Speculations on Weak ω-structures

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Abstract

There are some interesting definitions of algebraic structures that can be made with relatively little machinery beyond the usual definition of strict n-categories, which yield *something* like weak ω-categories, not that I'm claiming any particular equivalence. Of particular interest is how higher associativity laws and coherence properties possibly 'fall out' of composability of cells one dimension up.

The main question is, in what sense is might this definition be 'too strict', say, at dimensions 3 and above where not every weak category can be strictified?

General Disclaimer: I am not an expert in weak ω-category theory. It is entirely possible what is discussed here is not original, not correct, or both.

1 Preliminaries

The tower of n-categories is defined as usual by saying that a 0-category is a set, and an $n+1$-category is a category enriched over the evident category $n\mathbf{Cat}$ of n-categories for every $n \geq 0$.

A strict ω category is a globular set $(S_n)_{n\in\mathbb{N}}$ of n-cells such that every prefix of it forms a strict n-category in a coherent way.

We write the maps that return the codomain and domain of cells as c and d. We write ∂ as a variable varying over the set $\{c, d\}$. By the definition of globular sets, $cc = cd$ and $dd = dc$. For this reason, we can unambiguously write c^n and d^n for a string of n (co)domain operations, where the outermost is, respectively, codomain or domain. For the composition that is '$n-1$ times more horizontal' than 'purely vertical' composition, we write $\circ_n$. We write $\circ_1$ ('purely vertical' composition) sometimes as $\circ$ and sometimes as mere juxtaposition. It makes sense when the two cells being composed share a boundary which is n dimensions smaller than them: $f \circ_n g$ is well-defined if $c^n g = d^n f$. We have of course that and $d(f \circ g) = dg$ and $c(f \circ g) = cf$ and $\partial(f \circ_{n+1} g) = \partial f \circ_n \partial g$ for $n \geq 1$.

The notation $\mathrm{id}^n A$ is defined in the obvious way by $\mathrm{id}^0 A = A$ and $\mathrm{id}^{n+1} A = \mathrm{id}_{\mathrm{id}^n A}$.

2 Equivalence

Inside a strict ω-category, some arrows are very strongly invertible: those for which there is an infinite proof tree using the following rule.

$$\frac{d : A \to B \qquad d^* : B \to A \qquad m_A : dd^* \equiv \mathrm{id}_B \qquad m_B : d^*d \equiv \mathrm{id}_A}{d : A \equiv B}$$

That is, d is an equivalence between A and B if it has a pseudoinverse d^* such that both composites of d and d^* admit an equivalence to the appropriate identity arrow. That $\equiv$ is an equivalence relation can be easily shown by a coinductive argument.

3 Weak ω-structures

3.1 Definition

A *weak ω-structure* is a strict ω-category $(S_n)_{n\in\mathbb{N}}$ together with specified subsets $(W_n)_{n\in\mathbb{N}}$ such that

(i) If $w \in W_{n+1}$, then $\partial w \in W_n$, for all $n \in \mathbb{N}$.

(ii) If $s \in S_n$ then there exist $w \in W_n$ and $d \in S_{n+1}$ such that $d : s \equiv w$.

The intuition is that the cells W_n are the 'real' cells that belong to the structure, but that in order to talk about composition, we want to talk about pasting diagrams constructed out of those cells. The compositions of the strict n-category are the operations with which we construct pasting diagrams; they satisfy associative, interchange and unit laws on the nose. The axiom (i) is simple closure of the 'real' cells under domain and comdain, and (ii) is a sort of reflection principle that finds an image of the strict category in the weak structure, implying that every pasting diagram has a (not necessarily unique, but appropriately canonical) composite.

3.2 Composition

Naïvely, to take a composite in the weak structure, we just take the composite in the strict category and reflect: say[1] $f *_n g = h$ if $f \circ_n g \equiv h$. However, this doesn't quite seem appropriate for compositions that are at all horizontal, i.e. for $n \geq 2$. This is because the (co)domain $\partial(f *_{n+1} g)$ of $f *_{n+1} g$ should be $\partial f *_n \partial g$, not the result of *strict* composition $\partial f \circ_n \partial g$, which is what we get from the above recipe.

Instead we want to allow some 'slackening' equivalence cells to mediate between the strict composition and the weak composition of the domain and

[1] Really what is meant here despite the equals sign is a ternary relation on f, g, h. Weak composition is absolutely *not* a unique function that outputs h from f, g, but a relation which characterizes which h count as composites of f, g

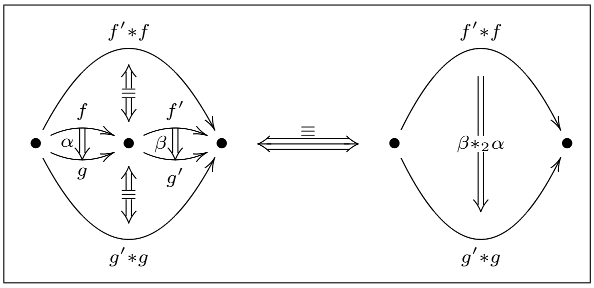

Figure 1: Weak Composition

codomain of the cells being horizontally composed. For example, for the $*_2$ ('once horizontal') composite of two cells, it should be any cell that is equivalent to the pasting diagram on the left side of Figure 1. The blame for this problem could be laid on $\equiv$ for only relating cells that have the same domain and codomain; we therefore generalize $\equiv$ to $\equiv_n$ (with $\equiv_1$ being $\equiv$) which is a relation between cells x, y that satisfy $\partial^n x = \partial^n y$. For $n \geq 1$, this is defined by

$$\frac{\alpha : d^n y \equiv d^n x \qquad \beta : c^n x \equiv c^n y \qquad d : \mathrm{id}_\beta^{n-1} \circ_n x \circ_n \mathrm{id}_\alpha^{n-1} \equiv_n y}{d : x \equiv_{n+1} y}$$

We can now officially define weak composition as a three-place relation by

$$\frac{f \circ_n g \equiv_n h}{*_n(f, g, h)}$$

We also should revise condition (ii) above to

(ii') If $s \in S_{n+m}$ and $\partial^m s \in W_n$, then there exist $w \in W_{n+m}$ and $d \in S_{n+m+1}$ such that $d : s \equiv_m w$.

(This will correctly guarantee that all composites at least *exist*)

Finally we can define the relation of what it means to be a weak identity:

$$\frac{f \equiv \mathrm{id}_A}{\mathrm{id}(A, f)}$$

Question How reasonable is it to make the following definition?

A *weak ω-category* is a globular set of cells $(W_n)_{n\in\mathbb{N}}$ together with relations $*_n(f, g, h)$ and $\mathrm{id}(A, f)$ that arises from the construction above, forgetting which S_n was used.

In any such structure, we do get some laws which are satsfied just by our ability to compose higher-dimensional cells. For instance, $*$ is weakly associative. For if we have some n-cells f, g, h sitting around, and we are able to compose

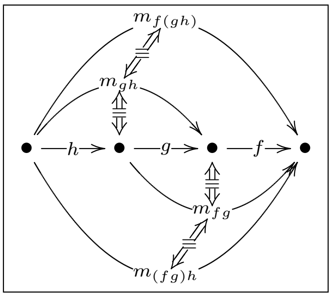

Figure 2: Weak Associativity

them and find out that $*(f, g, m_{fg})$ and $*(g, h, m_{gh})$ and $*(f, m_{gh}, m_{f(gh)})$ and $*(m_{fg}, h, m_{(fg)h})$, then we can fill in the diagram in Figure 2.

Note that the strict composite $f \circ g \circ h$ may not exist as a cell in W_n at all, but we can nonetheless build up (in S_{n+1}) the cell depicted by the whole diagram which stretches from $m_{f(gh)}$ to $m_{(fg)h}$, (which by assumption exist in W_n) and so that cell must have some reflection in W_{n+1}.

4 Conclusion

So this notion of weak ω-category is not completely obviously broken yet — we don't require associativity on the nose, for instance. But it's hard to picture what goes on at higher dimensions, and specifically I don't know how to formulate the following formally:

Question If we chop everything off at dimension $1, 2, 3$, is this at least equivalent to the commonly accepted definition of category, bicategory, tricategory?

An immediate obstacle is that composition is relational rather than functional. But I seem to recall that this was not unheard-of in the weak n-category literature.