Announcements

Office hours for Joel have changed this week:
3 p.m – 5 p.m on Wednesday (NOT Tuesday as usual)

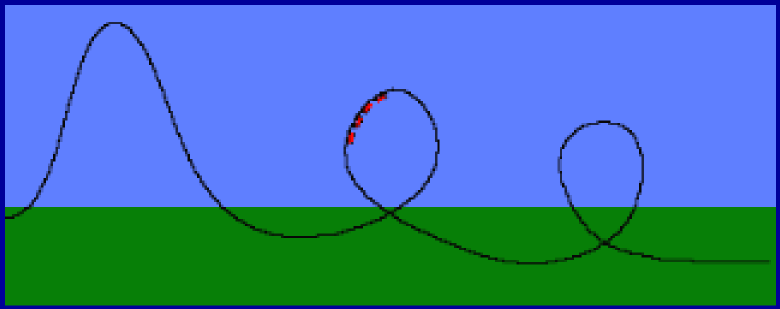
Michael is out this week

Written part of the second programming assignment is due this Thursday before the class or Friday before 9am in Jessica's mailbox

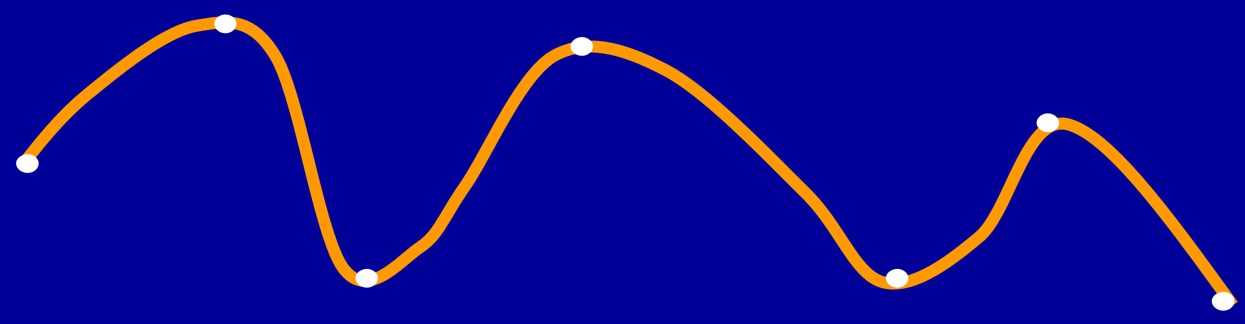
- Finish parametric curves (Splines)
- Physics of a mass point
- Basics of textures

Roller coaster

- Current programming assignment involves creating a 3D roller coaster animation
- We must model the 3D curve describing the roller coaster, but how?
- How to make the simulation obey the laws of gravity?

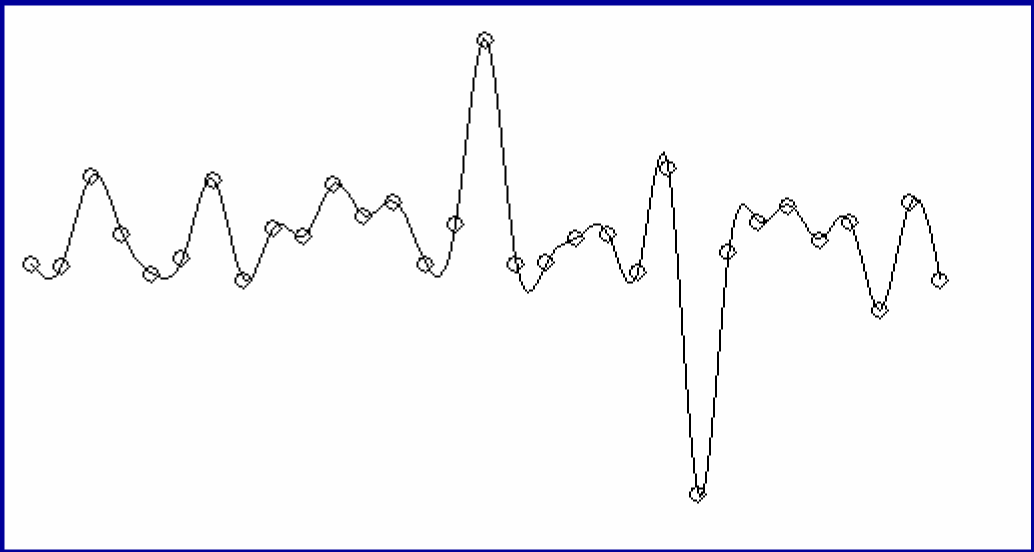


Model 3D curve for roller coaster



Splines: Piecewise Polynomials

- **A spline is a *piecewise polynomial* - many low degree polynomials are used to interpolate (pass through) the control points**
- ***Cubic piecewise* polynomials are the most common:**



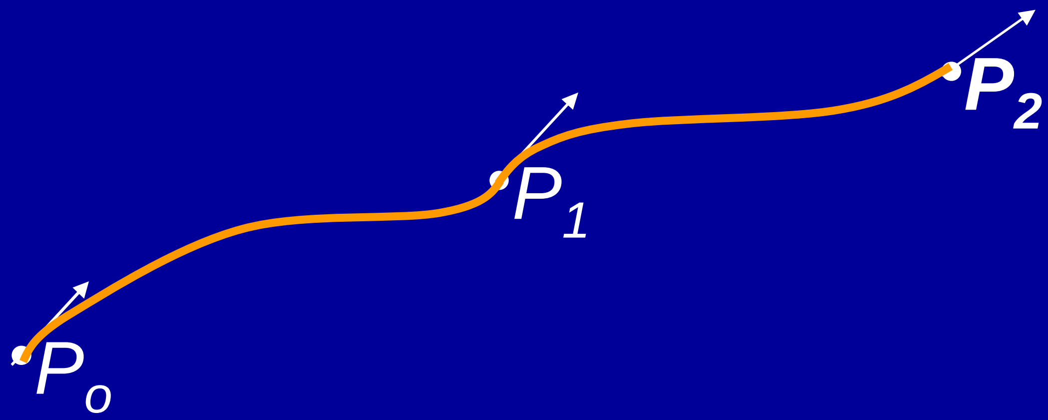
- **local control**
- **stability**
- **smoothness**
- **continuity**
- **easy to compute derivatives**

Splines

- **Types of splines:**
  - **Hermite Splines**
  - **Catmull-Rom Splines**
  - **Bezier Splines**
  - **Natural Cubic Splines**
  - **B-Splines**
  - **NURBS**

Hermite Curves

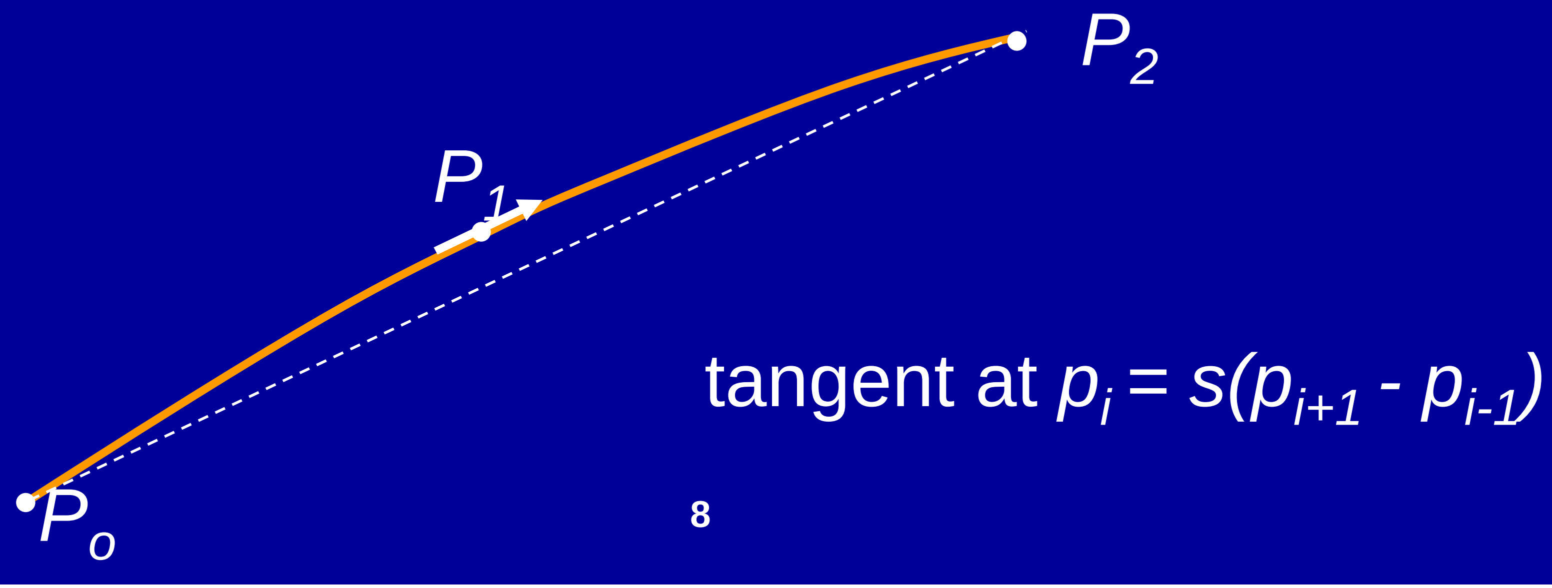
- **Cubic Hermite Splines**



That is, we want a way to specify the end points and the slope at the end points!

Catmull-Rom Splines

- **Use for the roller-coaster assignment**
- **With Hermite splines, the designer must specify all the tangent vectors**
- **Catmull-Rom: an interpolating cubic spline with *built-in C^1 continuity*.**



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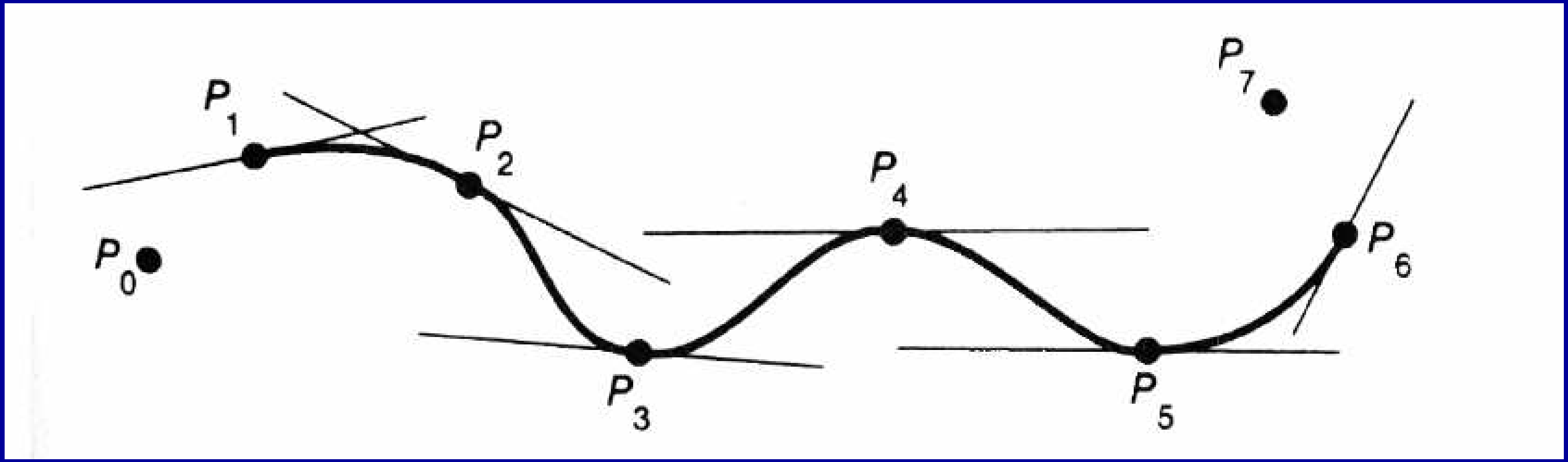


Catmull-Rom Spline Matrix

$$p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

CR basis **control vector**

- **Derived similarly to Hermite**
- **Parameter s is typically set to s=1/2.**

Cubic Curves in 3D

- **Three cubic polynomials, one for each coordinate**
  - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$
  - $y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
  - $z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$
- **In matrix notation**

$$[x(u) \quad y(u) \quad z(u)] = [u^3 \quad u^2 \quad u \quad 1] \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

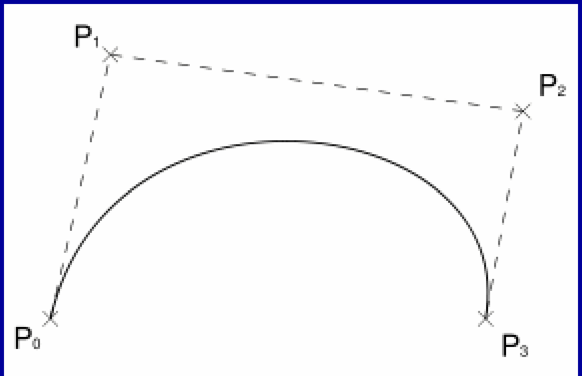
Catmull-Rom Spline Matrix in 3D

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

CR basis **control vector**

Bezier Curves*

- Another variant of the same game
- Instead of endpoints and tangents, four control points
  - points P0 and P3 are on the curve: P(u=0) = P0,   P(u=1) = P3
  - points P1 and P2 are off the curve
  - P'(u=0) = 3(P1-P0), P'(u=1) = 3(P3 – P2)
- Convex Hull property
  - curve contained within convex hull of control points
- Gives more control knobs (series of points) than Hermite
- Scale factor (3) is chosen to make "velocity" approximately constant



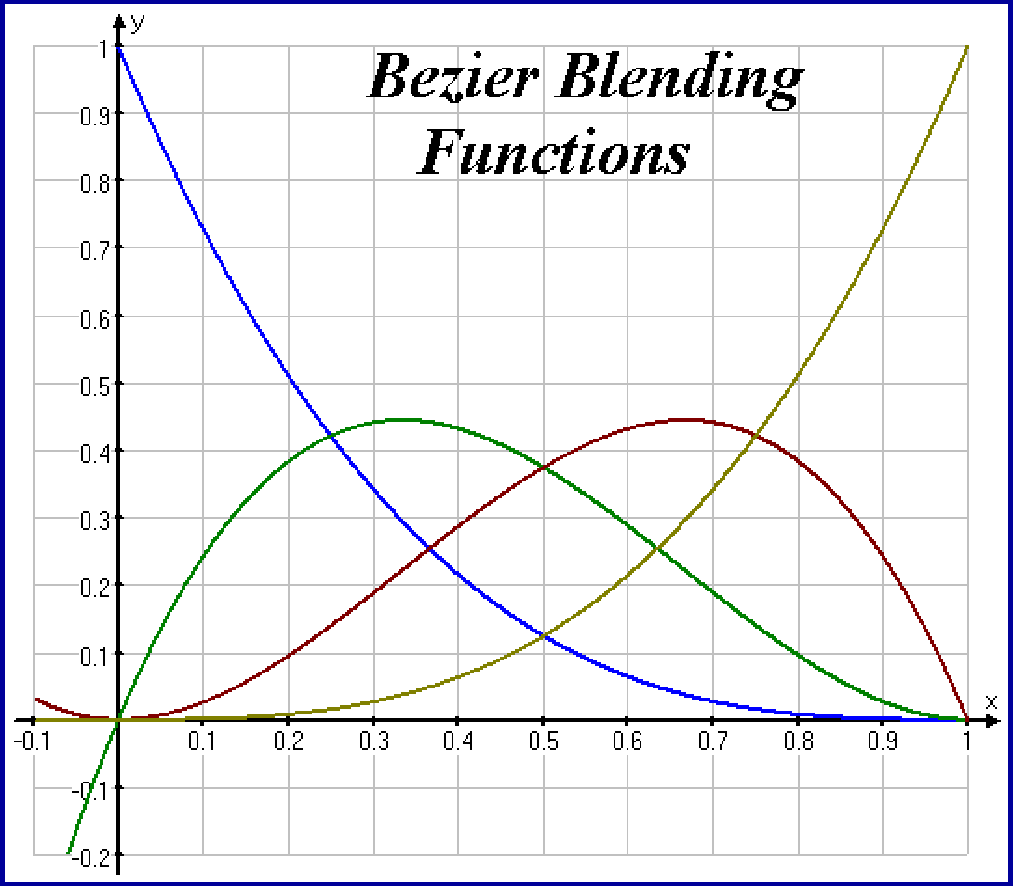
The Bezier Spline Matrix*

$$
\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}
$$

Bezier basis

Bezier control vector

Bezier Blending Functions*



$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

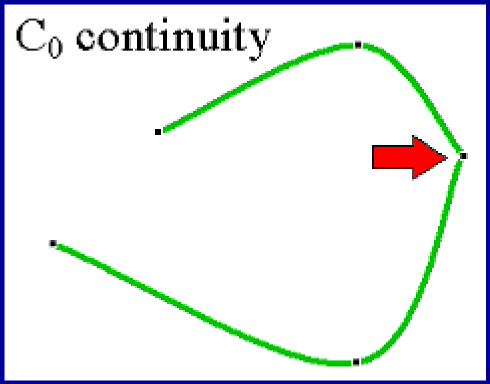
Also known as the order 4, degree 3 Bernstein polynomials

Nonnegative, sum to 1

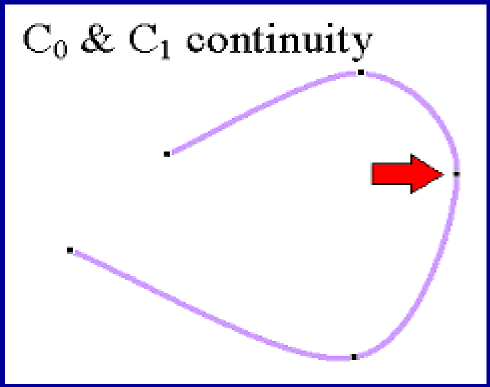
The entire curve lies inside the polyhedron bounded by the control points

Piecewise Polynomials

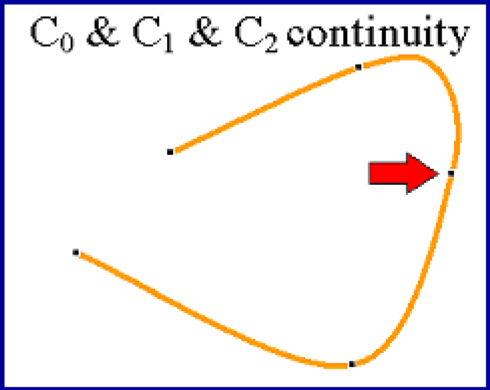
- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely



Continuous in position



Continuous in position and tangent vector



Continuous in position, tangent, and curvature

Splines with More Continuity?

- **How could we get C^2 continuity at control points?**

- **Possible answers:**
  - **Use higher degree polynomials**

    degree 4 = quartic, degree 5 = quintic, … but these get computationally expensive, and sometimes wiggly
  - **Give up local control à natural cubic splines**

    A change to any control point affects the entire curve
  - **Give up interpolation à cubic B-splines**

    Curve goes near, but not through, the control points

Comparison of Basic Cubic Splines

Type	Local Control	Continuity	Interpolation
Hermite	YES	C1	YES
Bezier	YES	C1	YES
Catmull-Rom	YES	C1	YES
Natural	NO	C2	YES
B-Splines	YES	C2	NO

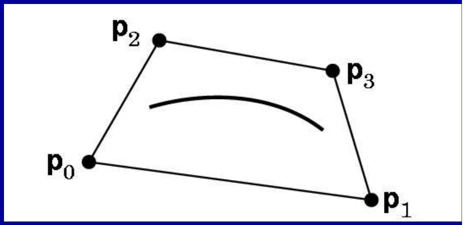
- **Summary**
  - **Can't get C2, interpolation and local control with cubics**

Natural Cubic Splines*

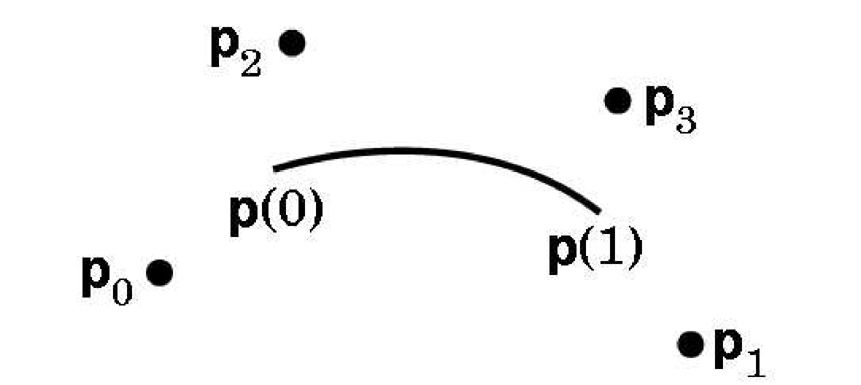
- **If you want 2nd derivatives at joints to match up, the resulting curves are called *natural cubic splines***
- **It's a simple computation to solve for the cubics' coefficients. (See *Numerical Recipes in C* book for code.)**
- **Finding all the right weights is a *global* calculation (solve tridiagonal linear system)**

B-Splines*

- **Give up interpolation**
  - **the curve passes *near* the control points**
  - **best generated with interactive placement (because it's hard to guess where the curve will go)**



- **Curve obeys the convex hull property**
- **C2 continuity and local control are good compensation for loss of interpolation**

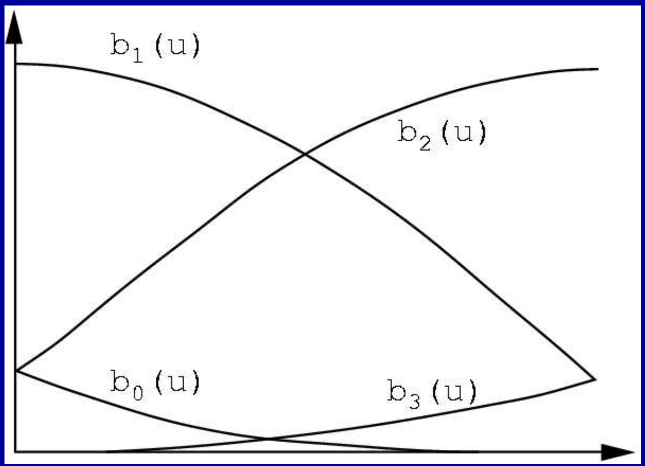


B-Spline Basis*

- We always need 3 more control points than spline pieces

$$M_{Bs} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$G_{Bsi} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



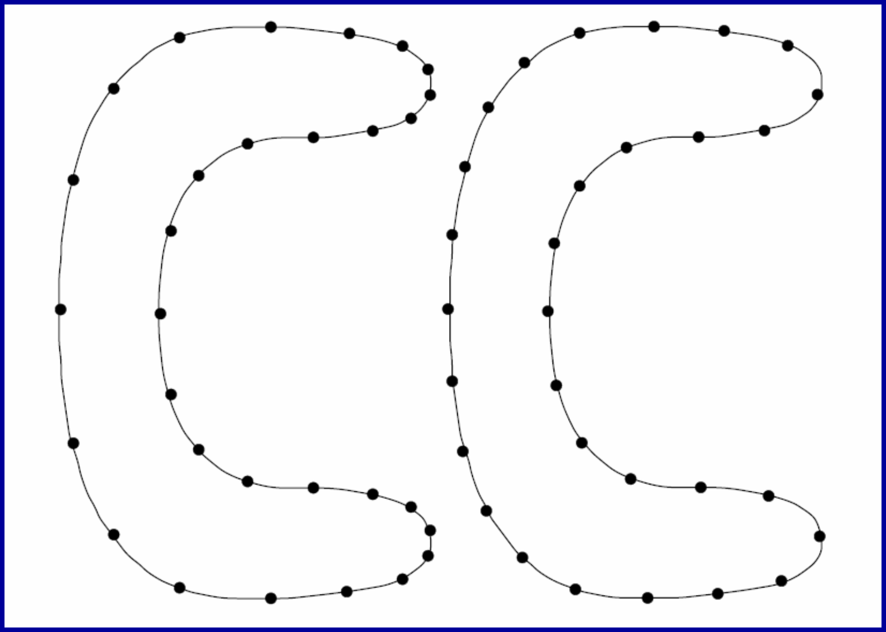
How to Draw Spline Curves

- **Basis matrix eqn. allows same code to draw any spline type**
- **<u>Method 1</u>: brute force**
  - **Calculate the coefficients**
  - **For each cubic segment, vary *u* from *0* to *1* (fixed step size)**
  - **Plug in *u* value, matrix multiply to compute position on curve**
  - **Draw line segment from last position to current position**

$$[x \;\; y \;\; z] = [u^3 \;\; u^2 \;\; u \;\; 1] \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\text{CR basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\text{control vector}}$$

How to Draw Spline Curves

- **What's wrong with this approach?**
  - **–Draws in even steps of u**
  - **–Even steps of u $\neq$ even steps of x**
  - **–Line length will vary over the curve**
  - **–Want to bound line length**
    - **»too long: curve looks jagged**
    - **»too short: curve is slow to draw**

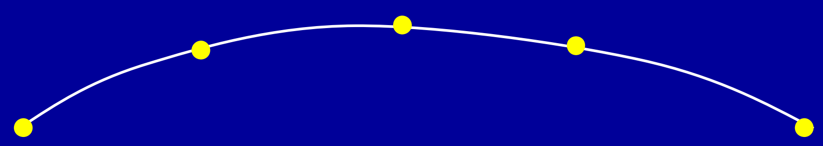


Drawing Splines, 2

- **Method 2: recursive subdivision - vary step size to draw short lines**

```
Subdivide(u0,u1,maxlinelength)
    umid = (u0 + u1)/2
    x0 = P(u0)
    x1 = P(u1)
    if |x1 - x0| > maxlinelength
       Subdivide(u0,umid,maxlinelength)
       Subdivide(umid,u1,maxlinelength)
    else drawline(x0,x1)
```

- **Variant on Method 2 - subdivide based on curvature**
  - **replace condition in "if" statement with straightness criterion**
  - **draws fewer lines in flatter regions of the curve**



In Summary...

- **Summary:**
  - **piecewise cubic is generally sufficient**
  - **define conditions on the curves and their continuity**

- **Things to know:**
  - **basic curve properties (what are the conditions, controls, and properties for each spline type)**
  - **generic matrix formula for uniform cubic splines $x(u) = uBG$**
  - **given definition derive a basis matrix**