedral group, **T**, is the subgroup of $GL_2(\mathbb{C})$ generated by

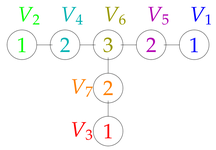
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix},$$

where $\epsilon = e^{2\pi i/8}$. **T** is order 24 and is isomorphic with the complex reflection group G_4, the subgroup of $GL_2(\mathbb{C})$ generated by

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad -\frac{\omega^2}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$.

From the character table of **T**, we can write down the tensor product rules under **T**, which correspond to the affine E_6 Dynkin diagram:

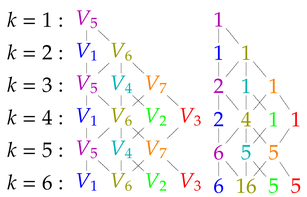


We use the **McKay correspondence** to generate this **Dynkin diagram**, which is a way to encode the tensor product rules. The tensor product rules for **T** are

$$V_1 \otimes V_5 = V_5, \qquad V_2 \otimes V_5 = V_4,$$
$$V_3 \otimes V_5 = V_7, \qquad V_4 \otimes V_5 = V_2 \oplus V_6,$$
$$V_5 \otimes V_5 = V_1 \oplus V_6, \qquad V_6 \otimes V_5 = V_4 \oplus V_5 \oplus V_7,$$
$$V_7 \otimes V_5 = V_3 \oplus V_6.$$

We generate the Dynkin diagram above by drawing an edge between V_i and V_j exactly when V_j appears in $V_i \otimes V_5$ (V_5 is the natural two-dimensional representation of **T**).

From this we can calculate the Bratteli diagram for **T** on $V^{\otimes k}$, the first six rows of which are



The dimension of $C_k^{\mathbf{T}}$ equals the sum of the multiplicities of the irreducibles in the kth row of the Bratteli diagram, so the first six dimensions are

$$1, 2, 6, 22, 86, 342.$$

We derived general formulae for the multiplicities of the irreducibles in the kth row of the Bratteli diagram and used these results to show that the dimension of the centralizer algebra is given in general by

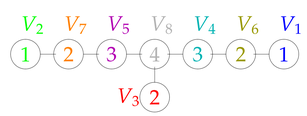
$$\dim C_k^{\mathbf{T}} = \frac{4^{k-1}+2}{3}.$$

The octahedral group

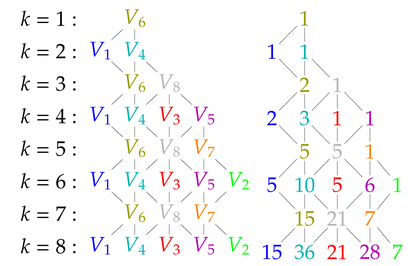
The octahedral group, **O**, is the subgroup of $GL_2(\mathbb{C})$ generated by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon \\ \epsilon^3 & \epsilon^7 \end{pmatrix}.$$

O has order 48. By taking the same approach to **O** as we took to **T**, we associate to it the affine E_7 Dynkin diagram.



The first eight rows of the Bratteli diagram for **O** on $V^{\otimes k}$ are then



We showed that the dimension of the centralizer algebra is given by

$$\dim C_k^{\mathbf{O}} = \frac{4^k + 6 \cdot 2^k + 8}{24}.$$

The first eight dimensions are

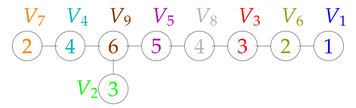
$$1, 2, 5, 15, 51, 187, 715, 2795.$$

The icosahedral group

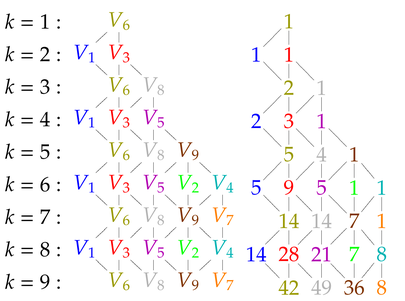
The icosahedral group, **I**, is the subgroup of $GL_2(\mathbb{C})$ generated by

$$\frac{1}{\sqrt{5}}\begin{pmatrix} \eta^4-\eta & \eta^2-\eta^3 \\ \eta^2-\eta^3 & \eta-\eta^4 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^2-\eta^4 & \eta^4-1 \\ 1-\eta & \eta^3-\eta \end{pmatrix},$$

where $\eta = e^{2\pi i/5}$. **I** is order 120. Via the McKay correspondence, we associate to **I** the affine E_8 diagram.



The first nine rows of the Bratteli diagram for **I** on $V^{\otimes k}$ are



We showed that the dimension of the centralizer algebra is

$$\dim C_k^{\mathbf{I}} = \frac{4^{k-1} + 3L_{2k} + 5}{15},$$

where L_k is the kth Lucas number ($L_1 = 1$, $L_2 = 3$, $L_{k+1} = L_{k-1} + L_k$). The first nine dimensions are

$$1, 2, 5, 14, 42, 133, 442, 1534, 5525.$$

Conclusions and conjectures

As discussed, we proved that the centralizer algebras C_k^G had dimensions given in the following table.

G	0	1	2	3	4	5	6	$\cdots$	k	$\cdots$	OEIS
T	1	1	2	6	22	86	342	$\cdots$	$\frac{4^{k-1}+2}{3}$	$\cdots$	A047849
O	1	1	2	5	15	51	187	$\cdots$	$\frac{4^k+6\cdot 2^k+8}{24}$	$\cdots$	A007581
I	1	1	2	5	14	42	133	$\cdots$	$\frac{4^{k-1}+3L_{2k}+5}{15}$	$\cdots$	not catalogued
$SL_2(\mathbb{C})$	1	1	2	5	14	42	132	$\cdots$	$C(k)$	$\cdots$	A000108

We conjecture that (for $G = \mathbf{T}, \mathbf{O}, \mathbf{I}$) the centralizer algebras C_k^G are generated by the permutations S_k along with a single other element b_G. This element b_G only appears for sufficiently large k ($k \geq s$, where s is the position from the right where the Dynkin diagram branches). For example, $b_{\mathbf{T}}$ first appears in $C_k^{\mathbf{T}}$ when $k = 3$; for $k < 3$, $C_k^{\mathbf{T}} = S_k$. We can form a basis for $C_3^{\mathbf{T}}$ by taking five linearly independent permutation matrices along with $b_{\mathbf{T}}$. We were able to determine values for $b_{\mathbf{T}}$ and $b_{\mathbf{O}}$ empirically. (Computing $b_{\mathbf{I}}$ was computationally infeasible.)

Similarly, we conjecture that C_{k+1}^G is generated by C_k along with s_k, the transposition that switches k and $k+1$.

This research has been funded in part by National Science Foundation grant DMS 0401098.