Focusing as Token-Passing

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Abstract

Focused proof-search is ordinarily presented as a proof theory unto itself. We show how it can also be presented in the form of a translation of the language of polarized propositions into an ordinary unfocused logic with two simple connectives that embody a bounded amount of focusing reasoning on the creation and consumption of linear atomic propositions. These atomic propositions function as tokens in a token-passing scheme that can be seen to embody the restrictions on proof-search that focusing imposes. For example, to begin another focus phase while one is in progress — which should not be possible — would require consuming a linear token that is not yet available.

1 Introduction

Focusing [And92] is a phenomenon discovered by Andreoli in the setting of reducing nondeterminism in proof search, but it is increasingly clear that it a deep property of all well-behaved logics.

One first observes that all the connectives of intuitionistic linear logic are either *invertible*, on the left of the turnstile, that is, their left sequent rule's conclusion implies its premise(s), or else invertible on the right. The left-invertible connectives are $\otimes, \oplus, 1, 0$, and the right-invertible connectives are $\&, \multimap, \top$.

To have a focusing discipline is to impose two sorts of requirements on proofs. An *inverting proof* is one in which, when read bottom-up[1] , invertible rules are *required* to be applied eagerly, preferring right decompositions to left, and performing left decompositions in order from right to left in the context.

A *focusing* proof, Andreoli's key novelty, goes farther still and demands not only that these invertible rules be applied eagerly, but also that the remaining rules are used in uninterrupted sequences working on a single proposition until they reach asynchronous decompositions again. In the context of focusing, these invertible rules are also named *asynchronous*, and the noninvertible rules named *synchronous*.

They key fact is that these restrictions are sound and complete. If a sequent has a proof, it is a focused proof, but very probably it has many *fewer* proofs,

[1] In general, *all* descriptions herein of things happening in some temporal order are to be understood relative to the bottom-up process of proof search.

and therefore focusing is practically useful for automated reasoning, because it eliminates many redundant and dead-end proofs.

The soundness of the focusing discipline is easy to see, since they amount to restrictions of ordinary proof. However, the completeness of focusing is traditionally more difficult, depending on unpleasant per-connective lemmas about how to permute their rules around inside larger derivations. In general, the amount of effort is quadratic in the number of logical connectives in the system, since one considers all ways one connective can permute past another.

Our present aim is to deliver a simple inductive proof of the completeness of focusing whose 'cognitive complexity' is linear in the number of connectives. More to the point, one which prescribes in a *modular* way what it is about each given connective that allows it to fit well into the larger focusing picture. We do this via a reduction of the language and proof theory of focusing to a token-passing discipline expressible in an unfocused first-order intuitionistic linear logic, so that violations in the focusing discipline correspond to attempts to consume linear tokens that are not yet available. The proof obligation for each connective is a constant number of cases threaded through each of the various lemmas below, which show it appropriately commutes with token-passing operations.

This work is more or less a sequel to [Ree08]. All of the intellectual debts confessed to therein apply equally well here.

2 Focused Language

The source language has polarized linear logic propositions, including polarized atoms:

Positive Propositions	P	$::=$	$\downarrow N \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid a^+$
Negative Propositions	N	$::=$	$\uparrow P \mid P \multimap N \mid N \mathbin{\&} N \mid \top \mid a^-$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, N \mid \Gamma, a_+$
Ordered Contexts	Ω	$::=$	$\cdot \mid \Omega, P$
Conclusions	Q	$::=$	$P \mid a_-$

The operators $\uparrow P$ and $\downarrow N$ are *polarity shifts*, which include each polarity of propositions in the other. Focus and inversion both stop when (and only when) a shift is reached. The expressions a^+, a^- are positive and negative *propositions*, while a_+, a_- are the *stabilized judgmental forms* of atoms, which are allowed in contexts and conclusions, respectively. This slightly unorthodox treatment of atoms has the benefit of making atoms *as propositions* behave more uniformly with respect to other propositions: every negative proposition, whether atomic or not, performs some asynchronous decomposition on the right before synchronously decomposing on the left, and vice-versa for positive propositions.

Ordinary contexts Γ are understood to be intrinsically unordered multisets despite necessarily being linearized on the written page, but contexts Ω are understood as ordered lists.

$$\frac{\Gamma;\Omega,P \vdash_f N}{\Gamma;\Omega \vdash_f P \multimap N}\multimap R \qquad \frac{\Gamma_1 \vdash_f [P] \qquad \Gamma_2[N] \vdash_f Q}{\Gamma_1,\Gamma_2[P \multimap N] \vdash_f Q}\multimap L \qquad \frac{\Gamma;\Omega \vdash_f N_1 \qquad \Gamma;\Omega \vdash_f N_2}{\Gamma;\Omega \vdash_f N_1 \mathbin{\&} N_2}\mathbin{\&}R$$

$$\frac{\Gamma[N_i] \vdash_f Q}{\Gamma[N_1 \mathbin{\&} N_2] \vdash_f Q}\mathbin{\&}L \qquad \frac{\Gamma_1 \vdash_f [P_1] \qquad \Gamma_2 \vdash_f [P_2]}{\Gamma_1,\Gamma_2 \vdash_f [P_1 \otimes P_2]}\otimes R \qquad \frac{\Gamma;\Omega,P_1,P_2 \vdash_f Q}{\Gamma;\Omega,P_1 \otimes P_2 \vdash_f Q}\otimes L$$

$$\frac{\Gamma \vdash_f [P_i]}{\Gamma \vdash_f [P_1 \oplus P_2]}\oplus R_i \qquad \frac{\Gamma;\Omega,P_1 \vdash_f Q \qquad \Gamma;\Omega,P_2 \vdash_f Q}{\Gamma;\Omega,P_1 \oplus P_2 \vdash_f Q}\oplus L \qquad \frac{}{\vdash_f [1]}1R \qquad \frac{\Gamma;\Omega \vdash_f Q}{\Gamma;\Omega,1 \vdash_f Q}1L$$

$$\frac{}{\Gamma;\Omega,0 \vdash_f Q}0L \qquad \frac{\Gamma;\cdot \vdash_f N}{\Gamma \vdash_f [\downarrow N]}\downarrow R \qquad \frac{\Gamma,N;\Omega \vdash_f Q}{\Gamma;\Omega,\downarrow N \vdash_f Q}\downarrow L \qquad \frac{\Gamma;\Omega \vdash_f P}{\Gamma;\Omega \vdash_f \uparrow P}\uparrow R \qquad \frac{\Gamma;P \vdash_f Q}{\Gamma[\uparrow P] \vdash_f Q}\uparrow L$$

$$\frac{}{a_+ \vdash_f [a^+]}a^+R \qquad \frac{\Gamma,a_+;\Omega \vdash_f Q}{\Gamma;\Omega,a^+ \vdash_f Q}a^+L \qquad \frac{\Gamma;\Omega \vdash_f a_-}{\Gamma;\Omega \vdash_f a^-}a^-R \qquad \frac{}{\Gamma[a^-] \vdash_f a_-}a^-L$$

$$\frac{\Gamma \vdash_f [P]}{\Gamma \vdash_f P}focR \qquad \frac{\Gamma[N] \vdash_f P}{\Gamma,N \vdash_f P}focL \qquad \frac{\Gamma \vdash_f P}{\Gamma;\cdot \vdash_f P}\cdot L$$

Figure 1: Focusing Proof Rules

The five judgments of the logic are

Stable	$\Gamma \vdash_f Q$
Right Inversion	$\Gamma;\Omega \vdash_f N$
Left Inversion	$\Gamma;\Omega \vdash_f Q$
Right Focus	$\Gamma \vdash_f [P]$
Left Focus	$\Gamma[N] \vdash_f Q$

and the proof rules for the focusing system are in Figure 1. The f decorating the turnstile is merely to distinguish these judgments from the unfocused linear logic below.

3 Unfocused Language

The unfocused language has syntax

Terms	t	$::=$	$\star \mid x$
Propositions	A, B	$::=$	$A \otimes A \mid A \oplus A \mid 1 \mid 0 \mid F_t A \mid \exists x.A(x) \mid$ $A \multimap A \mid A \mathbin{\&} A \mid \top \mid U_t A \mid \forall x.A(x) \mid p$
Atomic Propositions	p	$::=$	$a^+ \mid a^- \mid q(t)$
Contexts	Δ	$::=$	$\cdot \mid \Delta, A$

It is a straightforward first-order intuitionistic linear logic except for the connectives F_t and U_t. The first-order term language consists only of a distinguished constant $\star$, and first-order variables x. Atomic propositions p include

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B} \multimap R \qquad \frac{\Delta_1 \vdash A \qquad \Delta_2, B \vdash C}{\Delta_1, \Delta_2, A \multimap B \vdash C} \multimap L \qquad \frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \,\&\, B} \,\&R$$

$$\frac{\Delta, A_i \vdash C}{\Delta, A \,\&\, B \vdash C} \,\&L \qquad \frac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B} \otimes R \qquad \frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C} \otimes L \qquad \frac{\Delta \vdash A_i}{\Delta \vdash A \oplus B} \oplus R_i$$

$$\frac{\Delta, A \vdash C \qquad \Delta, B \vdash C}{\Delta, A \oplus B \vdash C} \oplus L \qquad \frac{}{\vdash 1} 1R \qquad \frac{\Delta \vdash C}{\Delta, 1 \vdash C} 1L \qquad \frac{}{\Delta, 0 \vdash C} 0L \qquad \frac{}{p \vdash p} hyp$$

$$\frac{\Delta, q(t) \vdash A}{\Delta \vdash U_t A} UR \qquad \frac{\Delta, A \vdash C}{\Delta, q(t), U_t A \vdash C} UL \qquad \frac{\Delta \vdash A}{\Delta, q(t) \vdash F_t A} FR \qquad \frac{\Delta, q(t), A \vdash C}{\Delta, F_t A \vdash C} FL$$

$$\frac{\Delta \vdash A(x)}{\Delta \vdash \forall x.A(x)} \forall R \qquad \frac{\Delta, A(t) \vdash C}{\Delta, \forall x.A(x) \vdash C} \forall L \qquad \frac{\Delta \vdash A(t)}{\Delta \vdash \exists x.A(x)} \exists R \qquad \frac{\Delta, A(x) \vdash C}{\Delta, \exists x.A(x) \vdash C} \exists L$$

Figure 2: Linear Logic Proof Rules

atoms a^+, a^- of both polarities from the focused language, and a distinguished atom $q(t)$ that takes one first-order argument. This latter constitutes the tokens in the token-passing mechanism. There is only one judgment, $\Delta \vdash A$, and its rules are in Figure 2.

It can be seen that the new connectives F_t and U_t are reducible to uses of existing connectives in the sense that

$$F_t A \dashv\vdash q(t) \otimes A$$

$$U_t A \dashv\vdash q(t) \multimap A$$

However, the left rule for $U_t A$ and the right rule for $F_t A$ requires that $q(t)$ be immediately present in the context. These connectives appear to embody just a little bit of focusing reasoning, for example since FR effectively forces the sequence of the two rules $\otimes R$ and *hyp* to happen in one uninterrupted sequence. One might fairly say that the moral of this work is exactly that these two 'microfocused' connectives F_t and U_t are all that is necessary to explain focusing globally.

We observe that the unfocused logic is internally sound and complete, in the sense that identity and cut admissibility results hold. The proofs of these results are thoroughly standard.

Lemma 3.1 *The following rules are admissible, i.e. whenever their premises are provable, the conclusion is provable.*

$$\frac{}{A \vdash A} id \qquad \frac{\Delta_1 \vdash A \qquad \Delta_2, A \vdash B}{\Delta_1, \Delta_2 \vdash B} cut$$

The fact of cut admissibility is used tacitly throughout many of the proofs below, when known provable equivalences are cut into sequents to yield new derivations.

4 Token-Passing Translation

The aim of this section is to specify the translation from the focused language to the unfocused language that preserves proof search behavior.

Let us first try to sketch out an intuition for how the translation works. Compared to ordinary sequent calculus derivations, focusing prohibits many proof attempts; those that don't eagerly decompose asynchronous connectives, and those that begin working on synchronous connective and then switch attention to something else. We will represent the ability of the prover to choose what to do next by a linear token, a linear hypothesis of a propositional atom, say $q(\star)$, being present in the context. So the translation of a typical stable sequent $\Gamma \vdash P$ (ignoring for now the possibility that the right-hand side is a judgmental atom a_+) in the focusing calculus will be something like

$$\Gamma^\bullet, q(\star) \vdash P^\bullet$$

where $\Gamma^\bullet$ and $P^\bullet$ are some translated versions of Γ and P respectively.

But now to focus on a negative proposition in Γ or the positive conclusion P is to give up our freedom to work on any proposition we like, so we should have to *spend* the token $q(\star)$. We want therefore to prefix every proposition in Γ with $U_\star$ (for the UL rule read bottom-up consumes the token $q(t)$) and prefix the proposition P with $F_\star$ (for the $F_t R$ rule read bottom-up likewise consumes a token). Therefore we instead now speculate that $\Gamma \vdash P$ is interpreted by

$$U_\star \Gamma^\bullet, q(\star) \vdash F_\star P^\bullet$$

Imagine now simulating the right focus rule $focR$, by applying the inference rule FR:

$$\frac{U_\star \Gamma^\bullet \vdash P^\bullet}{U_\star \Gamma^\bullet, q(\star) \vdash F_\star P^\bullet}$$

Even though we are working now in an unfocused logic, there is nothing we can do except continue to decompose $P^\bullet$, for every other proposition is guarded by a $U_\star$ whose decomposition requires the immediate presence of a missing linear token.

The entire translation is an elaboration of this basic idea, that proof search can be finely controlled by the presence of linear tokens. When polarity shifts that interrupt focus are reach, tokens are once again produced, simulating the renewed freedom in the focusing system to choose another proposition to begin working on. When asynchronous phases are begun, first-order quantifiers and the term argument to the atom q are used to ensure that the context of asynchronous positive hypotheses is decomposed in a unique order. One copy of the token $q(\star)$ is available when the sequent is stable (i.e. there is no active focus,

X	X^t	$X^\bullet$	$X^\triangleright$
$P_1 \otimes P_2$	$\exists x.U_x P_1^t \otimes P_2^x$	$P_1^\bullet \otimes P_2^\bullet$	$P_1^\triangleright \otimes P_2^\triangleright$
$P \multimap N$	$\forall x.U_x P^t \multimap N^x$	$P^\bullet \multimap N^\bullet$	$P^\triangleright \multimap N^\triangleright$
$P_1 \oplus P_2$	$P_1^t \oplus P_2^t$	$P_1^\bullet \oplus P_2^\bullet$	$P_1^\triangleright \oplus P_2^\triangleright$
$N_1 \mathbin{\&} N_2$	$N_1^t \mathbin{\&} N_2^t$	$N_1^\bullet \mathbin{\&} N_2^\bullet$	$N_1^\triangleright \mathbin{\&} N_2^\triangleright$
1	$q(t)$	1	1
0	0	0	0
$\top$	$\top$	$\top$	$\top$
$\downarrow N$	$F_t U_\star N^\bullet$	$N^\star$	$U_\star N^\triangleright$
$\uparrow P$	$U_t F_\star P^\bullet$	$P^\star$	$F_\star P^\triangleright$
a^+	$F_t a^+$	a^+	a^+
a^-	$U_t a^-$	a^-	a^-

Figure 3: Token-Passing Translation

and no available asynchronous decomposition to be done) and one copy of $q(x)$ for some variable x is available when we are in the middle of an asynchronous phase. In the middle of a synchronous phase, no tokens are available.

That being said, we now formally specify the translation. Let X stand for either a positive proposition P or a negative proposition N. The translation consists of two functions, the *asynchronous translation* X^t of X with respect to a first-order term t, and the *synchronous translation* $X^\bullet$ of X. The idea is that when translating a positive proposition on the right of the turnstile, i.e., where it is synchronous, we use the synchronous translation, and on the left, we use the asynchronous translation, and vice-versa for negative propositions. These functions are defined in Figure 3.

The intuition behind P^t on the left (resp. N^t on the right) is that it will do some asynchronous work and eventually yield the token $q(t)$. The intuition behind $P^\bullet$ on the right (resp. $N^\bullet$ on the left) is that it will do some synchronous work, pass to an asynchronous phase, and at the end of that, yield the token $q(\star)$. Note that the translation of a proposition appearing in the 'wrong place', for instance an occurrence of P^t on the *right* of the turnstile, is still, from the point of view of the unfocused host logic, a completely meaningful proposition — it will simply not have the tightly controlled proof search behavior we would expect it it to have if it were on the other side. The ability to work in this expanded language — which can smoothly express varying degrees of violation of focusing discipline — will prove to be of central importance below in Lemma 4.4 and its consequences. The translation $X^\triangleright$ appears as a convenient variant of $X^\bullet$ beginning with Lemma 4.5 below.

All three translations are lifted pointwise to contexts Γ. For an ordered context $\Omega = (P_n, \ldots, P_1)$, make the following definitions, for fresh term variables

$x_1, \ldots, x_{n-1}$.

$$\begin{array}{rcccccc} \Omega^\star & = & U_{x_{n-1}} P_n^\star, & U_{x_{n-2}} P_{n-1}^{x_{n-1}}, & \ldots, & U_{x_1} P_2^{x_2}, & P_1^{x_1} \\ U_t\Omega^\star & = & U_{x_{n-1}} P_n^\star, & U_{x_{n-2}} P_{n-1}^{x_{n-1}}, & \ldots, & U_{x_1} P_2^{x_2}, & U_t P_1^{x_1} \\ \Omega^\bullet & = & P_n^\bullet, & P_{n-1}^\bullet, & \ldots, & P_2^\bullet, & P_1^\bullet \\ \Omega^\triangleright & = & P_n^\triangleright, & P_{n-1}^\triangleright, & \ldots, & P_2^\triangleright, & P_1^\triangleright \end{array}$$

The mechanism in the definition of $\Omega^\star$ effectively simulates an ordered context in linear logic: only the rightmost proposition is initially available to decompose, which eventually yields a token $q(x_1)$ to be spent to unlock the second-to-rightmost proposition, which eventually yields a token $q(x_2)$ to unlock the next, and so on, until the leftmost yields a token $q(\star)$.

The expression $U_t\Omega^\star$ is an abuse of notation, but it becomes useful below. The difference between it and $\Omega^\star$ is that we have here prefixed U_t on the rightmost proposition P^{x_1}. Abusing notation slightly further define $F_\star(a_-)^\bullet = F_\star(a_-)^\triangleright = a^-$ and $U_\star(a_+)^\bullet = U_\star(a_+)^\triangleright = a^+$.

We now show that this translation is a faithful simulation of focusing.

Theorem 4.1 *Suppose x is a fresh term variable. The following isomorphisms of proofs hold:*

$$\begin{array}{rcc} \Gamma \vdash_f Q & \cong & U_\star\Gamma^\bullet, q(\star) \vdash F_\star Q^\bullet \\ \Gamma \vdash_f [P] & \cong & U_\star\Gamma^\bullet \vdash P^\bullet \\ \Gamma; [N] \vdash_f Q & \cong & U_\star\Gamma^\bullet, N^\bullet \vdash F_\star Q^\bullet \\ \Gamma; \Omega \vdash_f N & \cong & U_\star\Gamma^\bullet, U_x\Omega^\star \vdash N^x \\ \Gamma; \Omega \vdash_f Q & \cong & U_\star\Gamma^\bullet, \Omega^\star \vdash F_\star Q^\bullet \end{array}$$

That is, for any Γ, Q, there is a bijection between the set of proofs of $\Gamma \vdash_f Q$ and the set of proofs of $U_\star\Gamma^\bullet, q(\star) \vdash F_\star P^\bullet$, and so on.

Proof By defining the bijections inductively over the structure of the possible derivations.

For instance, consider a focused proof ending in $\Gamma \vdash_f Q$. It is either an application of the $focL$ rule, with a premise of the form $\Gamma'[N] \vdash_f Q$ where $\Gamma', N = \Gamma$, or an application of $focR$, with the premise $\Gamma \vdash_f [P]$ where $Q = P$. Likewise, any linear logic proof ending in $U_\star\Gamma^\bullet, q(\star) \vdash F_\star Q^\bullet$ is either an application of UL that (read bottom-up) consumes the token $q(\star)$ and has a premise of the form $U_\star(\Gamma')^\bullet, N^\bullet \vdash F_\star P^\bullet$ where $\Gamma', N = \Gamma$ or else an application of FR that consumes $q(\star)$ and has a premise $U_\star\Gamma^\bullet \vdash P^\bullet$. In either event, we apply the induction hypothesis to get a bijection of the appropriate families of subderivations. We can see that only possible derivations ending in $U_\star\Gamma^\bullet \vdash P^\bullet$ (resp. $(U_\star\Gamma^\bullet, N^\bullet \vdash F_\star Q^\bullet)$,$(U_\star\Gamma^\bullet, U_x\Omega^\star \vdash N^x)$,$(U_\star\Gamma^\bullet, \Omega^\star \vdash F_\star Q^\bullet)$) are those that decompose $P^\bullet$ (resp. $(N^\bullet)$,(N^x),(the rightmost proposition of $\Omega^\star$)), and that those decompositions correspond exactly to the focused decomposition rule.

The remaining cases we must consider are therefore divided up according to the inference rules of the focusing system. We show some representative cases.

Case:

$$\frac{\Gamma; \Omega \vdash_f P}{\Gamma; \Omega \vdash_f {\uparrow} P} {\uparrow} R$$

Its conclusion translates to

$$U_\star \Gamma^\bullet, U_x \Omega^\star \vdash U_x F_\star P^\bullet$$

which can, and can only (by reasoning about availability of tokens) be proved by a derivation ending with

$$\dfrac{\dfrac{U_\star \Gamma^\bullet, \Omega^\star \vdash F_\star P^\bullet}{U_\star \Gamma^\bullet, U_x \Omega^\star, q(x) \vdash F_\star P^\bullet}\, UL}{U_\star \Gamma^\bullet, U_x \Omega^\star \vdash U_x F_\star P^\bullet}\, UR$$

but the top line of this partial derivation is exactly the translation of the premise $\Gamma; \Omega \vdash P$ of the focusing rule.

Case:

$$\dfrac{\Gamma; \Omega, P_1, P_2 \vdash_f Q}{\Gamma; \Omega, P_1 \otimes P_2 \vdash_f Q}\, \otimes L$$

Its conclusion translates to

$$U_\star \Gamma^\bullet, U_y \Omega^\star, \exists y. U_y P_1^x \otimes P_2^y \vdash F_\star Q^\bullet$$

which can, and can only (by reasoning about availability of tokens) be proved by a derivation ending with

$$\dfrac{\dfrac{U_\star \Gamma^\bullet, U_x \Omega^\star, U_y P_1^x, P_2^y \vdash F_\star Q^\bullet}{U_\star \Gamma^\bullet, U_x \Omega^\star, U_y P_1^x \otimes P_2^y \vdash F_\star Q^\bullet}\, \otimes L}{U_\star \Gamma^\bullet, U_x \Omega^\star, \exists y. U_y P_1^x \otimes P_2^y \vdash F_\star Q^\bullet}\, \exists L$$

but the top line of this partial derivation is exactly the translation of the premise $\Gamma; \Omega, P_1, P_2 \vdash Q$ of the focusing rule up to renaming of term variables, for note that $U_x \Omega^\star, U_y P_1^x, P_2^y = (\Omega, P_1, P_2)^\star$.

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We observe several useful identities among the various linear logic operators. Lemma 4.2 are related to the triangle equalities of adjunctions in category theory (for F_t is a left adjoint to U_t) and Lemma 4.3 exhibit how the various positive and negative connectives commute with F_t and U_t.

Lemma 4.2

1. $F_t A \dashv\vdash F_t U_t F_t A$
2. $U_t A \dashv\vdash U_t F_t U_t A$
3. $F_t A \dashv\vdash \exists x. F_x U_x F_t A$

4. $U_t A \dashv\vdash \forall x. U_x F_x U_t A$

Lemma 4.3

1. $F_t(A_1 \otimes A_2) \dashv\vdash F_t A_1 \otimes A_2 \dashv\vdash A_1 \otimes F_t A_2$
2. $U_t(A \multimap B) \dashv\vdash F_t A \multimap B \dashv\vdash A \multimap U_t B$
3. $F_t(A_1 \oplus A_2) \dashv\vdash F_t A_1 \oplus F_t A_2$
4. $U_t(B_1 \,\&\, B_2) \dashv\vdash U_t B_1 \,\&\, U_t B_2$
5. $F_t 0 \dashv\vdash 0$
6. $U_t \top \dashv\vdash \top$

Next we state a central fact about the translation that shows that the asynchronous and synchronous translations of each connective are compatible. Given the prior intuitions that the asynchronous translation X^t of a proposition serves as a promise to *eventually* yield the token $q(t)$, this result says that we can alternatively, without risking provability, ask to receive the token now, *before* asynchronous decomposition of X. Conversely, where a translated context contained hypotheses such as $U_\star N^\bullet$ that require a token be spent immediately, the lemma says that we can equivalently allow the token to be spent later, by hypothesizing $N^\star$. In other words, we are showing that certain small violations of the token-passing protocol — small relaxations of focusing discipline — do not change provability of a sequent. It is by connecting together these small steps that we see further below that full focusing is sound and complete.

Lemma 4.4

$$P^t \dashv\vdash F_t P^\bullet$$

$$N^t \dashv\vdash U_t N^\bullet$$

Proof By straightforward induction on P, N, constructing small derivations in linear logic for each connective.

For example, consider $P = P_1 \otimes P_2$. We reason that

$(P_1 \otimes P_2)^t = \exists x. U_x P_1^t \otimes P_2^x$	
$\dashv\vdash \exists x. U_x P_1^t \otimes F_x P_2^\bullet$	i.h.
$\dashv\vdash \exists x. U_x F_t P_1^\bullet \otimes F_x P_2^\bullet$	i.h.
$\dashv\vdash \exists x. F_x U_x F_t P_1^\bullet \otimes P_2^\bullet$	Lemma 4.3
$\dashv\vdash (\exists x. F_x U_x F_t P_1^\bullet) \otimes P_2^\bullet$	
$\dashv\vdash F_t P_1^\bullet \otimes P_2^\bullet$	Lemma 4.2
$\dashv\vdash F_t(P_1^\bullet \otimes P_2^\bullet)$	Lemma 4.3
$\dashv\vdash F_t(P_1 \otimes P_2)^\bullet$	

■

It will become useful to switch over to thinking about *simple token-passing translation* $X^{\triangleright}$, which only ever uses the token $q(\star)$. It is now easy to see that it is provably equivalent to the existing translation.

Lemma 4.5

$$P^{\bullet} \dashv\vdash P^{\triangleright}$$

$$N^{\bullet} \dashv\vdash N^{\triangleright}$$

Proof By induction on P, N, applying Lemma 4.4 at the shift operators. ∎

Notice how this lemma provides an interpretation of $\downarrow$ as $U_\star$ and of $\uparrow$ as $F_\star$.

5 Focused Cut-Elimination and Identity

This section establishes the internal soundness and completeness of the focusing logic as a corollary of the translation above. It is not necessary to understand these results to understand Section 6 below.

First, we can see how all of these five stages of a focused proof in Theorem 4.1 correspond to unfocused sequents using only the simplified translation.

Corollary 5.1

$$\begin{array}{ccc}
U_\star\Gamma^{\bullet}, q(\star) \vdash F_\star Q^{\bullet} & \Leftrightarrow & U_\star\Gamma^{\triangleright}, q(\star) \vdash F_\star Q^{\triangleright} \\
U_\star\Gamma^{\bullet} \vdash P^{\bullet} & \Leftrightarrow & U_\star\Gamma^{\triangleright} \vdash P^{\triangleright} \\
U_\star\Gamma^{\bullet}, N^{\bullet} \vdash F_\star Q^{\bullet} & \Leftrightarrow & U_\star\Gamma^{\triangleright}, N^{\triangleright} \vdash F_\star Q^{\triangleright} \\
U_\star\Gamma^{\bullet}, U_x\Omega^{\star} \vdash N^{x} & \Leftrightarrow & U_\star\Gamma^{\triangleright}, q(\star), \Omega^{\triangleright} \vdash N^{\triangleright} \\
U_\star\Gamma^{\bullet}, \Omega^{\star} \vdash F_\star Q^{\bullet} & \Leftrightarrow & U_\star\Gamma^{\triangleright}, q(\star), \Omega^{\triangleright} \vdash F_\star Q^{\triangleright}
\end{array}$$

Proof For the first three cases, by repeated application of Lemma 4.5.

For the first of the two remaining cases, we must show (after using the fact that $N^x \dashv\vdash U_x N^{\triangleright}$ and therefore yields $q(x)$ into the context)

$$U_x\Omega^{\star}, q(x) \qquad q(\star), \Omega^{\triangleright}$$

are equivalent, assuming x is a fresh term variable. This can be accomplished inductively by showing that the contexts

$$U_x P^{t}, q(x) \qquad q(t), P^{\triangleright}$$

are equivalent, or in other words

$$\exists x. U_x P^{t} \otimes q(x) \dashv\vdash q(t) \otimes P^{\triangleright}$$

which amounts, by appealing to Lemma 4.4 and Lemma 4.5, to

$$\exists x. F_x U_x F_t P^{\triangleright} \dashv\vdash F_t P^{\triangleright}$$

which directly follows from Lemma 4.2.

For the second case, we must show that the contexts $\Omega^\star$ and $q(\star), \Omega^\triangleright$ are equivalent. The reasoning is largely the same. The rightmost proposition of $\Omega^\star$, call it $P_1^{x_1}$, is equivalent to the proposition $F_x P^\triangleright$ by Lemma 4.4 and Lemma 4.5, so it can create a token to transform the second-to-rightmost proposition $U_{x_1} P_2^{x_2}$ into $F_{x_1} U_{x_1} P_2^{x_2}$, which since x_1 is fresh is equivalent to $\exists x_1 . F_{x_1} U_{x_1} P_2^{x_2}$ which is equivalent to $P_2^{x_2}$ and we repeat until the last token yielded is $q(\star)$, and $\Omega^\star$ is progressively transformed into $\Omega^\triangleright$. ■

This then means that we can read off identity and cut admissibility results for the focusing logic, for they translate directly to invocations of cut and identity in the underlying unfocused logic.

Corollary 5.2

$$\frac{}{\uparrow P \vdash_f P} \qquad \frac{}{N \vdash_f \downarrow N}$$

$$\frac{\Gamma_1 \vdash_f [P] \qquad \Gamma_2; \Omega, P \vdash_f Q}{\Gamma_1, \Gamma_2; \Omega \vdash_f Q} \qquad \frac{\Gamma_1; \Omega \vdash_f N \qquad \Gamma_2[N] \vdash_f Q}{\Gamma_1, \Gamma_2; \Omega \vdash_f Q}$$

Proof Use Theorem 4.1 and Corollary 5.1, and the admissibility of identity and cut, on the following derivations.

$$\frac{\dfrac{}{F_\star P^\triangleright \vdash F_\star P^\triangleright}\,id}{U_\star F_\star P^\triangleright, q(\star) \vdash F_\star P^\triangleright}\,UL \qquad \frac{\dfrac{}{U_\star N^\triangleright \vdash U_\star N^\triangleright}\,id}{U_\star N^\triangleright, q(\star) \vdash F_\star U_\star N^\triangleright}\,UL$$

$$\frac{U_\star \Gamma_1^\triangleright \vdash P^\triangleright \quad U_\star \Gamma_2^\triangleright, q(\star), \Omega^\triangleright, P^\triangleright \vdash F_\star Q^\triangleright}{U_\star \Gamma_1^\triangleright, U_\star \Gamma_2^\triangleright, q(\star), \Omega^\triangleright \vdash Q^\triangleright}\,cut \qquad \frac{U_\star \Gamma_1^\triangleright, q(\star), \Omega^\triangleright \vdash N^\triangleright \quad U_\star \Gamma_2^\triangleright, N^\triangleright \vdash F_\star Q^\triangleright}{U_\star \Gamma_1^\triangleright, U_\star \Gamma_2^\triangleright, q(\star), \Omega^\triangleright \vdash Q^\triangleright}\,cut$$

■

6 Completeness of Focusing

Finally, we wish to show that focusing is complete relative to unfocused proofs.

Let Ξ stand for a context of propositions X of either polarity. Three further translations are defined in Figure 4. The covariant defocusing translation $X^{||}$ takes positive propositions to positive, and negative to negative. The contravariant defocusing translation $X^\sim$ takes positive to negative, and negative to positive. The erasure function X° takes polarized propositions of either polarity to unpolarized propositions A by simply erasing all shift operators. Further define

X	$\overleftarrow{X}$	$\overrightarrow{X}$
P	$P^\sim$	$P^{\|\|}$
N	$N^{\|\|}$	$N^\sim$

X	$X^{\sim}$	$X^{\|\|}$	X°
$P_1 \otimes P_2$	$\uparrow(\downarrow P_1^{\sim} \otimes \downarrow P_2^{\sim})$	$\downarrow\uparrow P_1^{\|\|} \otimes \downarrow\uparrow P_2^{\|\|}$	$P_1^{\circ} \otimes P_2^{\circ}$
$P \multimap N$	$\downarrow(\downarrow P^{\sim} \multimap \uparrow N^{\sim})$	$\downarrow\uparrow P^{\|\|} \multimap \uparrow\downarrow N^{\|\|}$	$P^{\circ} \multimap N^{\circ}$
$P_1 \oplus P_2$	$\uparrow(\downarrow P_1^{\sim} \oplus \downarrow P_2^{\sim})$	$\downarrow\uparrow P_1^{\|\|} \oplus \downarrow\uparrow P_2^{\|\|}$	$P_1^{\circ} \oplus P_2^{\circ}$
$N_1 \& N_2$	$\downarrow(\uparrow N_1^{\sim} \& \uparrow N_2^{\sim})$	$\uparrow\downarrow N_1^{\|\|} \& \uparrow\downarrow N_2^{\|\|}$	$N_1^{\circ} \& N_2^{\circ}$
1	$\uparrow 1$	1	1
0	$\downarrow 0$	0	0
$\top$	$\downarrow \top$	$\top$	$\top$
$\downarrow N$	$N^{\|\|}$	$N^{\sim}$	N°
$\uparrow P$	$P^{\|\|}$	$P^{\sim}$	P°
a^+	$\uparrow a^+$	a^+	a^+
a^-	$\downarrow a^-$	a^-	a^-

Figure 4: Pause and Erasure Translations

and lift these translations pointwise to Ξ. The function $\overleftarrow{X}$ always outputs negative propositions, and $\overrightarrow{X}$ always outputs positive.

The point of the defocusing translations is to insert so many shifts that focused proof on the result of translating a sequent is the same as unfocused proof of the polarization-erasure of that sequent. This fact is captured by the following result.

Lemma 6.1 *There is an isomorphism of proofs* $\Xi^{\circ} \vdash X^{\circ} \cong \overleftarrow{\Xi} \vdash_f \overrightarrow{X}$.

Proof By straightforward induction, paying attention to how focusing discipline constrains proofs of $\overleftarrow{\Xi} \vdash_f \overrightarrow{X}$. ■

Here is the key lemma, which says that any polarized proposition is essentially equivalent to the unfocused version of it.

Lemma 6.2

1. $F_\star P^{\triangleright} \dashv\vdash F_\star P^{||\triangleright}$
2. $U_\star N^{\triangleright} \dashv\vdash U_\star N^{||\triangleright}$
3. $F_\star U_\star N^{\triangleright} \dashv\vdash F_\star N^{\sim\triangleright}$
4. $U_\star F_\star P^{\triangleright} \dashv\vdash U_\star P^{\sim\triangleright}$

Proof By straightforward induction, building small linear logic derivations. For example, consider case 1 with $P = P_1 \otimes P_2$. Then we must show $F_\star(P_1 \otimes P_2)^{\triangleright} \dashv\vdash F_\star(P_1 \otimes P_2)^{||\triangleright}$. But this follows by reasoning

$F_\star(P_1 \otimes P_2)^\triangleright \dashv\vdash F_\star(P_1^\triangleright \otimes P_2^\triangleright)$	Def'n of $X^\triangleright$
$\dashv\vdash (F_\star P_1^\triangleright \otimes P_2^\triangleright)$	Lemma 4.3
$\dashv\vdash (F_\star P_1^{\|\|\triangleright} \otimes P_2^\triangleright)$	i.h.
$\dashv\vdash (F_\star U_\star F_\star P_1^{\|\|\triangleright} \otimes P_2^\triangleright)$	Lemma 4.3
$\dashv\vdash (U_\star F_\star P_1^{\|\|\triangleright} \otimes F_\star P_2^\triangleright)$	Lemma 4.3
$\dashv\vdash (U_\star F_\star P_1^{\|\|\triangleright} \otimes F_\star P_2^{\|\|\triangleright})$	i.h.
$\dashv\vdash (U_\star F_\star P_1^{\|\|\triangleright} \otimes F_\star U_\star F_\star P_2^{\|\|\triangleright})$	Lemma 4.3
$\dashv\vdash F_\star(U_\star F_\star P_1^{\|\|\triangleright} \otimes U_\star F_\star P_2^{\|\|\triangleright})$	Lemma 4.3
$\dashv\vdash F_\star(\downarrow\uparrow P_1^{\|\|} \otimes \downarrow\uparrow P_2^{\|\|})^\triangleright$	Def'n of $X^\triangleright$
$\dashv\vdash F_\star(P_1 \otimes P_2)^{\|\|\triangleright}$	Def'n of $X^{\|\|}$

■

Now we can state and prove the completeness of focusing for an entire stable sequent: if the sequent's erasure has an unfocused proof, then the sequent has a focused proof.

Theorem 6.3 *If $\Gamma^\circ \vdash P^\circ$, then $\Gamma \vdash_f P$.*

Proof Suppose $\Gamma^\circ \vdash P^\circ$. By Lemma 6.1, we have $\Gamma^{||} \vdash_f P^{||}$. By Theorem 4.1 and Lemma 4.5, we get $U_\star\Gamma^{||\triangleright}, q(\star) \vdash F_\star P^{||\triangleright}$. By repeated application of Lemma 6.2, we get $U_\star\Gamma^\triangleright, q(\star) \vdash F_\star P^\triangleright$. Going back in the opposite direction with Theorem 4.1 and Lemma 4.5 we arrive at $\Gamma \vdash_f P$ as required. ■

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